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A general approach for getting the commutator representations of the hierarchies of nonlinear evolution equations

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Abstract

A general approach for generating the commutator representations of the hierarchies of nonlinear evolution equations (NLEEs) is presented. For this approach, three concrete examples are given.

The inverse scattering transform (IST) method plays a very important role in the investigation of nonlinear evolution equations (NLEEs) [1-10]. This method has been successfully applied to the NLEEs, which are of great importance in physics. In the theory of integrable systems, it is significant for us to search for as many new integrable evolution equations as possible. Tu [11] presented a method for deriving the isospectral hierarchy of integrable evolution equations from a proper linear spectral problem and successfully obtained many isospectral hierarchies of integrable evolution equations [12-16]. For these methods there is one isospectral hierarchy of NLEEs associated with a proper given linear spectral problem.

In this Letter, directly starting from the spectral problem $L\psi=\lambda\psi$ or $\psi_x=U\psi$ (λ is a spectral parameter) and not requiring to consider its auxiliary problem $\psi_t=V\psi$, by making use of the spectral gradient method (SGM) which was invented by Fuchssteiner [17] and used by Fokas and Andersen [18] for obtaining hereditary symmetries for Hamiltonian systems more than ten years ago, we shall generate two different hierarchies of NLEEs connected with the same given spectral problem. Moreover, a general approach for obtaining the commutator (or Lax) representations of the hierarchies of NLEEs is given. In order to acquire the commutator representations of NLEEs it is crucial to find the operator solution of a key operator equation. We shall present three examples to show them. Here, it should be pointed out that the approach, used for producing two different hierarchies of NLEEs associated with the given spectral problem in this Letter, is a reformulation of the IST method.

In the following we give some fundamental symbols and notations. Let $x\in\Omega$ (Ω is the underlying interval $(-\infty,+\infty)$ or $(0,T)$ for decaying conditions at infinity or periodic conditions, respectively), $t\in\mathbb{R}$, $u=(u_1,\ldots,u_q)^{\mathrm{T}}$, $u_i=u_i(x,t)$, $1\leqslant i\leqslant q$. β is denoted by all complex (or real) functions $P[u]=P(x,t,u(x,t))$ which are $\mathbb{C}^\infty$-differentiable with respect to x, t. Let

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$$\beta^s=\{(p_1, ..., p_s)^{\mathrm{T}}|p_i\in\beta,\ 1\leqslant i\leqslant s\}\ ,$$

$$C_u[\lambda]=\left\{\sum_{k\in\mathbf{Z}} p_k[u]\lambda^k|\ \sum_{k\in\mathbf{Z}} \text{is a finite sum},\ p_k[u]\in\beta,\ \lambda\in\mathbb{C}\text{ or }\mathbb{R}\right\}.$$

We denote by $\partial^0(\sum_{k\in\mathbf{Z}} p_k[u]\lambda^k)$ the degree of the polynomial $\sum_{k\in\mathbf{Z}} p_k[u]\lambda^k$, i.e.

$$\partial^0\left(\sum_{k\in\mathbf{Z}} p_k[u]\lambda^k\right)=\max\left\{k|\ k\in\mathbb{Z},\ \sum_{k\in\mathbf{Z}} \text{is a finite sum}\right\}.$$

Conventions: $(\cdot)_x=\partial(\cdot)/\partial x$, $(\cdot)_t=\partial(\cdot)/\partial t$, $\partial=\partial/\partial x$, $\partial\partial^{-1}=\partial^{-1}\partial=1$.

Consider the spectral problem as follows,

$$\psi_x=U(u,\lambda)\psi\ ,\qquad U(u,\lambda)=(U_{ij})_{n\times n}\ ,\qquad U_{ij}=U_{ij}(u,\lambda)\in C_u[\lambda]\ ,\tag{1}$$

where u is a q-dimensional vector potential function, λ is a spectral parameter, $\psi=(\psi_1, ..., \psi_n)^{\mathrm{T}}\in\beta^n$, $U(u,\lambda)$ is an $n\times n$ matrix.

On the one hand, for the spectral problem (1) (especially as $\mathrm{Tr}\,U(u,\lambda)=0$), in the light of the methods proposed by Tu [19] and Cao [20] we can always obtain the functional gradient $\nabla_u\lambda=(\delta\lambda/\delta_{u1}, ..., \delta\lambda/\delta u_q)^{\mathrm{T}}$ ($\nabla_u\lambda\neq 0$) of the spectral parameter λ with respect to the vector potential function u. In general, $\nabla_u\lambda$ is related to the potential u, the spectral parameter λ and the corresponding spectral function ψ.

Suppose that there exist two $q\times q$ matrix integro-differential operators $K=K(u,\partial,\partial^{-1})$, $J=J(u,\partial,\partial^{-1})$ which are only related to u, ∂ and ∂^{-1} such that

$$K\nabla_u\lambda=\lambda^\theta J\nabla_u\lambda\ ,\tag{2}$$

where θ is an invariant constant connected with (1). K and J, which satisfy (2), are called the pair of Lenard operators of (1). Generally speaking, the pair of Lenard operators K, J are skew-symmetric and J is usually a symplectic or Hamiltonian operator. K, J are mainly obtained by (1), the actual expression of $\nabla_u\lambda$ and some delicate techniques. Here, the pair of Lenard operators K, J exactly constitute the recursion operator $\mathscr{L}=J^{-1}K$ in NLEEs solvable by the IST method.

Now, according to the IST method, we can directly define the two Lenard gradient recursive sequences of (1) as follows:

(i) The first Lenard gradient sequence $\{G_j\}$:

$$G_{-1}\in\mathrm{Ker}\,J=\{G\in\beta^q|JG=0\}\ ,\qquad KG_{j-1}=JG_j\ ,\qquad j=0,1,2,...\ .\tag{3}$$

(ii) The second Lenard gradient sequence $\{\hat{G}_j\}$:

$$\hat{G}_{-1}\in\mathrm{Ker}\,K=\{\hat{G}\in\beta^q|K\hat{G}=0\}\ ,\qquad J\hat{G}_{j-1}=K\hat{G}_j\ ,\qquad j=0,1,2,...\ .$$

$X_j(u)=JG_j$ and $\hat{X}_j(u)=K\hat{G}_j$ ($j=0,1,2,...$) are called the first and second vector field of (1), respectively. The following two hierarchies of equations

$$u_t=X_j(u)\ ,\qquad j=0,1,2,...\ ,\tag{5}$$

$$u_t=\hat{X}_j(u)\ ,\qquad j=0,1,2,...\ ,\tag{6}$$

are called the first and second hierarchy of NLEEs associated with (1), respectively.

On the other hand, write $\gamma=\max_{1\leqslant i,j\leqslant n}\partial^0(U_{ij})$. Then $U(u,\lambda)$ can be expressed as

$$U(u,\lambda)=U_\gamma(u)\lambda^\gamma+...\ ,\tag{7}$$

or

$$U(u,\lambda)=U_{-\gamma}(u)\lambda^{-\gamma}+...\ .\tag{7'}$$

Here $U_\gamma(u)$ (or $U_{-\gamma}(u)$) is an $n\times n$ matrix and each element of it belongs to β. For convenience, we discuss (7) below ((7′) can be discussed similarly).

Assume the $n\times n$ matrix $U_\gamma(u)$ is inverse. Then (1) can become

$$\bar{L}\psi=\lambda=\lambda^\gamma\psi\,, \tag{8}$$

where $\bar{L}=\bar{L}(u,\lambda,\partial)$ is an $n\times n$ matrix differential operator. Let $\lambda^{e-\gamma}$ be multiplied in the two sides of (8), then (8) reads

$$L\psi=\lambda^e\psi\,, \tag{9}$$

where the $n\times n$ matrix differential operator $L=\lambda^{e-\gamma}\bar{L}(u,\lambda,\partial)$ depends on u, λ, and ∂.

Definition [21]. The Gateaux derivative operator L_* of the above operator L is defined by

$$L_*(\xi)=\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}L(u+\epsilon\xi)\,,\qquad \xi\in\beta^q\,. \tag{10}$$

For any given vector function $G\in\beta^q$, we construct an operator equation of $V=V(G)$,

$$[V,L]=L_*(KG)L^{\alpha-1}-L_*(JG)L^\alpha\,. \tag{11}$$

Here [,] stands for the Lie bracket, K and J are the pair of Lenard operators of (1), L is determined by (9), α is a proper chosen constant according to (1). For some vector function $G\in\beta^q$, we use $V=V(G)$ to express the corresponding operator solution of (11). The following two theorems reveal the close connection between the commutator representations of the two hierarchies of NLEEs (5), (6) and the operator solutions of the operator equation (11).

Theorem 1. Let $\{G_j\}$, $\{\hat{G}_j\}$ be defined by (3), (4), respectively. Suppose that for any $\{G_j\}$, $\{\hat{G}_j\}$ ($j=-1,0,1,\dots$), there exist differential operators $V_j=V(G_j)$, $\hat{V}_j=V(\hat{G}_j)$ solving the operator equation (11) with $G=G_j$, $\hat{G}_j$. Then the operators

$$W_m=\sum_{j=0}^{m}V_{j-1}L^{m-j-\alpha+1}\,,\qquad \hat{W}_m=\sum_{j=0}^{m}\hat{V}_{j-1}L^{-m+j-\alpha} \tag{12}$$

satisfy the equations

$$[W_m,L]=L_*(X_m)\,,\qquad [L,\hat{W}_m]=L_*(\hat{X}_m)\,,\qquad m=0,1,2,\dots\,, \tag{13}$$

separately.

Proof. From (11), (3) and (4) we have

$$[V_j,L]=L_*(KG_j)L^{\alpha-1}-L_*(JG_j)L^\alpha=L_*(X_{j+1})L^{\alpha-1}-L_*(X_j)L^\alpha\,,\qquad j=-1,0,1,\dots\,,$$

$$[\hat{V}_j,L]=L_*(K\hat{G}_j)L^{\alpha-1}-L_*(J\hat{G}_j)L^\alpha=L_*(\hat{X}_j)L^{\alpha-1}-L_*(\hat{X}_{j+1})L^\alpha\,,\qquad j=-1,0,1,\dots\,.$$

Notice that $JG_{-1}=0$, $K\hat{G}_{-1}=0$, and $L_*(0)=0$. Thus

$$[W_m,L]=\left[\sum_{j=0}^{m}V_{j-1}L^{m-j-\alpha+1},L\right]$$

$$=\sum_{j=0}^{m}[V_{j-1},L]L^{m-j-\alpha+1}=\sum_{j=0}^{m}[L_*(X_j)L^{\alpha-1}-L_*(X_{j-1})L^\alpha]L^{m-j-\alpha+1}$$

$$= \sum_{j=0}^{m} [L_*(X_j)L^{m-j} - L_*(X_{j-1})L^{m-j+1}] = L_*(X_m)\,, \qquad m=0,1,2,\dots,$$

$$[L, \hat{W}_m] = -\sum_{j=0}^{m} [\hat{V}_{j-1}, L]L^{-m+j-\alpha} = -\sum_{j=0}^{m} [L_*(\hat{X}_{j-1})L^{\alpha-1} - L_*(\hat{X}_{j-1})L^{\alpha}]L^{-m+j-\alpha}$$

$$= -\sum_{j=0}^{m} [L_*(\hat{X}_{j-1})L^{-m+j-1} - L_*(\hat{X}_j)L^{-m+j}] = L_*(\hat{X}_m)\,, \qquad m=0,1,2,\dots\,.$$

The proof is completed.

Theorem 2. Suppose that the condition of Theorem 1 is satisfied. If the Gateaux derivative operator L_* is an injective homomorphism, then the two evolution equations $u_t = X_m(u)$, $u_t = \hat{X}_m(u)$ $(m=0,1,2,\dots)$ of (1) possess the following commutator representations,

$$L_t = [W_m, L] = \left[\sum_{j=0}^{m} V_{j-1}L^{m-j-\alpha+1}, L\right], \qquad m=0,1,2,\dots, \tag{14}$$

$$L_t = [L, \hat{W}_m] = \left[L, \sum_{j=0}^{m} \hat{V}_{j-1}L^{-m+j-\alpha}\right], \qquad m=0,1,2,\dots, \tag{15}$$

respectively.

Proof. Notice that $L_t = L_*(u_t)$. From Theorem 1 we obtain

$$L_t - [W_m, L] = L_*(u_t) - L_*(X_m) = L_*(u_t - X_m)\,, \qquad L_t - [L, \hat{W}_m] = L_*(u_t) - L_*(\hat{X}_m) = L_*(u_t - \hat{X}_m)\,.$$

In addition, because L_* is injective, we have $u_t = X_m(u)$, $u_t = \hat{X}_m(u)$ if and only if $L_t = [W_m, L]$, $L_t = [L, \hat{W}_m]$, respectively, which are the desired results.

Immediately, from the relation $[W_m, L] = L_*(X_m)$, $[L, \hat{W}_m] = L_*(\hat{X}_m)$ $(m=0,1,2,\dots)$ and noting that L_* is injective, we have

Corollary. The potential $u = (u_1, \dots, u_q)^{\mathrm{T}}$ is a finite gap, that is, it satisfies the two nonlinear stationary equations

$$\sum_{k=0}^{N} \alpha_k X_{N-k} = 0\,, \qquad \sum_{k=0}^{N} \beta_k \hat{X}_{N-k} = 0 \qquad (N \geqslant 0)\,, \tag{16}$$

respectively, if and only if

$$\left[\sum_{k=0}^{N} \alpha_k W_{N-k}, L\right] = 0\,, \qquad \left[\sum_{k=0}^{N} \beta_k \hat{X}_{N-k}, L\right] = 0 \qquad (N \geqslant 0)\,, \tag{17}$$

where α_k, β_k $(N \geqslant k \geqslant 0)$ are some constants.

Thus according to the above skeleton, in order to secure the commutator representations of the two hierarchies (5), (6) it is crucial to find the operator solution $V = V(G)$ of operator equation (11) for any given vector function $G \in \beta^q$. Now, by making use of the above approach, in the following we study three spectral problems, give their corresponding two different hierarchies of NLEEs and construct the commutator representations of those hierarchies.

Example 1. Consider the spectral problem studied by Geng [22],

$$\psi_x=\begin{pmatrix}1 & \lambda_{u-1}\\ \lambda_{u+1} & -1\end{pmatrix}\psi\,. \tag{18}$$

From (18) it is not difficult to calculate that

$$\nabla_u\lambda=\delta\lambda/\delta u=\lambda(\psi_1^2-\psi_2^2)\left(\int_\Omega u(\psi_2^2-\psi_1^2)\,\mathrm{d}x\right)^{-1}. \tag{19}$$

Because of $\partial^2\nabla_u\lambda=-2\lambda u\partial\nabla_u\lambda$, only choosing

$$K=\partial^s\,,\qquad J=-2\partial u\partial\,, \tag{20}$$

as the pair of Lenard operators of (18), we are sure to get

$$K\nabla_u\lambda=\lambda^\theta J\nabla_u\lambda\,,\qquad \theta=1\,. \tag{21}$$

We recursively define the two Lenard gradient sequences of (18) as follows: (i) The first Lenard's gradient sequence $\{G_j\}$:

$$G_{-1}=\partial^{-1}u^{-1}\in\operatorname{Ker}J\,,\qquad KG_{j-1}=JG_j\,,\qquad j=0,1,2,\ldots\,. \tag{22}$$

(ii) The second Lenard gradient sequence $\{\hat{G}_j\}$:

$$\hat{G}_{-1}=ax^2+bx+c\in\operatorname{Ker}K\,,\qquad \forall a,b,c\in\mathbb{R}\text{ or }\mathbb{C}\,,\qquad J\hat{G}_{j-1}=K\hat{G}_j\,,\qquad j=0,1,2,\ldots\,. \tag{23}$$

The first vector field $X_j=JG_j$ yields the first hierarchy of NLEEs of (18)

$$u_t=X_j(u)\,,\qquad j=0,1,2,\ldots\,, \tag{24}$$

with two representative equations

$$u_t=X_0(u)\equiv-(u^{-2}u_x)_x\,,\qquad u_t=X_1(u)\equiv-\tfrac{1}{4}(u^{-2})_{xxx}\,. \tag{25}$$

The first is the remarkable nonlinear diffusion equation and has important applications in plasma physics, solid state physics and other fields, such as metallurgy and polymer science [23–25].

The second vector field $\hat{X}_j=K\hat{G}_j$ produces the second hierarchy of NLEEs of (18)

$$u_t=\hat{X}_j(u)\,,\qquad j=0,1,2,\ldots\,, \tag{26}$$

with two representative equations

$$u_t=\hat{X}_0(u)\equiv-2[2a(xu)_x+bu_x]\,,\qquad u_t=\hat{X}_1(u)\equiv-4[(2ax+b)u^2+2au_x\partial^{-1}(xu)+bu_x\partial^{-1}u]\,. \tag{27}$$

For $a=0$, $b=1$, the latter can be reduced to the semi-classical limit KdV equation $v_t=-4v_xv$ via the transformation $u=v_x$.

Eq. (18) is equivalent to

$$L\psi=\lambda^\gamma\psi\,,\qquad L=\frac{1}{u}\begin{pmatrix}-1 & \partial+1\\ \partial-1 & 1\end{pmatrix},\qquad \gamma=1\,. \tag{28}$$

The Gateaux derivative operator L_* is

$$L_*(\xi)=-\frac{\xi}{u}L\,,\qquad \forall\xi\in\beta\,, \tag{29}$$

and L_*: $\xi\mapsto L_*(\xi)$ is obviously an injective homomorphism. Here $\theta=\gamma=1$. Now, for any given function $G\in\beta$, we consider the operator equation of $V=V(G)$,

$$[V, L] = L_*(KG)L^{-1} - L_*(JG)\,, \tag{30}$$

which corresponds to letting $\alpha = 0$ in (11).

Theorem 3. Let L, L_*, K, J be defined by (28), (29), (20), respectively. Then for any function $G \in \beta$, the operator equation (30) has the operator solution

$$V = V(G) = \begin{pmatrix} 2G_x & G_{xx} - 2G_x \\ G_{xx} + 2G_x & -2G_x \end{pmatrix} + \begin{pmatrix} 0 & 2uG_x \\ 2uG_x & 0 \end{pmatrix} L\,. \tag{31}$$

Proof. Let $V = V_0 + V_1 L$, $L = L_0 + L_1 \partial$, where

$$V_0 = \begin{pmatrix} 2G_x & G_{xx} - 2G_x \\ G_{xx} + 2G_x & -2G_x \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & 2uG_x \\ 2uG_x & 0 \end{pmatrix};$$

$$L_0 = \frac{1}{u}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad L_1 = \frac{1}{u}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{32}$$

Make the commutator $[V, L]$ (notice $\partial = L_1^{-1}(L - L_0)$):

$$\begin{aligned} [V, L] &= [V_0, L] + [V_1, L]L = [V_0, L_0] - L_1 V_{0x} + [V_0, L_1]\partial + ([V_1, L_0] - L_1 V_{1x} + [V_1, L_1]\partial)L \\ &= [V_0, L_0] - L_1 V_{0x} - [V_0, L_1]L_1^{-1}L_0 \\ &\quad + ([V_0, L_1]L_1^{-1} + [V_1, L_0] - L_1 V_{1x} - [V_1, L_1]L_1^{-1}L_0)L + [V_1, L_1]L_1^{-1}L^2\,. \end{aligned} \tag{33}$$

Furthermore, substituting (32) into (33), by direct calculation we can obtain $[V_1, L_1] = 0$,

$$[V_0, L_0] - L_1 V_{0x} - [V_0, L_1]L_1^{-1}L_0 = -\frac{G_{xxx}}{u} = -\frac{KG}{u} = L_*(KG)L^{-1}\,,$$

$$([V_0, L_1]L_1^{-1} + [V_1, L_0] - L_1 V_{1x})L = -\frac{2(uG_x)_x}{u}L = \frac{JG}{u}L = -L_*(JG)\,.$$

This shows that the operator V defined by (31) is an operator solution $V = V(G)$ of Eq. (30). The proof is completed.

So, from Theorem 2 we immediately know that the first hierarchy of (18), $u_t = X_m(u)$, and the second hierarchy of NLEEs of (18), $u_t = \hat{X}_j(u)$, have the commutator representations

$$L_t = [W_m, L]\,, \quad m = 0, 1, 2, \ldots,$$

$$W_m = \sum_{j=0}^{m}\left\{\begin{pmatrix} 2G_{j-1,x} & G_{j-1,xx} - 2G_{j-1,x} \\ G_{j-1,xx} + 2G_{j-1,x} & -2G_{j-1,x} \end{pmatrix} L^{m-j+1} + \begin{pmatrix} 0 & 2uG_{j-1,x} \\ 2uG_{j-1,x} & 0 \end{pmatrix} L^{m-j+2}\right\}, \tag{34}$$

$$L_t = [L, \hat{W}_m]\,, \quad m = 0, 1, 2, \ldots,$$

$$\hat{W}_m = \sum_{j=0}^{m}\left\{\begin{pmatrix} 2\hat{G}_{j-1,x} & \hat{G}_{j-1,xx} - 2\hat{G}_{j-1,x} \\ \hat{G}_{j-1,xx} + 2\hat{G}_{j-1,x} & -2\hat{G}_{j-1,x} \end{pmatrix} L^{-m+j} + \begin{pmatrix} 0 & 2u\hat{G}_{j-1,x} \\ 2u\hat{G}_{j-1,x} & 0 \end{pmatrix} L^{-m+j+1}\right\}, \tag{35}$$

respectively.

Example 2. Consider the spectral problem proposed by Cao and Geng [26],

$$\psi_x = \begin{pmatrix} \lambda u & \lambda v+\lambda^2 \\ \epsilon(\lambda v-\lambda^2) & -\lambda u \end{pmatrix}\psi , \qquad \epsilon=\pm 1 , \tag{36}$$

$$\nabla_{(u,v)}\lambda = \begin{pmatrix} \delta\lambda/\delta u \\ \delta\lambda/\delta v \end{pmatrix} = \begin{pmatrix} 2\lambda\psi_1\psi_2 \\ \lambda\psi_2^2-\epsilon\lambda\psi_1^2 \end{pmatrix}\left(\int_\Omega (\epsilon v\psi_1^2-2\epsilon\lambda\psi_1^2-2u\psi_1\psi_2-v\psi_2^2-2\lambda\psi_2^2)\,\mathrm{d}x\right)^{-1} . \tag{37}$$

Noticing the relation $\partial(\psi_1^2+\epsilon\psi_2^2)=2(2\lambda v\psi_1\psi_2-\epsilon\lambda u\psi_2^2+\lambda u\psi_1^2)$, we should choose the pair of Lenard operators of (36) K, J as

$$K=\begin{pmatrix} \epsilon\partial & 0 \\ 0 & \partial \end{pmatrix}, \qquad J=\begin{pmatrix} 4v\partial^{-1}v & 2\epsilon-4\epsilon v\partial^{-1}u \\ -2\epsilon-4\epsilon u\partial^{-1}v & 4u\partial^{-1}u \end{pmatrix}. \tag{38}$$

Then we have

$$K\nabla_{(u,v)}\lambda=\lambda^\theta J\nabla_{(u,v)}\lambda , \qquad \theta=2 . \tag{39}$$

The two hierarchies of NLEEs of (36) are determined in the following procedure:
(i) The first Lenard gradient sequence $\{G_j\}$ is defined by

$$G_{-1}=(\epsilon u, v)^{\mathrm{T}}\in\mathrm{Ker}\,J , \qquad KG_{j-1}=JG_j , \qquad j=0, 1, 2, \ldots . \tag{40}$$

The first hierarchy of NLEEs of (36) is

$$(u, v)_t^{\mathrm{T}}=X_j(u, v)=JG_j , \qquad j=0, 1, 2, \ldots , \tag{41}$$

which produces the C–G hierarchy of equations [26] with the representative equations

$$(u, v)_t^{\mathrm{T}}=X_1(u, v)\equiv(-v_{xx}+(\epsilon u^3+uv^2)_x, \epsilon u_{xx}+(\epsilon u^2v+v^3)_x)^{\mathrm{T}} . \tag{42}$$

(ii) The second Lenard gradient sequence $\{\hat{G}_j\}$ is defined by

$$\hat{G}_{-1}=(0, 0)^{\mathrm{T}}\in\mathrm{Ker}\,K , \qquad (\partial^{-1}v-\epsilon\partial^{-1}u)0=\tfrac{1}{4}\epsilon , \qquad J\hat{G}_{j-1}=K\hat{G}_j , \qquad j=0, 1, 2, \ldots . \tag{43}$$

The second hierarchy NLEEs of (36) is given by

$$(u, v)_t^{\mathrm{T}}=\hat{X}_j(u, v)=K\hat{G}_j , \qquad j=0, 1, 2, \ldots , \tag{44}$$

with the representative equation

$$\begin{aligned}(u, v)_t^{\mathrm{T}}=\hat{X}_1(u, v)\equiv{}&(4v(\partial^{-1}v\partial^{-1}v+\epsilon\partial^{-1}u\partial^{-1}u)-2\epsilon\partial^{-1}u, \\ &-4\epsilon u(\partial^{-1}v\partial^{-1}v+\epsilon\partial^{-1}u\partial^{-1}u)-2\epsilon\partial^{-1}v)^{\mathrm{T}} ,\end{aligned} \tag{45}$$

which can be reduced to

$$\tilde{u}_{xt}=2\tilde{v}_x(\tilde{v}^2+\epsilon\tilde{u}^2)-2\epsilon\tilde{u} , \qquad \tilde{v}_{xt}=-2\epsilon\tilde{u}_x(\tilde{v}^2+\epsilon\tilde{u}^2)-2\epsilon\tilde{v} , \tag{46}$$

via the transformation $u=\tilde{u}_x$, $v=\tilde{v}_x$.
Eq. (36) can be rewritten as

$$L\psi=\lambda^\gamma\psi , \qquad L=\begin{pmatrix} \lambda v & -\epsilon\lambda u-\epsilon\partial \\ -\lambda u+\partial & -\lambda v \end{pmatrix}, \qquad \gamma=2 , \qquad \epsilon=\pm 1 . \tag{47}$$

Here $\theta=\gamma=2$.

$$L_*(\xi)=\begin{pmatrix} \xi_2 & -\epsilon\xi_1 \\ -\xi_1 & -\xi_2 \end{pmatrix}L^{1/2} , \qquad \xi=(\xi_1, \xi_2)^{\mathrm{T}}\in\beta^2 , \qquad L_* \text{ is injective} . \tag{48}$$

Choose $\alpha=\frac{1}{2}$, then by using a method similar to Theorem 3 we can prove

Theorem 4. For any vector function $G=(G^{(1)},G^{(2)})^{\mathrm{T}}\in\beta^2$, the following operator equation of $V=V(G)$,

$$[V,L]=L_*(KG)L^{-1/2}-L_*(JG)L^{1/2}, \tag{49}$$

possesses the operator solution

$$V=V(G)=\begin{pmatrix}\epsilon G^{(1)} & G^{(2)}\\ \epsilon G^{(2)} & -\epsilon G^{(1)}\end{pmatrix}+\begin{pmatrix}0 & 2\partial^{-1}(uG^{(2)}-\epsilon vG^{(1)})\\ -2\epsilon\partial^{-1}(uG^{(2)}-\epsilon vG^{(1)}) & 0\end{pmatrix}L^{1/2}. \tag{50}$$

In (49), L, L_*, K, J are expressed by (47), (48), (38), respectively.

Hence, the two hierarchies of NLEEs of (36) $(u,v)^{\mathrm{T}}_t=X_m(u,v)$, $(u,v)^{\mathrm{T}}_t=\hat{X}_m(u,v)$ have the commutator representations

$$L_t=[W_m,L],\qquad m=0,1,2,\ldots,\qquad W_m$$
$$=\sum_{j=0}^{m}\left\{\begin{pmatrix}\epsilon G^{(1)}_{j-1} & G^{(2)}_{j-1}\\ \epsilon G^{(2)}_{j-1} & -\epsilon G^{(1)}_{j-1}\end{pmatrix}L^{m-j+1/2}+\begin{pmatrix}0 & 2\partial^{-1}(uG^{(2)}_{j-1}-\epsilon vG^{(1)}_{j-1})\\ -2\epsilon\partial^{-1}(uG^{(2)}_{j-1}-\epsilon vG^{(1)}_{j-1}) & 0\end{pmatrix}L^{m-j+1}\right\}, \tag{51}$$

$$L_t=[L,\hat{W}_m],\qquad m=0,1,2,\ldots,\qquad \hat{W}_m$$
$$=\sum_{j=0}^{m}\left\{\begin{pmatrix}\epsilon \hat{G}^{(1)}_{j-1} & \hat{G}^{(2)}_{j-1}\\ \epsilon \hat{G}^{(2)}_{j-1} & -\epsilon \hat{G}^{(1)}_{j-1}\end{pmatrix}L^{-m+j-1/2}+\begin{pmatrix}0 & 2\partial^{-1}(u\hat{G}^{(2)}_{j-1}-\epsilon v\hat{G}^{(1)}_{j-1})\\ -2\epsilon\partial^{-1}(u\hat{G}^{(2)}_{j-1}-\epsilon v\hat{G}^{(1)}_{j-1}) & 0\end{pmatrix}L^{-m+j}\right\}, \tag{52}$$

respectively. Here $G_{j-1}=(G^{(1)}_{j-1},G^{(2)}_{j-1})^{\mathrm{T}}$, $\hat{G}_{j-1}=(\hat{G}^{(1)}_{j-1},\hat{G}^{(2)}_{j-1})^{\mathrm{T}}$ $(j=0,1,2,\ldots)$ are the first and second Lenard sequence of (36) separately.

Example 3. Consider the spectral problem presented by Boiti and Tu [27],

$$\psi_x=\begin{pmatrix}-\mathrm{i}\lambda+\mathrm{i}\lambda^{-1}s & u+\mathrm{i}\lambda^{-1}v\\ u-\mathrm{i}\lambda^{-1}v & \mathrm{i}\lambda-\mathrm{i}\lambda^{-1}s\end{pmatrix}\psi, \tag{53}$$

$$\nabla_{(u,v,w)}\lambda=\begin{pmatrix}\delta\lambda/\delta u\\ \delta\lambda/\delta v\\ \delta\lambda/\delta s\end{pmatrix}=\begin{pmatrix}-\psi_1^2+\psi_2^2\\ \mathrm{i}\lambda^{-1}(\psi_1^2+\psi_2^2)\\ 2\mathrm{i}\lambda^{-1}\psi_1\psi_2\end{pmatrix}\left(\int_\Omega(\mathrm{i}\lambda^{-2}v\psi_1^2+2\mathrm{i}\psi_1\psi_2+2\mathrm{i}\lambda^{-2}s\psi_1\psi_2+\mathrm{i}\lambda^{-2}v\psi_2^2)\,\mathrm{d}x\right)^{-1}. \tag{54}$$

The pair of Lenard operators is chosen as

$$K=\begin{pmatrix}\partial & 2s & 2v\\ -2s & 0 & 0\\ -2v & 0 & 0\end{pmatrix},\qquad J=\begin{pmatrix}0 & 2 & 0\\ -2 & -\partial & u\\ 0 & -2u & \partial\end{pmatrix}. \tag{55}$$

Thus, we have

$$K\nabla_{(u,v,s)}\lambda=\lambda^{\theta}J\nabla_{(u,v,s)}\lambda,\qquad \theta=2. \tag{56}$$

The two Lenard gradient sequences are defined by

$$G_{-1}=(u,0,1)^{\mathrm{T}}\in\operatorname{Ker}J,\qquad KG_{j-1}=JG_j,\qquad j=0,1,2,\ldots, \tag{57}$$

$$\hat{G}_{-1}=(0,v,-s)^{\mathrm{T}}\in\operatorname{Ker}K,\qquad J\hat{G}_{j-1}=K\hat{G}_j,\qquad j=0,1,2,\ldots. \tag{58}$$

The first Lenard sequence $\{G_j\}$ yields the first hierarchy of NLEEs of (53)

$$(u, v, s)_t^{\mathrm{T}} = X_j(u, v, s) = JG_j, \qquad j=0, 1, 2, \dots, \tag{59}$$

which are exactly the Boiti–Tu hierarchy of equations [27] with the representative equation

$$u_t = u_x + 2v, \qquad v_t = -2us, \qquad s_t = -2uv. \tag{60}$$

The second hierarchy of NLEEs of (53) is given by the second Lenard sequence $\{\hat{G}_j\}$, i.e.

$$(u, v, s)_t^{\mathrm{T}} = \hat{X}_j(u, v, s) = K\hat{G}_j, \qquad j=0, 1, 2, \dots, \tag{61}$$

with the representative equation

$$u_t = 2v, \qquad v_t = -v_x - 2us, \qquad s_t = -2uv. \tag{62}$$

Eq. (53) can become

$$\bar{L}\psi = \lambda^\gamma \psi, \qquad \bar{L} = \begin{pmatrix} \mathrm{i}\partial + \lambda^{-1}s & -\mathrm{i}u + \lambda^{-1}v \\ \mathrm{i}u + \lambda^{-1}v & -\mathrm{i}\partial + \lambda^{-1}s \end{pmatrix}, \qquad \gamma = 1. \tag{63}$$

Here $\theta - \gamma = 1$, so we should choose the spectral operator L as follows

$$L = \lambda \bar{L} = \begin{pmatrix} \mathrm{i}\lambda\partial + s & -\mathrm{i}\lambda u + v \\ \mathrm{i}\lambda u + v & -\mathrm{i}\lambda\partial + s \end{pmatrix}, \tag{64}$$

and we have

$$L\psi = \lambda^2 \psi. \tag{65}$$

$$L_*(\xi) = \begin{pmatrix} \xi_3 & \xi_2 \\ \xi_2 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{i}\xi_1 \\ \mathrm{i}\xi_1 & 0 \end{pmatrix} L^{1/2}, \qquad \forall \xi = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}} \in \beta^3, \quad L_* \text{ is injective}. \tag{66}$$

Let $a = \frac{3}{2}$. Through a lengthy calculation it is not difficult for us to obtain.

Theorem 5. Let L, L_*, K, J be expressed by (64), (66), (55), respectively. Then for any given vector function $G = (G^{(1)}, G^{(2)}, G^{(3)})^{\mathrm{T}} \in \beta^3$, the operator equation

$$[V, L] = L_*(KG) L^{1/2} - L_*(JG) L^{3/2}, \tag{67}$$

possesses the operator solution

$$V = V(G) = \begin{pmatrix} 0 & G^{(1)} \\ G^{(1)} & 0 \end{pmatrix} + \begin{pmatrix} -\mathrm{i}G^{(3)} & \mathrm{i}G^{(2)} \\ -\mathrm{i}G^{(2)} & \mathrm{i}G^{(3)} \end{pmatrix} L^{1/2}. \tag{68}$$

So, the two hierarchies of (53) $(u, v, s)_t^{\mathrm{T}} = X_m(u, v, s)$, $(u, v, s)_t^{\mathrm{T}} = \hat{X}_m(u, v, s)$ have the commutator representations

$$L_t = [W_m, L], \qquad m = 0, 1, 2, \dots,$$
$$W_m = \sum_{j=0}^{m} \left\{ \begin{pmatrix} 0 & G_{j-1}^{(1)} \\ G_{j-1}^{(1)} & 0 \end{pmatrix} L^{m-j-1/2} + \begin{pmatrix} -\mathrm{i}G_{j-1}^{(3)} & \mathrm{i}G_{j-1}^{(2)} \\ -\mathrm{i}G_{j-1}^{(2)} & \mathrm{i}G_{j-1}^{(3)} \end{pmatrix} L^{m-j} \right\}, \tag{69}$$

$$L_t = [L, \hat{W}_m], \qquad m = 0, 1, 2, \dots,$$
$$\hat{W}_m = \sum_{j=0}^{m} \left\{ \begin{pmatrix} 0 & \hat{G}_{j-1}^{(1)} \\ \hat{G}_{j-1}^{(1)} & 0 \end{pmatrix} L^{-m+j-3/2} + \begin{pmatrix} -\mathrm{i}\hat{G}_{j-1}^{(3)} & i\hat{G}_{j-1}^{(2)} \\ -\mathrm{i}\hat{G}_{j-1}^{(2)} & \mathrm{i}\hat{G}_{j-1}^{(3)} \end{pmatrix} L^{-m+j-1} \right\}, \tag{70}$$

respectively. Here $G_{j-1} = (G_{j-1}^{(1)}, G_{j-1}^{(2)}, G_{j-1}^{(3)})^{\mathrm{T}}$, $\hat{G}_{j-1} = (\hat{G}_{j-1}^{(1)}, \hat{G}_{j-1}^{(2)}, \hat{G}_{j-1}^{(3)})^{\mathrm{T}}$ ($j = 0, 1, 2, \dots$) are the first and second Lenard sequence of (53) separately.

Remark. In the case of the AKNS (and also of the KdV) hierarchy, the two hierarchies generated in formulas (5) and (6) reduce to one. Its commutator representations (or Lax representations) were obtained long ago.

By using our effective approach described above, we naturally think about looking for the commutator representations of the hierarchies of NLEEs associated with other spectral problems, which are left to papers.

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