A finite-dimensional integrable system and the involutive solutions of the higher-order Heisenberg spin chain equations

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Abstract

By the use of the spectral problem nonlinearization method, a finite-dimensional integrable system and the involutive solutions of the higher-order Heisenberg spin chain equations are presented. In particular, the involutive solution of the well-known Heisenberg spin chain equation

\[ u_t = \frac{i}{2} (u \partial_x v - v \partial_x u), \quad v_t = \frac{i}{2} (w \partial_x v - v \partial_x w) \quad (w^2 + uv = 1) \]

is obtained.

In the middle of the 1970s, the continuous Heisenberg spin chain aroused considerable interest [1-4]. Tjon and Wright obtained the explicit formula for the single-soliton solution in the isotropic case [3]. Takhtajan studied the integration of the continuous Heisenberg spin chain equation through the inverse scattering transform method and obtained its Lax representation [4]. Afterwards, Chen and Li gave the higher-order Heisenberg spin chain equations [5]. All of these studies about the Heisenberg spin chain are admirable. However, within the author's knowledge, there have not been any reports on the solution representations of the higher-order Heisenberg spin chain equations.

In this Letter, using the spectral problem and Lax pair nonlinearization method [6,7], which was first suggested by Cao [8] in 1988 and was successfully applied to produce completely integrable finite-dimensional Hamiltonian systems in the Liouville sense, we first give a finite-dimensional integrable Hamiltonian system associated with the Heisenberg spin chain, and then through this completely integrable system in the Liouville sense and its involutive system, we present the solution representations of the higher-order Heisenberg spin chain equations. In particular, the solution representation of the well-known Heisenberg spin chain equation

\[ u_t = \frac{i}{2} (u \partial_x w - w \partial_x u), \quad v_t = \frac{i}{2} (w \partial_x v - v \partial_x w) \quad (w^2 + uv = 1) \]

is obtained.

Consider the Heisenberg spectral problem [4]

\[ \psi_x = -i \lambda S \psi, \quad S = \begin{pmatrix} w & u \\ v & -w \end{pmatrix}, \quad w^2 + uv = 1, \tag{1} \]

in which \( y = (y_1, y_2)^T \), \( u \) and \( v \) are two potentials, \( \lambda \) is a spectral parameter. Let \( \lambda_j \) (1 \( \leq j \leq N \)) be \( N \) different spectral parameters, and \( y = (q_j, p_j)^T \) be the associated spectral functions. Define \( A_j = (\lambda_p^j, -\lambda_q^j)^T \). Then \( A_j \) satisfies

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\[ KA_j = \lambda_j J A_j, \quad (2) \]

where \( K \) and \( J \) are two operators \((\partial = \partial / \partial X, \partial \partial^{-1} = \partial^{-1} \partial = 1)\),

\[ K = i \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad J = \begin{pmatrix} u \partial^{-1}(u/w) \partial & u \partial^{-1}(v/w) \partial + 2w \\ -u \partial^{-1}(u/w) \partial - 2w & -v \partial^{-1}(v/w) \partial \end{pmatrix}, \quad (3) \]

which are called the pair of Lenard operators of the spectral problem (1).

Now, we recursively define the Lenard gradient sequence \( \{ G_j \} \) of (1) as follows,

\[ G_0 = \alpha (u, v) \in \text{Ker } J, \quad \alpha = \text{const}, \quad KG_j = JG_{j+1}, \quad j = 0, 1, 2, \ldots \quad (4) \]

It is easy to see the recursion operator

\[ \mathcal{L} = J^{-1}K \]

\[ \mathcal{L} = \frac{1}{2i} \begin{pmatrix} (1/w) \partial - \frac{1}{2} \partial^{-1} u \partial (1/w) \partial & \frac{1}{2} \partial^{-1} v \partial (1/w) \partial \\ -\frac{1}{2} \partial^{-1} u \partial (1/w) \partial & - (1/w) \partial + \frac{1}{2} \partial^{-1} v \partial (1/w) \partial \end{pmatrix}. \quad (5) \]

The Lenard recursive sequence \( \{ G_j \} \) can be calculated through (4) and (5). \( X_m = KG_m \) \((m = 0, 1, 2, \ldots)\) are called the Heisenberg vector fields of (1), which yield the hierarchy of nonlinear evolution equations associated with (1),

\[ (u, v) = X_m(u, v) = KG_m = K \mathcal{L}^m G_0, \quad m = 0, 1, 2, \ldots \quad (6) \]

The first few terms in the hierarchy (6) are

\[ \begin{pmatrix} u \\ v \end{pmatrix} = X_0(u, v) = i \alpha \begin{pmatrix} u_x \\ v_x \end{pmatrix} \quad (7) \]

\[ \begin{pmatrix} u \\ v \end{pmatrix} = X_1(u, v) = \frac{i}{2} \alpha \begin{pmatrix} w_{xx}u - u_{xx}w \\ v_{xx}w - w_{xx}v \end{pmatrix} \quad (8) \]

and

\[ \begin{pmatrix} u \\ v \end{pmatrix} = X_2(u, v) = -\frac{i}{4} \alpha \begin{pmatrix} u_{xxx} + \frac{3}{2} \left[ (u + v) v_x + w^2 \right]_x \\ v_{xxx} + \frac{3}{2} \left[ v (u_x v_x + w_x^2) \right]_x \end{pmatrix}. \quad (9) \]

Here, Eq. (8) is exactly the famous Heisenberg spin chain equation [4] when \( \alpha = -i \). Thus, (6) stands for the hierarchy of Heisenberg spin chain equations. It is not difficult to see that the Heisenberg hierarchy (6) possesses the Lax representations

\[ L\psi = \lambda \psi, \quad \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} w \\ -u \end{pmatrix} \partial, \quad w^2 + uv = 1, \quad (10) \]

\[ y_m = w_m \partial = - i \sum_{j=0}^{m} \left( \frac{1}{2} \partial^{-1} \left[ (u/w) G_j^{(1)} + (v/w) G_j^{(2)} \right] G_j^{(1)} \right) \lambda^{m+1-j} \psi, \quad (11) \]

where \( G_j = (G_j^{(1)}, G_j^{(2)})^T \) is determined by (4).

Now, we introduce a constraint relation [7] between the eigenfunctions and the potentials of (1),

\[ G_0 |_{\alpha = -1} = \sum_{j=1}^{N} A_j, \quad (12) \]
which is equivalent to
\[ u = -\langle Aq, q \rangle, \quad v = \langle Ap, p \rangle. \] (13)

Hence,
\[ w = \sqrt{1 - uv} = \sqrt{1 + \langle Aq, q \rangle \langle Ap, p \rangle}. \] Here, \( p = (p_1, ..., p_N)^T, q = (q_1, ..., q_N)^T, A = \text{diag}(\lambda_1, ..., \lambda_N), \langle \rangle \) stands for the standard inner-product in \( \mathbb{R}^N \).

Under the constraint (13), (1) is nonlinearized as
\[ q_x = -i\sqrt{1 + \langle Aq, q \rangle \langle Ap, p \rangle} Aq + i\langle Aq, q \rangle Aq, \quad p_x = i\langle Ap, p \rangle Aq + i\sqrt{1 + \langle Aq, q \rangle \langle Ap, p \rangle} Ap. \] (14)

**Proposition 1.** Suppose \((p, q)^T\) satisfies Eq. (14), then
\[ (\sqrt{1 + \langle Aq, q \rangle \langle Ap, p \rangle} - \langle Aq, p \rangle) \ast = 0. \] (15)

**Proof.**
\[ \langle Ap, q \rangle_x = i(\langle Aq, q \rangle \langle A^2p, p \rangle - \langle Ap, p \rangle \langle A^2q, q \rangle), \]
\[ \langle Aq, q \rangle_x = 2i(-\sqrt{1 + \langle Ap, p \rangle \langle Aq, q \rangle \langle A^2p, p \rangle} + \langle Ap, p \rangle \langle A^2q, q \rangle), \]
\[ \langle Ap, p \rangle_x = 2i(-\langle Ap, p \rangle \langle A^2p, q \rangle + \sqrt{1 + \langle Ap, p \rangle \langle Aq, q \rangle \langle A^2p, p \rangle}), \]
\[ (\sqrt{1 + \langle Ap, p \rangle \langle Aq, q \rangle} \langle Aq, q \rangle)^{-1/2}(\langle Aq, q \rangle \langle Ap, p \rangle_x + \langle Ap, p \rangle \langle Aq, q \rangle_x) \]
\[ = \langle Ap, q \rangle_x. \] (15) implies \( \sqrt{1 + \langle Ap, p \rangle \langle Aq, q \rangle} = \langle Aq, q \rangle + \beta, \beta = \text{const.} \) Let \( \beta = 0 \), then (14) can be rewritten as
\[ q_x = -i\langle Ap, p \rangle Aq + i\langle Aq, q \rangle Aq, \quad p_x = -i\langle Ap, p \rangle Aq + i\langle Ap, q \rangle Aq. \] (16)

**Proposition 2.** (16) can be expressed in the Hamiltonian form
\[ (H): \quad q_x = \partial H / \partial p, \quad p_x = -\partial H / \partial q, \] (17)
with the Hamiltonian function
\[ H = \frac{1}{2}i \langle Aq, q \rangle \langle Ap, p \rangle - \frac{1}{2}i \langle Ap, q \rangle^2. \] (18)

**Proof.** This is obvious.

In order to show the integrability of the Hamiltonian system (17), we introduce a set of functions \( F_m \) as follows,
\[ F_m = \frac{1}{2}i \sum_{j=0}^m \langle A^{j+1}q, q \rangle \langle A^{m+1-j}p, p \rangle - \langle A^{j+1}p, q \rangle \langle A^{m+1-j}p, q \rangle, \] (19)

Note \( H = F_0 \).

The Poisson bracket of two Hamiltonian functions \( F, C \) in the symplectic space \((\mathbb{R}^{2N}, dp \wedge dq)\) is defined by [9]
\[ (F, G) = \sum_{j=1}^N \left( \frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right) = \langle F, G_p \rangle - \langle F_p, G_q \rangle. \] (20)

\( F, G \) are called involutive if \( (F, G) = 0 \).
According to formula (20), through some calculations, we can easily get

**Proposition 3.**

\[(F_m, F_n) = 0, \forall m, n \in \mathbb{Z}^+.\]  

(21)

**Proposition 4.** \((H, F_m) = 0, \forall m \in \mathbb{Z}^+,\) thus the Hamiltonian system (17) is completely integrable in the Liouville sense and its involutive system is \(\{F_m\}\).

**Proposition 5.** \(F_m\) defined by (19) is actually generated through nonlinearization of the time part (11) of the Lax pair for the Heisenberg hierarchy (6) under the constraint (13).

**Proof.** Acting with the recursive operator \(L^j\) upon (12), and noticing (2) and (4), we have

\[
G_j = L^jG_0 |_{\alpha = 1} = \sum_{k=1}^{N} \lambda_k^j A_k = \left( \frac{\langle A^{j+1}p, p \rangle}{-\langle A^{j+1}q, q \rangle} \right).
\]

(22)

In virtue of (13), (14), (22) and \( w = \sqrt{1 + \langle Ap, p \rangle \langle Aq, q \rangle} = \langle Ap, q \rangle \), we can deduce

\[
-\frac{1}{2} \partial^{-1} \left( \frac{u}{w} G_j^{(1)} + \frac{v}{w} G_j^{(2)} \right) = \langle A^{j+1}p, q \rangle.
\]

(23)

So, under the constraint (13), the time part (11) is nonlinearized as

\[
q_{jn} = -i \sum_{j=0}^{m} \left( \langle A^{j+1}p, q \rangle A^{m+1-j}q - \langle A^{j+1}q, q \rangle A^{m+1-j}p \right),
\]

\[
p_{jn} = -i \sum_{j=0}^{m} \left( \langle A^{j+1}p, p \rangle A^{m+1-j}q - \langle A^{j+1}q, p \rangle A^{m+1-j}p \right).
\]

(24)

After expressing (24) in Hamiltonian form, we immediately know its Hamiltonian function is none other than \(F_m\). The proof is complete.

**Proposition 6.** Let \((q, p)^T\) be a solution of the Hamiltonian system (17). Then \(u = -\langle Aq, q \rangle, v = \langle Ap, p \rangle\) and \(w = \sqrt{1 + \langle Ap, p \rangle \langle Aq, q \rangle} = \langle Ap, q \rangle\) satisfy a stationary Heisenberg evolution equation

\[
K L^N G_0 |_{\alpha = 1} + \sum_{j=0}^{N-1} \alpha_{N-j} K L^j G_0 |_{\alpha = 1} = 0,
\]

(25)

where the \(\alpha_j\) are determined by \(\lambda_1, ..., \lambda_N\).

**Proof.** Consider the polynomial \((\alpha_0 = 1)\)

\[
p(\lambda) = \prod_{i=1}^{N} (\lambda - \lambda_i) = \alpha_0^N + \alpha_1 \lambda^{N-1} + ... + \alpha_N \cdot
\]

(26)

Acting with the operator \(K \sum_{j=0}^{N} \alpha_{N-j}\) upon (22) and using (26), we get (25). The Poisson bracket \((H, F_m) = 0\) implies the Hamiltonian systems \((H)\) and \((F_m)\) are consistent, and their solution operators \(g_\delta, g_m\) of the corresponding initial-value problems commute (see Ref. [9]). Denote the involutive solution [10] of the compatible equations \((H)\) and \((F_m)\)
\[q_{im} = \frac{\partial F_m}{\partial p}, \quad p_{im} = -\frac{\partial F_m}{\partial q} \] by
\[
(q(x, t_m), p(x, t_m)) = g^\alpha g_m(q(0, 0), p(0, 0)), \tag{27}
\]
which are smooth functions of \((x, t_m)\).

**Proposition 7.** Let \((q(x, t_m), p(x, t_m))^T\) be an involutive solution of the compatible systems \((H)\) and \((F_m)\). Then
\[
u(x, t_m) = -\langle Aq, q \rangle, \quad w(x, t_m) = \sqrt{1 + \langle Ap, p \rangle \langle Aq, q \rangle} = \langle Ap, q \rangle \tag{28}
\]
satisfy the higher-order Heisenberg evolution equation
\[
\begin{pmatrix}
u \\ \nu \end{pmatrix} = -iK\varphi^m G_0 |_{\alpha=1}, \quad G_0 |_{\alpha=1} = (v, u)^T, \quad m = 0, 1, 2, \ldots. \tag{29}
\]

**Proof.** On the one hand, substituting (24) into the following two equalities, we have
\[
\begin{align*}
\partial u &= -2\langle Aq, q \rangle = 2i(\langle Ap, q \rangle \langle A^{m+2} q, q \rangle - \langle A^{m+2} p, q \rangle \langle Aq, q \rangle), \\
\partial v &= 2\langle Ap, p \rangle = 2i(\langle Ap, q \rangle \langle A^{m+2} p, p \rangle - \langle A^{m+2} p, q \rangle \langle Ap, p \rangle).
\end{align*}
\]

On the other hand, from (3), (22) and (16), we get
\[
\begin{align*}
-K\varphi^m G_0 |_{\alpha=1} &= 2\left(\begin{pmatrix}
-\langle A^{m+1} q, q \rangle \\
\langle A^{m+1} q, p \rangle
\end{pmatrix}\right) = 2\left(\begin{pmatrix}
-\langle A^{m+1} q, -i\langle Ap, q \rangle Aq + i\langle Ap, q \rangle Ap \rangle \\
\langle A^{m+1} p, p \rangle + i\langle Ap, q \rangle Ap + i\langle Ap, q \rangle Ap \rangle
\end{pmatrix}\right).
\end{align*}
\]
Thus, Proposition 7 holds.

As applications of Proposition 7, we give two examples below.

**Example 1.** When \(m = 1\), (29) exactly becomes the well known Heisenberg spin chain equation (HSCE) (i.e. Eq. (8) as \(\alpha = -i\))
\[
\begin{align*}
u &= \frac{1}{2}i(\nu xxw - w xxu), \\
v &= \frac{1}{2}(w xxv - v xxw). \tag{30}
\end{align*}
\]
So, according to Proposition 7 and (27), we can know that the HSCE (30) possesses the solution representation
\[
\begin{align*}
u(x, t_1) &= -\langle Ag g^\dagger q(0, 0), g q^\dagger q(0, 0) \rangle, \\
v(x, t_1) &= \langle Ag g^\dagger p(0, 0), g q^\dagger p(0, 0) \rangle,
\end{align*}
\]
\[
\begin{align*}
\omega(x, t_1) &= \left[1 + \langle Ag g^\dagger p(0, 0), g q^\dagger p(0, 0) \rangle \langle Ag g^\dagger q(0, 0), g q^\dagger q(0, 0) \rangle \right]^{1/2} \\
&= \langle Ag g^\dagger p(0, 0), g q^\dagger q(0, 0) \rangle,
\end{align*}
\]
where \(g^\dagger q\) and \(g^\dagger p\) stand for the solution operators of the initial-value problems of Hamiltonian systems \((H)\) and \((F_1)\), respectively.

**Example 2.** In (29), setting \(m = 2\), we may easily obtain
\[
\begin{pmatrix}
u \\ \nu \end{pmatrix} = -iK\varphi^2 G_0 |_{\alpha=1} = -\frac{1}{4}\begin{pmatrix}
u xx xx + \frac{3}{2} \nu u xx v x + w xx^2 \end{pmatrix}, \tag{32}
\]
which is none other than Eq. (9) when \(\alpha = -i\).
Thus, Eq. (32) (i.e. the third-order Heisenberg spin chain equation) has the solution representation
\[ u(x, t_2) = -\langle Ag_0^2 q(0, 0), g_0^2 q(0, 0) \rangle, \quad v(x, t_2) = \langle Ag_0^2 p(0, 0), g_0^2 p(0, 0) \rangle, \]
\[ w(x, t_2) = \left[ 1 + \langle Ag_0^2 p(0, 0), g_0^2 p(0, 0) \rangle \langle Ag_0^2 q(0, 0), g_0^2 q(0, 0) \rangle \right]^{1/2} \]
\[ = \langle Ag_0^2 p(0, 0), g_0^2 q(0, 0) \rangle, \quad (33) \]
where \( g_0^2 \) and \( g_0^2 \) are the solution operators of the initial-value problems of the systems \((H)\) and \((F_2)\), respectively.

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