ELSEVIER

12 September 1994

PHYSICS LETTERS A

Physics Letters A 192 (1994) 316–322

Completely integrable system related to a new hierarchy of isospectral evolution equations

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Received 25 May 1994; accepted for publication 6 July 1994
Communicated by A.R. Bishop

Abstract

A new spectral problem is proposed, and the associated hierarchy of nonlinear evolution equations (NLEEs) are constructed by using the spectral gradient method. The Lax representations of this hierarchy are established through solving a key operator equation. Under a constraint between the potentials and the eigenfunctions, the spectral problem is nonlinearized as a finite-dimensional completely integrable system in the Liouville sense. Finally, the involutive solutions of the hierarchy of NLEEs are presented.

1. Introduction

An important and very active topic in the theory of integrability is to search for as many as possible new integrable systems. In recent years, the discoveries of bi-Hamiltonian structures of finite-dimensional dynamical systems [1,2], the nonlinearization theory for the Lax equations of the soliton systems [3,4], and the finite-dimensional restricted flows of the underlying infinite systems [5] seem to be the most important accomplishment in the theory of integrable systems. The above three discoveries lie in the existence of Lax representations for a given hierarchy of NLEEs. Hence, to find the Lax representation of the hierarchy of NLEEs is of great importance. It is well-known that the inverse scattering transform (IST) method plays a very important part in the study and discussion of NLEEs [6,7]. The major features of the evolution equations integrable by the IST are decided by the associated spectral problem. However, the main difficulty is how to connect NLEEs with the suitable spectral problem. Thus it is interesting for us to find a new spectral problem and the corresponding hierarchy of NLEEs.

In the present paper we first introduce a new spectral problem

$$y_x = My\,, \qquad M=\begin{pmatrix} \lambda-\frac{1}{2}uv & u \\ \lambda v & -\lambda+\frac{1}{2}uv \end{pmatrix}, \tag{1}$$

where u, v are two different potentials, λ is a constant spectral parameter, $\beta=$const, and construct the hierarchy of isospectral NLEEs associated with (1) by the spectral gradient method (SGM) [8], which is applied to

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0375-9601/94/$07.00
SSDI 0375-9601(94)00571-0

effectively obtain the hereditary symmetries of soliton equations, and simply derive the hierarchies of NLEEs associated with given spectral problems [9,10]. Then we establish the commutator (or Lax) representations of this hierarchy of NLEEs through solving a key operator equation. Moreover, using the nonlinearization approach of the spectral problem proposed in Refs. [3,4,11,12], we get a set of involutive function systems F_m ($m=0, 1, 2, \ldots$) in explicit form, which guarantees that the nonlinearization of spectral problem (1) is completely integrable in the Liouville sense under a constraint between the potentials and the eigenfunctions. Finally, the involutive solutions [13] of the hierarchy of NLEEs are given.

2. The hierarchy of NLEEs and Lax representations

Let λ be an eigenvalue, and $y=(y_1, y_2)^{\mathrm{T}}$ be the corresponding eigenfunction of (1). It is easy to calculate the spectral gradient $\nabla_{(u,v)}\lambda$ of the eigenvalue λ with respect to the potentials u, v,

$$\nabla_{(u,v)}\lambda \triangleq \begin{pmatrix} \delta\lambda/\delta u \\ \delta\lambda/\delta v \end{pmatrix} = \begin{pmatrix} y_2^2 - v y_1 y_2 \\ -\lambda y_1^2 - u y_1 y_2 \end{pmatrix}. \tag{2}$$

According to the spectral gradient method (SGM) [8,9] we seek for the so-called pair of Lenard operators $K=K(u, v)$, $J=J(u, v)$ which satisfy the equality $K\nabla_{(u,v)}\lambda=\lambda J\nabla_{(u,v)}\lambda$. Through some calculations and modifications, we find that only letting

$$K=\begin{pmatrix} \partial u\partial^{-1}u - u\partial^{-1}u\partial & \partial - \partial u\partial^{-1}v - u\partial^{-1}v\partial + uv \\ \partial + \partial v\partial^{-1}u + v\partial^{-1}u\partial - uv & -\partial v\partial^{-1}v + v\partial^{-1}v\partial \end{pmatrix},$$

$$J=-2\begin{pmatrix} -u\partial^{-1}u & -1+u\partial^{-1}v \\ 1+v\partial^{-1}u & -v\partial^{-1}v \end{pmatrix}, \qquad \partial=\partial/\partial x, \qquad \partial\partial^{-1}=\partial^{-1}\partial=1, \tag{3}$$

which are obviously skew-symmetric, we are sure to have

$$K\nabla_{(u,v)}\lambda=\lambda J\nabla_{(u,v)}\lambda. \tag{4}$$

The operators K, J defined by (3) are called a pair of Lenard operators of (1).

Now, choose $G_0=(v, u)^{\mathrm{T}}\in \mathrm{Ker}\, J$, and recursively define the Lenard gradient sequences G_j as follows

$$KG_{j-1}=JG_j, \qquad j=1, 2, \ldots. \tag{5}$$

It is not difficult to see that $G_j=G_j(u, v)=(G_j^{(1)}(u, v), G_j^{(2)}(u, v))^{\mathrm{T}}$ can be calculated one by one.

The nonlinear equations produced by the Lenard gradient sequences $G_m=G_m(u, v)$,

$$(u, v)_t^{\mathrm{T}}=KG_m=K\mathscr{L}^m G_0, \qquad m=0, 1, 2, \ldots, \tag{6}$$

are called a hierarchy of NLEEs associated with (1), the operator $\mathscr{L}$,

$$\mathscr{L}=J^{-1}K=\tfrac{1}{2}\begin{pmatrix} -\partial - \partial v\partial^{-1}u + uv^2\partial^{-1}u + uv & \partial v\partial^{-1}v - uv^2\partial^{-1}v \\ \partial u\partial^{-1}u + u^2v\partial^{-1}u & \partial - \partial u\partial^{-1}v - u^2v\partial^{-1}v + uv \end{pmatrix}, \tag{7}$$

is called the recursion operator of (6).

The first system in the hierarchy (6) is trivial. The second system of NLEEs in the hierarchy (6) is

$$u_t=\tfrac{1}{2}u_{xx}+\tfrac{1}{2}u^2v_x-\tfrac{1}{4}u^3v^2, \qquad v_t=-\tfrac{1}{2}v_{xx}+\tfrac{1}{2}v^2u_x+\tfrac{1}{4}v^3u^2, \tag{8}$$

whose physical properties are unknown.

In order to establish the Lax representations of the hierarchy (6), we rewrite the spectral problem (1) as

$$Ly=\lambda y\,,\qquad L=L(u,v)=\begin{pmatrix}\partial+\frac{1}{2}uv & -u\\ v\partial+\frac{1}{2}uv^2 & -\partial-\frac{1}{2}uv\end{pmatrix},\qquad \partial=\partial/\partial x\,. \tag{9}$$

Proposition 1. The Gateaux derivative operator L_{*w} of the spectral operator L defined by (9) in the direction ξ is

$$L_{*w}(\xi)\triangleq\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}L(w+\epsilon\xi)=\begin{pmatrix}\frac{1}{2}(u\xi_2+v\xi_1) & -\xi_1\\ \frac{1}{2}v(u\xi_2+v\xi_1) & \frac{1}{2}(u\xi_2-v\xi_1)\end{pmatrix}+\begin{pmatrix}0 & 0\\ \xi_2 & 0\end{pmatrix}L \tag{10}$$

and L_{*w} (simply written as L_* below) is an injective homomorphism, where $w=(u,v)^{\mathrm{T}}$, $\xi=(\xi_1,\xi_2)^{\mathrm{T}}$.

Proof. Direct calculation.

Consider the commutator $[V,L]=VL-LV$ of the following two operators,

$$V=V_0+V_1L\,,\qquad L=L_0+L_1\partial\,,$$

where

$$L_0=\begin{pmatrix}\frac{1}{2}uv & -u\\ \frac{1}{2}uv^2 & -\frac{1}{2}uv\end{pmatrix},\qquad L_1=\begin{pmatrix}1 & 0\\ v & -1\end{pmatrix}. \tag{11}$$

v_0 and v_1 are two function matrixes to be determined.

Let $G^{(1)}$ and $G^{(2)}$ be two arbitrary given smooth functions, and $G=(G^{(1)},G^{(2)})^{\mathrm{T}}$. We hope $[V,L]=L_*(KG)-L_*(JG)L$, where K, J and L_* are defined by (3) and (10), respectively. Through a lengthy calculation, we get

Proposition 2. Let $G^{(1)}$ and $G^{(2)}$ be two arbitrary given smooth functions, and $G=(G^{(1)},G^{(2)})^{\mathrm{T}}$. Then the operator equation of $V=V(G)$ determined by the pair of Lenard operators K, J and the spectral operator $L=L(u,v)$,

$$[V,L]=L_*(KG)-L_*(JG)L\,, \tag{12}$$

possesses the operator solution

$$V=V(G)=\begin{pmatrix}A(G & B(G)\\ 0 & -A(G)\end{pmatrix}+\begin{pmatrix}0 & 0\\ C(G) & 0\end{pmatrix}L\,, \tag{13}$$

with

$$A(G)=-\tfrac{1}{2}[\partial^{-1}(uG_x^{(1)}+vG_x^{(2)})+uv\partial^{-1}(uG^{(1)}-vG^{(2)})]\,,$$
$$B(G)=G^{(2)}+u\partial^{-1}(uG^{(1)}-vG^{(2)})\,,\qquad C(G)=G^{(1)}+v\partial^{-1}(uG^{(1)}-vG^{(2)})\,. \tag{14}$$

Proposition 3. Let $G_j=(G_j^{(1)},G_j^{(2)})^{\mathrm{T}}$ be the Lenard gradient sequences, and $V_j=V(G_j)$ be defined by (13) with $G=G_j$. Then the operator $W_m=\sum_{j=0}^m V_jL^{m-j}$ satisfies the relation

$$[W_m,L]=L_*(KG_m)\,,\qquad m=0,1,2,\ldots, \tag{15}$$

i.e. W_m $(m=0,1,2,\ldots)$ are a sequence of Lax operators [14] for the spectral problem (9).

Proof. $[W_m,L]=\sum_{j=0}^m[V_j,L]L^{m-j}=\sum_{j=0}^m[L_*(KG_j)L^{m-j}-L_*(JG_j)L^{m-j+1}]=L_*(KG_m)$.

Remark. The Lax operators W_m ($m=0, 1, 2, ...$) can compose a Lie operator algebra, which is going to be reported in another paper.

Proposition 4. The hierarchy (8) of NLees possesses the Lax representations

$$L_t=[W_m, L], \qquad m=0, 1, 2, \ldots . \tag{16}$$

Proof. Noticing $L_t=L_*(u_t, v_t)$ and Proposition 3, we have

$$L_t-[W_m, L]=L_*(u_t, v_t)-L_*(KG_m)=L_*((u, v)_t^{\mathrm{T}}-KG_m) .$$

In addition, L_* is injective, which implies $(u, v)_t^{\mathrm{T}}=KG_m \Leftrightarrow L_t=[W_m, L]$.

Corollary 1. The hierarchy (6) is the natural compatibility condition of $y_x=My$ and $y_t=W_m y=\sum_{j=0}^{m} V_j\lambda^{m-j}y$.

From (15), we have the following result immediately.

Corollary 2. The potentials u and v satisfy a stationary NLEE,

$$K\mathscr{L}^N G_0+\sum_{k=0}^{N-1}\alpha_{N-k}K\mathscr{L}^k G_0=0 \qquad (N\geqslant 0) , \tag{17}$$

if and only if

$$\left[W_N+\sum_{k=0}^{N-1}\alpha_{N-k}W_k, L\right]=0 \qquad (N\geqslant 0) , \tag{18}$$

where α_k ($1\leqslant k\leqslant N$) are some constants, $G_0=(v, u)^{\mathrm{T}}$.

3. Nonlinearization of the spectral problem (1) and its complete integrability

Let λ_j ($j=1, 2, ..., N$) and $y=(q_j, p_j)^{\mathrm{T}}$ be N different spectral parameters and corresponding eigenfunctions of (1). Consider the constraint relation [4] $G_0=-\sum_{j=1}^{N}\nabla_{(u,v)}\lambda_j$, i.e.

$$u=\frac{\langle\Lambda q, q\rangle}{1-\langle p, q\rangle}, \qquad v=\frac{-\langle p, p\rangle}{1-\langle p, q\rangle} . \tag{19}$$

Hence, under (19), (1) is nonlinearized to be

$$q_x=\Lambda q+\frac{\langle\Lambda q, q\rangle}{1-\langle p, q\rangle}p+\frac{\langle p, p\rangle\langle\Lambda q, q\rangle}{2(1-\langle p, q\rangle)^2}q, \qquad p_x=-\Lambda p-\frac{\langle p, p\rangle}{1-\langle p, q\rangle}\Lambda q-\frac{\langle p, p\rangle\langle\Lambda q, q\rangle}{2(1-\langle p, q\rangle)^2}p, \tag{20}$$

which can be expressed as the Hamiltonian system

$$(H): \quad q_x=\frac{\partial H}{\partial p}, \qquad p_x=-\frac{\partial H}{\partial q}, \tag{21}$$

with the Hamiltonian function

$$H=\langle p, q\rangle+\frac{\langle p, p\rangle\langle\Lambda q, q\rangle}{2(1-\langle p, q\rangle)} . \tag{22}$$

In order to prove the integrability of (21) in the Liouville sense, we now construct a set of functions F_m as follows,

$$F_m=(1-\langle p,q\rangle)\langle \Lambda^{m+1}p,q\rangle+\tfrac{1}{2}\langle \Lambda^{m+1}q,q\rangle\langle p,p\rangle+\tfrac{1}{2}\sum_{j=0}^{m-1}\begin{vmatrix}\langle \Lambda^{j+1}q,q\rangle & \langle \Lambda^{j+1}q,p\rangle\\ \langle \Lambda^{m-j}p,q\rangle & \langle \Lambda^{m-j}p,p\rangle\end{vmatrix},$$

$$m=0,1,2,\ldots, \tag{23}$$

where $p=(p_1,\ldots,p_N)^{\mathrm{T}}$, $q=(q_1,\ldots,q_N)^{\mathrm{T}}$, $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$, $\langle\ ,\ \rangle$ is the standard inner product in $\mathbb{R}^N$.

In the symplectic space $(\mathbb{R}^{2N}, \mathrm{d}p\wedge\mathrm{d}q)$, the Poisson bracket (E,F) of two functions E, F is defined by [15]

$$(E,F)=\sum_{j=1}^{N}\left(\frac{\partial E}{\partial q_j}\frac{\partial F}{\partial p_j}-\frac{\partial E}{\partial p_j}\frac{\partial F}{\partial q_j}\right)=\langle E_q,F_p\rangle-\langle E_p,F_q\rangle\,. \tag{24}$$

E and F are called involutive if $(E,F)=0$.

According to (24), it is not difficult for us to obtain

Proposition 5. $(F_m,F_n)=0$, $\forall\ m,n\in\mathbb{Z}^+$

Proposition 6. The Hamiltonian system (21) is completely integrable in the Liouville sense.

Proof. Through some direct calculations, we have $(H,F_m)=0$. Thus the desired result is correct.

Proposition 7. Let $(q,p)^{\mathrm{T}}$ be a solution of the Hamiltonian system (21). Then u and v determined by (19) are the solution of a stationary system,

$$K\mathscr{L}^N G_0+\sum_{k=0}^{N-1}\gamma_{N-k}K\mathscr{L}^k G_0=0\,, \tag{26}$$

where the constants γ_j $(1\leqslant j\leqslant N)$ are determined by $\lambda_1,\ldots,\lambda_N$.

Proof. We get (26) after considering the polynomial

$$p(\lambda)=\prod_{j=1}^{N}(\lambda-\lambda_j)=\lambda^N+\gamma_1\lambda^{N-1}+\ldots+\gamma_N$$

and the relation

$$\mathscr{L}^k G_0=-\sum_{j=1}^{N}\lambda_j^k\nabla_{(u,v)}\lambda_j\,. \tag{27}$$

4. The involutive solutions of the hierarchy (6)

Denote the phase flows of the Hamiltonian systems (H) and (F_m): $q_{t_m}=\partial F_m/\partial p$, $p_{t_m}=-\partial F_m/\partial q$, by g_H^x, $g_m^{t_m}$ respectively. $(H,F_m)=0$ implies the compatibility of (H) and (F_m), and the commutativity of g_H^x and $g_m^{t_m}$ [15].

Define

$$\begin{pmatrix}q(x,t_m)\\ p(x,t_m)\end{pmatrix}=g_H^x g_m^{t_m}\begin{pmatrix}q(0,0)\\ p(0,0)\end{pmatrix}, \tag{28}$$

which is called the involutive solution [13] of the compatible systems (H) and (F_m).

Proposition 8. Let $(q(x,t_m),p(x,t_m))^{\mathrm{T}}$ be an involutive solution of the consistent systems (H) and (F_m). Then

$$u(x,t_m)=\frac{\langle Aq(x,t_m),q(x,t_m)\rangle}{1-\langle p(x,t_m),q(x,t_m)\rangle}\,,\qquad u(x,t_m)=\frac{-\langle p(x,t_m),p(x,t_m)\rangle}{1-\langle p(x,t_m),q(x,t_m)\rangle} \tag{29}$$

are a solution of the NLEEs

$$(u,v)^{\mathrm{T}}_{t_m}=K\mathscr{L}^m G_0=J\mathscr{L}^{m+1}G_0\,,\qquad m=0,1,2,\ldots\,. \tag{30}$$

Proof.

$$\frac{\partial u}{\partial t_m}=\frac{2\langle Aq,q_{t_m}\rangle(1-\langle p,q\rangle)+\langle Aq,q\rangle(\langle p,q_{t_m}\rangle+\langle q,p_{t_m}\rangle)}{(1-\langle p,q\rangle)^2}\,,$$

$$\frac{\partial v}{\partial t_m}=\frac{-2\langle p,p_{t_m}\rangle(1-\langle p,q\rangle)-\langle p,p\rangle(\langle p,q_{t_m}\rangle+\langle q,p_{t_m}\rangle)}{(1-\langle p,q\rangle)^2}\,.$$

Substituting the expressions of $\partial F_m/\partial p$ and $\partial F_m/\partial q$ into the above two equalities, we obtain

$$u_{t_m}=2\langle A^{m+2}q,q\rangle-2\frac{\langle Aq,q\rangle}{1-\langle p,q\rangle}\langle A^{m+1}p,q\rangle\,,$$

$$v_{t_m}=2\langle A^{m+1}p,p\rangle-2\frac{\langle p,p\rangle}{1-\langle p,q\rangle}\langle A^{m+1}p,q\rangle\,. \tag{31}$$

On the other hand, in the light of (27), (29) and $\partial^{-1}(u\langle A^{m+1}p,p\rangle+v\langle A^{m+2}q,q\rangle)=\langle A^{m+1}p,q\rangle$, we have

$$J\mathscr{L}^{m+1}G_0=2\begin{pmatrix}u\partial^{-1}u & 1-u\partial^{-1}v\\ -1-v\partial^{-1}u & v\partial^{-1}v\end{pmatrix}\begin{pmatrix}-\langle A^{m+1}p,p\rangle+v\langle A^{m+1}p,q\rangle\\ \langle A^{m+2}q,q\rangle+u\langle A^{m+1}p,q\rangle\end{pmatrix}$$

$$=2\begin{pmatrix}\langle A^{m+2}q,q\rangle-[\langle Aq,q\rangle/(1-\langle p,q\rangle)]\langle A^{m+1}p,q\rangle\\ \langle A^{m+1}p,p\rangle-[\langle p,p\rangle/(1-\langle p,q\rangle)]\langle A^{m+1}p,q\rangle\end{pmatrix}. \tag{32}$$

(31) and (32) imply (30).

From Proposition 6, we can directly get an involutive solution of equation (8).

Proposition 9.

$$u(x,t_1)=\frac{\langle Aq(x,t_1),q(x,t_1)\rangle}{1-\langle p(x,t_1),q(x,t_1)\rangle}\,,\qquad v(x,t_1)=\frac{-\langle p(x,t_1),p(x,t_1)\rangle}{1-\langle p(x,t_1),q(x,t_1)\rangle}$$

satisfy Eq. (8), where $(q(x,t_1),p(x,t_1))^{\mathrm{T}}$ is the involutive solution of the compatible systems (H) and (F_1).

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