An Almost Linear Time 2.8334-Approximation Algorithm for the Disc Covering Problem ⋆

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Abstract. The disc covering problem asks to cover a set of points on the plane with a minimum number of fix-sized discs. We develop an $O(n(\log n)^2(\log\log n)^2)$ deterministic time 2.8334-approximation algorithm for this problem. Previous approximation algorithms [7, 3, 6], when used to achieve the same approximation ratio for the disc covering problem, will have much higher time complexity than our algorithms.

1 Introduction

The disc covering problem is to find a minimum number of discs of a prescribed radius r to cover a given set of points on the plane. This problem has many applications in areas such as image processing, wireless communication and patten recognition. It was proved to be NP-hard [4]. The first approximation algorithm was derived by Hochbaum and Maass [7]. Their algorithm has computational time $n^{2\lfloor l\sqrt{2}\rfloor^2}$ for approximation ratio $(1+\frac{1}{l})^2$, where $l > 0$ is an integer accuracy control parameter. This approximation algorithm was further improved to $n^{6\lfloor l\sqrt{2}\rfloor}$ in [3, 6]. The high computational complexity of those polynomial time approximation algorithms make them impractical for implementation in practice. Therefore, it is interesting, but challenging, to design faster polynomial time approximation algorithms for the disc covering problem.

In this paper we try to find faster approximation algorithms for the disc covering problem with some reasonably small approximation ratios. We derive an almost linear time approximation algorithm for the disc covering problem. This algorithm runs in $O(n(\log n)^2(\log\log n)^2)$ time with a 2.8334-approximation ratio. Previous approximation algorithms [7, 3, 6] will have much higher time complexity than our algorithms, when they are used to achieve the same 2.8334-approximation ratio for the disc covering problem. An $O(n)$ time $3(1+\beta)$-approximation algorithm for the two-dimensional disc covering problem was shown in [5]. We generalize this linear time approximation algorithm to any fixed d-dimensional space by using the concept of Borsuk number, which is the

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minimum number of d-dimensional balls of radius r to fill a d-dimensional ball of a radius that is slightly larger than r.

We develop some novel method to cover the points in the local region, which is roughly occupied by one disc. Instead of covering each local region by three discs like [5], we let two local regions share one disc in some cases. Our method involves the nontrivial algorithm by Chan [1] for covering points with two fixed size discs, and another interesting algorithm by Meggido and Supowit [9] for covering points with one fixed size disc.

2 Notations and Shifting Strategy

Given a set of input points P and a radius r, Let $o(P)$ denote the minimum number of discs of radius r to cover all the points in P. For any given approximation algorithm A, which outputs $A(P)$ many discs of radius r to cover P, we shall have $A(P) \geq o(P)$. The approximation ratio of the algorithm A is defined as $max_P \frac{A(P)}{o(P)}$. Let $C_r(p)$ be a disc with a radius r and centered at the point p.

For two points p_1, p_2 in the d-dimensional Euclidean space R^d, $dist(p_1, p_2)$ is the Euclidean distance between p_1 and p_2. For a set $A \subseteq R^d$, $dist(p_1, A) = min_{q \in A} dist(p_1, q)$.

A point in R^d is a grid point if all of its coordinates are integers. For a d-dimensional point $p = (i_1, i_2, \cdots, i_d)$ and $a > 0$, define $grid_a(p)$ to be the set $\{(x_1, x_2, \cdots, x_d) | i_j - \frac{a}{2} \leq x_j < i_j + \frac{a}{2}, j = 1, 2, \cdots, d\}$, which is a half open and half close a^d-volume d-dimensional cubic region. For $a_1, \cdots, a_d > 0$, a $(a_1, \cdots, a_d)$-grid point is a point $(i_1 a_1, \cdots, i_d a_d)$ for some integers $i_1, \cdots, i_d$. For a ball B in R^d, let $r(B)$ denote the radius of B, $center(B)$ denote the center of B, and $extend_\delta(B)$ be the ball with the same center as B but with a larger radius $(1+\delta) r(B)$ for $\delta > 0$.

We will use the shifting method developed by Hochbaum and Maass [7] to handle some subcases of our algorithms. For completeness, we give the description of the shifting method to deal with the disc covering problem. Let $l > 0$ be the integer parameter to control the accuracy of approximation. Assume that all the points in the input set P are in a region B, and discs of radius r are used to cover P. The region B is partitioned into vertical strips of width $2r$, $B_1, B_2, \cdots, B_k$. Without loss of generality, we assume that the union of every two consecutive strips intersects P (otherwise, the covering problem can be decomposed into two independent covering problems). This indicates that the number of strips is $O(|P|)$. Group every l consecutive strips into a wider strip of width $2rl$. In other words, each wider strip is $L_i = B_i \cup B_{i+1} \cdots \cup B_{i+l-1}$ for $i = 1, \cdots, k-l+1$, and $L_i = B_i \cup B_{i+1} \cdots \cup B_k$ for $i = k-l+2, \cdots, k$. We also define $L_i^0 = B_1 \cup B_2 \cdots B_{i-1}$, which is the union of first $i-1$ blocks. The i-th shifted case has a set of wider strips $P_i = \{L_i^0, L_i, L_{i+l}, L_{i+2l}, \cdots, L_{i+t_i l}\}$, forming a partition for B ($B = L_i^0 \cup L_i \cup L_{i+l} \cup \cdots L_{i+tl}$).

Define $opt_P(D)$ to be the set of the discs in an optimal solution for covering the points of the set P in the region D. Let $d_i = \sum_{L \in P_i} |opt_P(L)|$. The crucial property of the shifting method [7] is that $\sum_{i=1}^{l} d_i \leq (1+l)|opt_P(B)|$. This implies that $\min_{1\le i\le l} d_i \le (1+\frac{1}{l})|opt_P(B)|$. Assume we have a local algorithm A for solving each local area L_i with approximation ratio AP_A. The solution of the algorithm A for the partition P_i is $s_i = \sum_{L\in P_i} A(L) \le AP_A \cdot d_i$. The shifting method SA applies the algorithm A for each partition P_i, $1 \le i \le l$, and outputs the result $SA(B) = \min_{i=1}^{l} s_i$. Therefore, $SA(B) \le (1+\frac{1}{l}) \cdot AP_A \cdot opt_P(B)$.

Theorem 1 ([7]). *Assume that a local algorithm A has an approximation ratio AP_A for the disc covering problem. Then, the approximation ratio AP_{SA} of the shifting method utilizing A satisfies $AP_{SA} \le (1+\frac{1}{l})AP_A$.*

For the d-dimensional ball covering problem, repeating the shifting method at the directions of d-axis, we can get the following result:

Theorem 2 ([7]). *Assume that a local algorithm A has an approximation ratio AP_A for the disc covering problem in the region of volume l^d. Then, the approximation ratio AP_{SA} of the shifting method utilizing A satisfies $AP_{SA} \le (1+\frac{1}{l})^d$. Furthermore, its computational time is $O(ndlT_d(l))$, where $T_d(l)$ is the computational time for the optimal solution in a local d-dimensional cubic region of volume $(2rl)^d$.*

3 Borsuk Number and Disc Covering

For any given dimension $d > 0$ and any given radius $r > 0$, let the Borsuk number $B(d)$ be the minimum number of d-dimensional balls of radius r in R^d that can fill a d-dimensional ball of radius $r+\delta$ in R^d for some $\delta > 0$. It is well-known that $B(2) = 3$ and $B(3) = 4$. Given a set P of points in R^d, the d-dimensional disc (or ball) covering problem is to find a minimum number of d-dimensional discs (or balls) of radius r to cover all the points in P.

Lemma 1. *For any given dimension $d > 0$ and any fixed parameter $\delta > 0$, there is an $O(n)$ time algorithm that, given a set of n points P in R^d, returns a set of points $Sketch_\delta(P) \subseteq P$ such that for every $(\delta,\cdots,\delta)$-grid point q, $grid_\delta(q) \cap P \ne \emptyset$ iff $|grid_\delta(q) \cap Sketch_\delta(P)| = 1$.*

Proof. Set $Sketch_\delta(P) = \emptyset$. Unmark all the $(\delta,\delta,\cdots,\delta)$-grid points. For each point p in P, find the $(\delta,\delta,\cdots,\delta)$-grid point q such that $p \in grid_\delta(q)$. If q is not marked, add p to $Sketch_\delta(P)$ and mark q. This takes $O(n)$ time.

Theorem 3. *Given a fixed dimension $d > 0$, a constant $\beta > 0$ and a radius $r > 0$, for any set P of n points in R^d, we have two algorithms for covering $Sketch_\delta(P)$ for some constant $\delta > 0$ and P, respectively:*

1. *There exists an $O(n)$ time $(1+\beta)$-approximation algorithm for covering all the points in $Sketch_\delta(P)$ with discs of radius r.*
2. *There exists an $O(n)$ time $B(d)(1+\beta)$-approximation algorithm for covering all the points in P with discs of radius r.*

Proof. Select an integer l such that $(1 + \frac{1}{l})^d \leq 1 + \beta$. Assume that $\delta_1 > 0$ is the constant such that a d-dimensional ball of radius $r(1 + \delta_1)$ can be filled by $B(d)$ many d-dimensional balls of radius r. Let $\delta = \frac{r\delta_1}{\sqrt{d}}$. Let Q be the set $Sketch_\delta(P)$ derived from Lemma 1. Apply the shifting method to find the $(1+\beta)$-approximation to the minimum number of balls to cover all the points in Q. By Theorem 2, we can get the $(1 + \beta)$-approximation for covering the points of Q in $O(nlT(l))$ steps, where $T(l)$ is the time in the $(2rl)^d$ region that has at most $(\frac{2rl}{\delta} + 1)^d$ $(\delta, \cdots, \delta)$-grid points. Therefore, it has at most $(\frac{2rl}{\delta} + 3)^d$ points in Q. We use d points to determine the position of a ball in d-dimensional space. Finding the optimal covering for the points of Q in a $(\frac{2rl}{\delta})^d$ region can be done in $O((\frac{2rl}{\delta} + 3)^{2d}) = O((\frac{2\sqrt{d}l}{\delta_1} + 3)^{2d})$ steps for fixed d. This completes the proof for first part of the theorem.

To prove the second the part of the theorem, we continue with the set of balls, denoted by S, obtained by the algorithm for the first part for covering $Sketch_\delta(P)$. By the construction of $Sketch_\delta(P)$, every point p in P is either covered by a ball in S, or it is not covered but is within distance $\sqrt{d}\delta$ to some ball S. We replace each ball D in S by a ball D' of radius $r + \sqrt{d}\delta$ and centered at $center(D)$, i.e., $D' = extend_{\sqrt{d}\delta}(D)$. Let S' denote the new set of those larger balls. Obviously, balls in S' covers P. By the choice of δ, $r+\sqrt{d}\delta = r(1+\delta_1)$. Thus, every ball in S' can be filled with $B(d)$ balls of radius r. Therefore, replacing each ball D' in S' with $B(d)$ ball of radius r that fill D' yields a set of balls of radius r for covering all the points in P. This completes the proof for the second part of the part of the theorem. □

4 A 2.8334-Approximation Algorithm for 2D Covering

We present our main result in this section. We derive a 2.8334 approximation algorithmfor the 2D disc covering problem with almost linear computational time. We will use the linear time algorithm for finding the minimum disc to cover a set of points by Meggido and Supowit [9]. We also use the $O(n(\log n)^2(\log\log n)^2)$-time algorithm developed by Chan, who improved the previous $O(n(\log n)^9)$-time deterministic algorithm by Sharir [10] to check if a set of points on the plane can by covered by two discs. An $O(n(\log n)^2)$time randomized algorithm to check if a set of points on the plane can by covered by two discs was developed by Epstein [2].

Lemma 2. *Let $r > 0$ be a real number. For any constant $1 \geq \alpha > 0$, there are three constants $\alpha \geq \alpha_1, \alpha_2, \alpha_3 > 0$ such that for every disc D_1 of radius $r' = (1 + \beta)r$ with $\beta \leq \alpha_1$ and every disc D_2 of radius r, if $2r' - \frac{r}{2} \leq dist(center(D_1), center(D_2)) \leq 2r' - \alpha_2 r$, then the line through their intersection points has distance at most $r' - \alpha_3 r$ to $center(D_1)$.*

Proof. We first compute the two intersection points of the two discs D_1 and D_2. Without loss of generality, we assume that the center of D_1 is at the origin $(0, 0)$ and the center of D_2 is at x-axis $(d, 0)$, where $d = dist(center(D_1), center(D_2)) \leq$

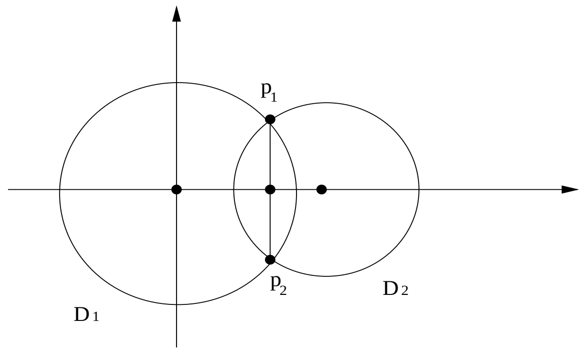

Fig. 1. Two Discs with Intersection

$2r' - \alpha_2 r$. See Figure 1 for an illustration. The two intersection points are at $p_1 = (x, y)$ and $p_2 = (x, -y)$. It is easy to see that $r'^2 - x^2 = r^2 - (d - x)^2$. Thus, we have $x = \frac{r'^2 - r^2 + d^2}{2d}$. It is easy to see that x is maximal when $d = 2r' - \alpha_2 r$. Thus, x is at most

$$\frac{r'^2 - r^2}{2d} + \frac{d}{2} = \frac{(r' - r)(r' + r)}{2d} + \frac{d}{2} \tag{1}$$

$$= \frac{\beta r(2 + \beta)r}{2(2(1 + \beta)r - \alpha_2 r)} + \frac{2(1 + \beta)r - \alpha_2 r}{2} \tag{2}$$

$$= (1 + \frac{\beta(2 + \beta)}{2(2(1 + \beta) - \alpha_2)} + \beta - \frac{\alpha_2}{2})r \tag{3}$$

$$\leq (1 + \frac{3\beta}{2} + \beta - \frac{\alpha_2}{2})r. \tag{4}$$

We use that fact $0 < \beta, \alpha_2 \leq 1$ in the transition from (3) to (4), which gives that $\frac{\beta(2+\beta)}{2(2(1+\beta)-\alpha_2)} \leq \frac{\beta(2+1)}{2(2(1+0)-1)} = \frac{3\beta}{2}$. Assign $\alpha_1 = \frac{\alpha}{4}$ and $\alpha_2 = \alpha$, and $\alpha_3 = \frac{\alpha}{8}$. Thus we have $x \leq (1 + \beta + \frac{3\alpha_1}{2} - \frac{\alpha_2}{2})r \leq (1 + \beta - \frac{\alpha}{8})r = r' - \alpha_3 r$. □

Lemma 3. *Let $r > 0$ and $\beta > 0$. There exist constants ϵ and δ with $\beta \geq \delta > \epsilon > 0$ such that if D is a disc of radius $r' = (1 + \epsilon)r$ and L is a line of the distance $d \leq r' - \delta r$ to the center of D, then the larger part of D on one of two sides of L can be covered by two discs of radius r.*

Proof. We select positive constants ϵ, δ and t that satisfy the conditions below:

$$t = 5 \tag{5}$$

$$\delta = 12t^2\epsilon^{\frac{2}{3}} \tag{6}$$

$$\frac{t}{2} > t^2\epsilon + (12t^2)^2\epsilon^{\frac{1}{3}} \tag{7}$$

$$\min(\beta, \frac{1}{4}) > \delta, \epsilon \tag{8}$$

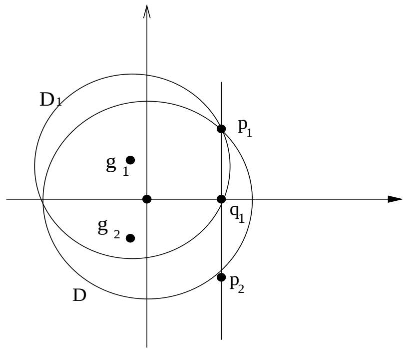

Fig. 2. A Disc with an Intersection Line

It is easy to see the existence of those three constants. Without loss of generality, assume that the center of D is at the origin $(0,0)$ and line L is parallel to y-axis. Let $p_1 = (x, y)$ and $p_2 = (x, -y)$ be the intersection points of disc D and line L (see Figure 2). Since the center of D has distance at most $r' - \delta r$ to line L,

$$x \le r' - \delta r. \tag{9}$$

We put a disc D_1 of radius r with center at point $g_1 = (-t\epsilon r, \delta r)$. We put the second disc D_2 of radius r with center at $g_2 = (-t\epsilon r, -\delta r)$.

We use $L_{\le x}$ to denote the half plane on left side of line L. We will prove that for every point q in the area of $D \cap L_{\le x}$ has either $dist(q, g_1) \le r$ or $dist(q, g_2) \le r$. Let $q = (x_1, y_1)$ be a point on the boundary of D with $-r' \le x_1 \le x$ and $y_1 \ge 0$. Clearly, $x_1^2 + y_1^2 = r'^2$.

$$\begin{aligned}
dist(q, g_1)^2 &= (x_1 + t\epsilon r)^2 + (y_1 - \delta r)^2 & (10)\\
&= x_1^2 + 2t\epsilon r x_1 + (t\epsilon r)^2 + y_1^2 - 2(\delta r)y_1 + (\delta r)^2 & (11)\\
&= r'^2 + 2t\epsilon r x_1 + (t\epsilon r)^2 + (\delta r)^2 - 2(\delta r)y_1 & (12)\\
&= (r + \epsilon r)^2 + 2t\epsilon r x_1 + (t\epsilon r)^2 + (\delta r)^2 - 2\delta r y_1 & (13)\\
&= r^2 + 2\epsilon r^2 + \epsilon^2 r^2 + 2t\epsilon r x_1 + (t\epsilon r)^2 + (\delta r)^2 - 2\delta r y_1 & (14)
\end{aligned}$$

Case 1. $-\frac{r}{2} < x_1 \le x \le r' - \delta r$. This condition implies that $r' + x_1 > \frac{r}{2}$ and $r' - x_1 \ge r' - x \ge \delta r$. Therefore,

$$y_1 = \sqrt{r'^2 - x_1^2} = \sqrt{(r' - x_1)(r' + x_1)} \ge \sqrt{\frac{\delta}{2}} r. \tag{15}$$

Now we prove that the distance between q and g_1 is bounded by r. By (14),

$$\begin{aligned}
dist(q, g_1)^2 &= r^2 + 2\epsilon r^2 + \epsilon^2 r^2 + 2t\epsilon r x_1 + (t\epsilon r)^2 + (\delta r)^2 - 2\delta r y_1 & (16)\\
&\le r^2 + 2\epsilon r^2 + \epsilon^2 r^2 + 2t\epsilon r x_1 + (t\epsilon r)^2 + (\delta r)^2 - 2\sqrt{\frac{1}{2}}\delta^{\frac{3}{2}} r^2 & (17)
\end{aligned}$$

$$\le r^2 + 2\epsilon r^2 + \epsilon^2 r^2 + 2t\epsilon r^2 + (t\epsilon r)^2 + (\delta r)^2 - \sqrt{2}\delta^{\frac{3}{2}} r^2 \tag{18}$$
$$\le r^2 + (2\epsilon + \epsilon^2 + 2t\epsilon + (t\epsilon)^2 + \delta^2 - \sqrt{2}\delta^{\frac{3}{2}})r^2 \tag{19}$$
$$\le r^2 + (6t^2\epsilon - \frac{\sqrt{2}\delta^{\frac{3}{2}}}{2})r^2 \tag{20}$$
$$< r^2 \tag{21}$$

Transition from (16) to (17) is from (15). In transition (19) to (20), we use the fact thats $\epsilon, \epsilon^2, t\epsilon, t^2\epsilon^2 \le t^2\epsilon$ and $\delta^2 \le \frac{\delta^{\frac{3}{2}}}{2} \le \frac{\sqrt{2}\delta^{\frac{3}{2}}}{2}$. Those are from conditions (6) to (8). Transition (20) to (21) is from (6).

Case 2. $-r' \le x_1 \le -\frac{r}{2}$. We have that $2t\epsilon r x_1 \le -t\epsilon r^2$. By (14),

$$dist(q, g_1)^2 \le r^2 + 2\epsilon r^2 + \epsilon^2 r^2 + 2t\epsilon r x_1 + (t\epsilon r)^2 + (\delta r)^2 - 2\delta r y_1 \tag{22}$$
$$\le r^2 + 2\epsilon r^2 + \epsilon^2 r^2 - t\epsilon r^2 + (t\epsilon r)^2 + (\delta r)^2 \tag{23}$$
$$\le r^2 + (t^2\epsilon^2 + \delta^2 - \frac{t\epsilon}{2})r^2 \tag{24}$$
$$\le r^2 + (t^2\epsilon^2 + (12t^2)^2\epsilon^{\frac{4}{3}} - \frac{t\epsilon}{2})r^2 \tag{25}$$
$$\le r^2 + \epsilon(t^2\epsilon + (12t^2)^2\epsilon^{\frac{1}{3}} - \frac{t}{2})r^2 \tag{26}$$
$$< r^2 \tag{27}$$

In transition (23) to (24), we use that fact that $2 + \epsilon \le \frac{t}{2}$, which is derived from (8). Transition (24) to (25) follow from (6). (26) to (27) is from the condition (7).

Case 3. $q_1 = (x, 0)$. Notice that x satisfies (9). We have that

$$dist(q_1, g_1) = (x + t\epsilon r)^2 + (\delta r)^2 \tag{28}$$
$$= x^2 + 2t\epsilon r x + (t\epsilon r)^2 + (\delta r)^2 \tag{29}$$
$$\le (r' - \delta r)^2 + 2t\epsilon r x + (t\epsilon r)^2 + (\delta r)^2 \tag{30}$$
$$\le r'^2 - 2\delta r r' + \delta^2 r^2 + 2t\epsilon r^2 + (t\epsilon r)^2 + (\delta r)^2 \tag{31}$$
$$\le (1+\epsilon)^2 r^2 - 2\delta r^2 + \delta^2 r^2 + 2t\epsilon r^2 + (t\epsilon r)^2 + (\delta r)^2 \tag{32}$$
$$\le r^2 + 2\epsilon r^2 + (\epsilon r)^2 - 2\delta r^2 + 2t\epsilon r^2 + (t\epsilon r)^2 + 2(\delta r)^2 \tag{33}$$
$$\le r^2 + (6t^2\epsilon - \delta)r^2 \tag{34}$$
$$< r^2. \tag{35}$$

For transition (33) to (34), we use the facts that $\epsilon, \epsilon^2, t\epsilon, t^2\epsilon^2 \le t^2\epsilon$ and $2\delta^2 \le \delta$, which are derived from conditions (6) to (8). Transition (34) to (35) is due to condition (6).

Let $B(D \cap L_{\le x})$ be the set of all the points (x_1, y_1) on the boundary of the top half of disc D with $y_1 \ge 0$ and $-r' \le x_1 \le x$. Notice $q_1 = (x, 0)$ as in Case 3. For every point $p = (u, v)$ in the top half of $D \cap L_{\le x}$, i.e., $u^2 + v^2 \le r'^2, -r' \le u \le x$, and $0 \le v$, we have

$$dist(p, g_1) \le \max(\max_{q \in B(D \cap L_{\le x})}(dist(q, g_1), dist(q_1, g_1)) \le r.$$

Thus, we have proved that D_1 covers the top half of $D \cap L_{\leq x}$. Similarly, we can also prove that D_2 covers the bottom half of $D \cap L_{\leq x}$. Therefore, D_1 and D_2 completely cover $D \cap L_{\leq x}$. $\square$

Lemma 4. *Let $r > 0$ be any given real number. For every constant $\alpha \in (0,1)$, there exist constants ϵ and ρ in the interval $[0,\alpha]$ such that for every two discs D_1, D_2 of radius $(1+\epsilon)r$, if $dist(center(D_1), center(D_2)) \leq 4(1+\epsilon)r - \rho r$, then they can be covered by five discs of radius r.*

Proof. We prove the lemma by Lemmas 2 and 3. By Lemma 2, we have three constants α_1, α_2 and α_3 such that for every $0 < \epsilon_1 < \alpha_1$, for any disc A_1 of radius $(1+\epsilon_1)r$ and any disc A_2 of radius r, if the distance of their centers is within the range $[2(1+\epsilon_1)r - \frac{r}{2}, 2(1+\epsilon_1)r - \alpha_2 r]$, the line through the intersection points between A_1 and A_2 has distance at most $(1+\epsilon_1)r - \alpha_3 r$ to the center of A_1. Let $\beta = min\{\frac{1}{20}, \alpha_1, \alpha_3\}$ and $\rho = 2\alpha_2$. By Lemma 3, there are constants ϵ and δ such that $\beta \geq \delta > \epsilon > 0$ satisfy the condition in Lemma 3.

Case 1: The distance of the two centers of D_1 and D_2 is at least $4(1+\epsilon)r - r$. Let p be the middle point on the line through the centers of D_1 and D_2. By the condition of the lemma, $dist(p, center(D_1)) \leq 2(1+\epsilon)r - \frac{\rho}{2}r \leq 2(1+\epsilon)r - \alpha_2 r$. So, $dist(p, center(D_1)) \in [2(1+\epsilon)r - \frac{r}{2}, 2(1+\epsilon)r - \alpha_2 r]$. Let B be the disc of radius r and of center at the middle point p. By Lemma 2, the line through the intersection points between D_1 and B has distance at most $(1+\epsilon)r - \alpha_3 r \leq (1+\epsilon)r - \delta r$ to the center of D_1. By Lemma 3, $D_1 - B$ can be covered by two discs of radius r. Similarly $D_2 - B$ can also be covered by two discs of radius r. Therefore, $D_1 \cup D_2$ can be covered by at most five discs.

Case 2: The distance of the two centers of D_1 and D_2 is less than $4(1+\epsilon)r - r$. We consider two subcases:

Subcase 1: Disc D_1 and disc D_2 have intersection points p_1 and p_2 and the center of D_1 has distance at most $r' - \delta r$ to the line L through p_1 and p_2. By Lemma 3, the larger part of D_1 at one side of L can be covered by two discs of radius r. The part of D_2 at other side of L can be also covered by two discs of radius r. Therefore, $D_1 \cup D_2$ can be covered by four discs.

Subcase 2: Disc D_1 and disc D_2 have no intersection or the center of D_1 has distance $\geq r' - \delta r$ to the line L through the intersection points of D_1 and D_2. We put disc D with its center at the median on the line segment connecting the centers of D_1 and D_2. Disc D has enough intersection with both D_1 and D_2 (the center of D to the line through the intersection points between D and D_1 (D_2) will be small enough) so that by Lemma 3, both $D_1 - D$ can be covered by two discs of radius r and $D_2 - D$ can be covered by two discs of radius r. Therefore, $D_1 \cup D_2$ can be covered by five discs. $\square$

Theorem 4. *There exists an $O(n(\log n)^2(\log\log n)^2)$-time 2.8334-approximation algorithm for the disc covering problem on the plane.*

Proof. Let r be the radius of discs for the covering problem. Let γ be a positive real constant such that every disc of radius $\leq (1+\gamma)r$ can be covered by three discs of radius r. Let $\alpha = \gamma$. Select ϵ and ρ according to Lemma 4. We set $a = \frac{1}{3}$

and $\eta = \frac{\epsilon r}{\sqrt{2}}$. Select the constant $\beta > 0$ small enough such that $\frac{6-a}{2}(1+\beta) \leq 2.8334$ and $2+\frac{5a}{2}(1+\beta) \leq 2.8334$. Such a β exists because $\frac{6-a}{2} = 2+\frac{5a}{2} < 2.8334$. Let r' be equal to $r(1+\epsilon)$.

Algorithm
Input: A set of n points P on the plane.
(1) With the parameter β, use the algorithm of the first part of Theorem 3 to find a set of discs S of radius r to cover $Sketch_\eta(P)$ (see Lemma 1).
(2) Let $T = \emptyset$ and $U = S$.
(3) For each disc $D \in U$
(4) begin
(5) if (there is a disc $D' \in S$ with $dist(center(D), center(D')) \leq 4r' - \rho r$) then
(6) begin
(7) let $U = U - \{D, D'\}$, and
(8) let $T = T \cup \{(D, D')\}$.
(9) end (if)
(10) end (for)
(11) Let V be the set of discs in the pairs of T.
(12) For each $(D, D') \in T$, cover $extend_\eta(D)$ and $extend_\eta(D')$ with at most 5 discs of radius r.
(13) For each disc in $D \in U$, cover all the points in $extend_\eta(D) \cap P$ with a minimal number of discs (at most 3 discs are needed).
End of Algorithm

Let m be the total number of discs in the set S obtained by the algorithm in the first part of Theorem 3 for covering $Sketch_\eta(P)$. Let S, T and U be the sets after running the algorithm above. Recall in section 2 that we define $o(A)$, for any set A of points, as the minimal number of discs of radius r for covering all the points in A. As $Sketch_\eta(P)$ is a subset of P and $\frac{m}{o(Sketch_\eta(P))} \leq 1+\beta$, we have $o(Sketch_\eta(P)) \leq o(P)$ and

$$o(P) \geq \frac{m}{1+\beta}. \tag{36}$$

Let t be the number of pairs of discs of distance $4r' - \rho r$ that have been identified by the algorithm. Those pairs are put into T.

Case 1. $2t \geq am$. In other words, $|V| \geq am$. The algorithm outputs at most $\frac{5}{2}am + 3(1-a)m = \frac{6-a}{2}m$ discs of radius r to cover P. The approximation ratio is $\leq \frac{6-a}{2}m/\frac{m}{(1+\beta)} = \frac{6-a}{2}(1+\beta) \leq 2.8334$.

Case 2. $2t < am$. In other words, $|V| < am$. So, $|U| \geq (1-a)m$. Notice that any two discs in U has distance $> 4r' - \rho r$. Let $m_i (i = 1, 2, 3)$ be the number of discs D in U such that $extend_\eta(D) \cap P$ requires i discs of radius r to cover. Over all, the algorithm needs at least $\frac{m_1+2m_2+3m_3}{2}$ discs of radius r to cover all $extend_\eta(D) \cap P$ for every D in U, since each disc can be shared by at most two adjacent regions ($extend_\eta(D_1)$ and $extend_\eta(D_2)$ for discs D_1 and D_2 in U). On the other hand, we can cover all points in P with $(m_1+2m_2+3m_3)+\frac{5}{2}am$ discs of radius r, and these many discs have been identified by the algorithm. Combining with (36), we have $o(P) \geq \max(\frac{m}{1+\beta}, \frac{m_1+2m_2+3m_3}{2})$. The approximation ratio of the algorithm is at most $\frac{(m_1+2m_2+3m_3)+\frac{5}{2}am}{o(P)} \leq \frac{(m_1+2m_2+3m_3)+\frac{5}{2}am}{\max(\frac{m}{1+\beta}, \frac{m_1+2m_2+3m_3}{2})} \leq \frac{(m_1+2m_2+3m_3)}{\frac{m_1+2m_2+3m_3}{2}} + \frac{\frac{5}{2}am}{\frac{m}{1+\beta}} \leq 2 + \frac{5a}{2}(1+\beta) \leq 2.8334$.

By Theorem 3, the shifting part at step (1) in the algorithm for covering $Sketch_\eta(P)$ takes $O(n)$ time. We can find the smallest circle to cover a set of points in linear time [9]. We need $O(z(\log z)^2(\log\log z)^2)$ time to check if a set of z points on the plane can be covered by two discs of radius r [1]. It is easy to see that each disc only intersects with $O(1)$ other discs. This shows that steps (3) to (10) in the algorithm takes $O(|S|) = O(n)$ time. In summary, for each disc D in U, it takes at most $O(z(\log z)^2(\log\log z)^2)$ time to find a minimal number of discs of radius r to cover $extend_\eta(D)\cap P$, where $z = |extend_\eta(D)\cap P|$. Since each point stays in $O(1)$ discs in S, the total time of step (9) is $O(n(\log n)^2(\log\log n)^2)$. □

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