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A three-component Camassa-Holm system with cubic nonlinearity and peakons

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In this paper, we propose a three-component Camassa-Holm (3CH) system with cubic nonlinearity and peaked solitons (peakons). The 3CH model is proven to be integrable in the sense of Lax pair, Hamiltonian structure, and conservation laws. We show that this system admits peakons and multi-peakon solutions. Additionally, reductions of the 3CH system are investigated so that a new integrable perturbed CH equation with cubic nonlinearity is generated to possess peakon solutions.

Keywords: Three-component Camassa-Holm equation; Peakon; Lax pair; Conservation laws.

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1. Introduction

In the past two decades, the Camassa-Holm (CH) equation [4]

$$m_t - 2mu_x - m_x u = 0, \quad m = u - u_{xx} + k, \tag{1.1}$$

with k being an arbitrary constant, derived by Camassa and Holm [4] as a shallow water wave model, has attracted much attention in the theory of soliton and integrable system. The CH equation was first included in the work of Fuchssteiner and Fokas on hereditary symmetries as a very special case [18]. Since the work of Camassa and Holm [4], more diverse studies on this equation have remarkably been developed [2, 5, 7, 10, 13, 15, 16, 20, 23, 28, 32, 33]. The most interesting feature of the CH equation (1.1) is that it admits peaked soliton (peakon) solutions in the case $k = 0$. A peakon is a kind of weak solution in some Sobolev space with corner at its crest. The stability and interaction of peakons were discussed in several references [1, 3, 8, 9, 26].

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As extension of the CH peakon equation, other integrable peakon models have also been found, such as the Degasperis-Procesi (DP) equation [11, 12, 29]

$$m_t + 3mu_x + m_x u = 0, \quad m = u - u_{xx}, \tag{1.2}$$

the cubic nonlinear peakon equation [15, 20, 32, 34, 35]

$$m_t = \frac{1}{2}\left[m(u^2 - u_x^2)\right]_x, \quad m = u - u_{xx}, \tag{1.3}$$

and the Novikov's cubic nonlinear equation [25, 31]

$$m_t = u^2 m_x + 3uu_x m, \quad m = u - u_{xx}. \tag{1.4}$$

Then, a naturally interesting theme is to study integrable multi-component peakon equations. For example, in [6, 14, 24, 32] the authors proposed a two-component generalization of the CH equation. In [21, 38, 42, 43], two-component extensions of the cubic nonlinear equations (1.3) and (1.4) were investigated, while in [17, 22, 36] three-component extensions of the CH equation are derived.

In this paper, we propose the following three-component system

$$\begin{cases} m_{11,t} =\frac{1}{2}[m_{11}(u_{11}^2 - u_{11,x}^2 + u_{12}u_{21} - u_{12,x}u_{21,x})]_x \\ \qquad + \frac{1}{2}m_{12}(u_{11,x}u_{21} - u_{11}u_{21,x}) - \frac{1}{2}m_{21}(u_{11}u_{12,x} - u_{11,x}u_{12}), \\ m_{12,t} =\frac{1}{2}[m_{12}(u_{11}^2 - u_{11,x}^2 + u_{12}u_{21} - u_{12,x}u_{21,x})]_x \\ \qquad + m_{11}(u_{11}u_{12,x} - u_{11,x}u_{12}) + \frac{1}{2}m_{12}(u_{12,x}u_{21} - u_{12}u_{21,x}), \\ m_{21,t} =\frac{1}{2}[m_{21}(u_{11}^2 - u_{11,x}^2 + u_{12}u_{21} - u_{12,x}u_{21,x})]_x \\ \qquad + m_{11}(u_{11}u_{21,x} - u_{11,x}u_{21}) + \frac{1}{2}m_{21}(u_{12}u_{21,x} - u_{12,x}u_{21}), \\ m_{11} =u_{11} - u_{11,xx}, \quad m_{12} = u_{12} - u_{12,xx}, \quad m_{21} = u_{21} - u_{21,xx}. \end{cases} \tag{1.5}$$

Apparently, this system is reduced to the CH equation (1.1) as $u_{11} = 0,\ u_{21} = 2$ and to the cubic nonlinear CH equation (1.3) as $u_{12} = u_{21} = 0$. Therefore, it is a three-component formation based on the CH equation (1.1) and the cubic nonlinear CH equation (1.3), and we may call equation (1.5) the 3CH model. We show that the 3CH system is Hamiltonian and possesses a Lax pair and infinitely many conservation laws. Furthermore, this three-component system admits the single peakon of traveling wave type as well as multi-peakon solutions. Additionally, we pay attention to reductions of the 3CH system so that a new integrable perturbed CH equation with cubic nonlinearity is generated to possess peakon solutions.

The whole paper is organized as follows. In section 2, a Lax pair, Hamiltonian structure, and conservation laws of equation (1.5) are presented. In section 3, the single-peakon and multi-peakon solutions of equation (1.5) are given. Section 4 studies the reductions of system (1.5). Some conclusions and discussions are described in section 5.

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2. Lax pair, Hamiltonian form and conservation laws

Let us first introduce the $sl(2)$ valued matrices u and m as follows:

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix}, \quad m = \begin{pmatrix} u_{11} - u_{11,xx} & u_{12} - u_{12,xx} \\ u_{21} - u_{21,xx} & -u_{11} + u_{11,xx} \end{pmatrix} \triangleq \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & -m_{11} \end{pmatrix}. \tag{2.1}$$

Using this notation, equation (1.5) can be expressed in a nice matrix equation form

$$m_t = \frac{1}{2}[m(u^2 - u_x^2)]_x + \frac{1}{4}[m(uu_x - u_x u) - (uu_x - u_x u)m], \quad m = u - u_{xx}, \tag{2.2}$$

where u and m are the $sl(2)$ matrices (2.1).

Consider a pair of linear spectral problems

$$\phi_x = U\phi, \quad \phi_t = V\phi, \tag{2.3}$$

with

$$\begin{aligned} &\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T, \\ &U = \frac{1}{2}\begin{pmatrix} -I_2 & \lambda m \\ \lambda m & I_2 \end{pmatrix} \triangleq \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \\ &V = \frac{1}{2}\begin{pmatrix} \lambda^{-2} I_2 - \frac{1}{2}(u^2 - u_x^2 + uu_x - u_x u) & -\lambda^{-1}(u - u_x) + \frac{1}{2}\lambda m(u^2 - u_x^2) \\ -\lambda^{-1}(u + u_x) + \frac{1}{2}\lambda m(u^2 - u_x^2) & -\lambda^{-2} I_2 + \frac{1}{2}(u^2 - u_x^2 + u_x u - uu_x) \end{pmatrix} \\ &\quad \triangleq \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \end{aligned} \tag{2.4}$$

where λ is a spectral parameter, I_2 is the 2×2 identity matrix, and u and m are the $sl(2)$ valued matrices (2.1).

The compatibility condition of (2.3) generates

$$U_t - V_x + [U, V] = 0. \tag{2.5}$$

Substituting the expressions of U and V given by (2.4) into (2.5), we find that (2.5) is nothing but the matrix equation (2.2). Hence, (2.3) exactly gives the Lax pair of equation (1.5).

Motivated by the 2×2 Hamiltonian operators we proposed in [42] and by a tough guesswork, we figure out the following 3×3 operator

$$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}, \tag{2.6}$$

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where

$$\begin{aligned}
J_{11} =&\partial m_{11}\partial^{-1}m_{11}\partial+\frac{1}{2}m_{12}\partial^{-1}m_{21}+\frac{1}{2}m_{21}\partial^{-1}m_{12},\\
J_{12} =&\partial m_{11}\partial^{-1}m_{12}\partial-m_{12}\partial^{-1}m_{11},\\
J_{13} =&\partial m_{11}\partial^{-1}m_{21}\partial-m_{21}\partial^{-1}m_{11},\\
J_{21} =&-J_{12}^{*}=\partial m_{12}\partial^{-1}m_{11}\partial-m_{11}\partial^{-1}m_{12},\\
J_{22} =&\partial m_{12}\partial^{-1}m_{12}\partial-m_{12}\partial^{-1}m_{12},\\
J_{23} =&\partial m_{12}\partial^{-1}m_{21}\partial+2m_{11}\partial^{-1}m_{11}+m_{12}\partial^{-1}m_{21},\\
J_{31} =&-J_{13}^{*}=\partial m_{21}\partial^{-1}m_{11}\partial-m_{11}\partial^{-1}m_{21},\\
J_{32} =&-J_{23}^{*}=\partial m_{21}\partial^{-1}m_{12}\partial+2m_{11}\partial^{-1}m_{11}+m_{21}\partial^{-1}m_{12},\\
J_{33} =&\partial m_{21}\partial^{-1}m_{21}\partial-m_{21}\partial^{-1}m_{21}.
\end{aligned} \tag{2.7}$$

It is easy to check that J is skew-symmetric. By a direct but tedious calculation, we can prove the Jacobi identity

$$\langle\alpha,J'[J\beta]\gamma\rangle+\langle\beta,J'[J\gamma]\alpha\rangle+\langle\gamma,J'[J\alpha]\beta\rangle=0, \tag{2.8}$$

where the prime-sign stands for the Gâteaux derivative of an operator [20], and

$$\alpha=(\alpha_1,\alpha_2,\alpha_3)^T,\quad \beta=(\beta_1,\beta_2,\beta_3)^T,\quad \gamma=(\gamma_1,\gamma_2,\gamma_3)^T, \tag{2.9}$$

are arbitrary testing three-dimensional vectors. Thus J is a Hamiltonian operator.

Proposition 2.1. *Equation (1.5) can be rewritten in the following Hamiltonian form*

$$(m_{11,t},\ m_{12,t},\ m_{21,t})^T=J\left(\frac{\delta H}{\delta m_{11}},\ \frac{\delta H}{\delta m_{12}},\ \frac{\delta H}{\delta m_{21}}\right)^T, \tag{2.10}$$

where J is given by (2.6), and

$$H_1=\frac{1}{2}\int_{-\infty}^{+\infty}(u_{11}^2+u_{12}u_{21}+u_{11,x}^2+u_{12,x}u_{21,x})dx. \tag{2.11}$$

We believe that the 3CH equation (1.5) could be cast into a bi-Hamiltonian system. But we didn't find another Hamiltonian operator yet that is compatible with the Hamiltonian operator (2.6). This is mainly due to complexity of the 3CH system (1.5) with three-component.

Next, let us construct conservation laws of equation (1.5) with the method developed in [40,41]. We consider

$$\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}_x=\begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix},\quad \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}_t=\begin{pmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}, \tag{2.12}$$

where Φ_1, Φ_2, U_{ij} and V_{ij}, $1\le i,j\le 2$, are all 2×2 matrices. Let $\Omega=\Phi_2\Phi_1^{-1}$, then we may check that Ω satisfies the following matrix Riccati equation

$$\Omega_x=U_{21}+U_{22}\Omega-\Omega U_{11}-\Omega U_{12}\Omega. \tag{2.13}$$

From the compatibility condition of (2.12), we arrive at the conservation law

$$[tr(U_{11}+U_{12}\Omega)]_t=[tr(V_{11}+V_{12}\Omega)]_x, \tag{2.14}$$

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where $tr(A)$ denotes the trace of a matrix A.

Substituting the expressions U_{ij} and V_{ij}, $1 \leq i, j \leq 2$, given by (2.4) into (2.13) and (2.14), we immediately obtain the Riccati equation and conservation law for our equation (1.5)

$$\Omega_x = \frac{1}{2}\lambda m + \Omega - \frac{1}{2}\lambda \Omega m \Omega, \tag{2.15}$$

$$[tr(m\Omega)]_t = \left[tr\left(-\lambda^{-2}(u-u_x)\Omega - \frac{1}{2}\lambda^{-1}(u^2-u_x^2) + \frac{1}{2}m(u^2-u_x^2)\Omega\right)\right]_x. \tag{2.16}$$

Equation (2.16) shows that $tr(m\Omega)$ is a generating function of the conserved densities. To derive the explicit forms of conserved densities, we expand $m\Omega$ in terms of negative powers of λ as below:

$$m\Omega = \sum_{j=0}^{\infty} \omega_j \lambda^{-j}. \tag{2.17}$$

Substituting (2.17) into (2.15) and equating the coefficients of powers of λ, we arrive at

$$\begin{aligned} \omega_0 &= (m_{11}^2 + m_{12}m_{21})^{\frac{1}{2}} I_2, \qquad \omega_1 = \omega_0^{-1}[\omega_0 - m(m^{-1}\omega_0)_x], \\ \omega_{j+1} &= \omega_0^{-1}\left[\omega_j - \frac{1}{2}\sum_{i+k=j+1,\ 1\leq i,k\leq j} \omega_i\omega_k - m(m^{-1}\omega_j)_x\right], \quad j \geq 1. \end{aligned} \tag{2.18}$$

Inserting (2.17) and (2.18) into (2.16), we finally obtain the following infinitely many conserved densities ρ_j and the associated fluxes F_j:

$$\begin{aligned} &\rho_0 = tr(\omega_0) = 2(m_{11}^2 + m_{12}m_{21})^{\frac{1}{2}}, \\ &F_0 = \frac{1}{2}tr[(u^2-u_x^2)\omega_0] = (u_{11}^2 - u_{11,x}^2 + u_{12}u_{21} - u_{12,x}u_{21,x})(m_{11}^2 + m_{12}m_{21})^{\frac{1}{2}}, \\ &\rho_1 = tr(\omega_1),\ \ F_1 = \frac{1}{2}tr[-(u^2-u_x^2) + (u^2-u_x^2)\omega_1], \\ &\rho_2 = tr(\omega_2),\ \ F_2 = tr[-(u-u_x)m^{-1}\omega_0 + \frac{1}{2}(u^2-u_x^2)\omega_2], \\ &\rho_{j+1} = tr(\omega_{j+1}),\ \ F_{j+1} = tr[-(u-u_x)m^{-1}\omega_{j-1} + \frac{1}{2}(u^2-u_x^2)\omega_{j+1}], \quad j \geq 2, \end{aligned} \tag{2.19}$$

where ω_j is given by (2.18).

3. Peakon solutions

Let us suppose that a single peakon solution of (1.5) has the following form

$$u_{11} = c_{11}e^{-|x-ct|}, \quad u_{12} = c_{12}e^{-|x-ct|}, \quad u_{21} = c_{21}e^{-|x-ct|}, \tag{3.1}$$

where the constants c_{11}, c_{12} and c_{21} are to be determined. The first order derivatives of u_{11},u_{12} and u_{21} do not exist at $x = ct$. Thus (3.1) can not be a solution of equation (1.5) in the classical sense.

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However, with the help of distribution theory we have

$$\begin{aligned} u_{11,x} &= -c_{11} sgn(x-ct)e^{-|x-ct|}, \quad m_{11} = 2c_{11}\delta(x-ct), \\ u_{12,x} &= -c_{12} sgn(x-ct)e^{-|x-ct|}, \quad m_{12} = 2c_{12}\delta(x-ct), \\ u_{21,x} &= -c_{21} sgn(x-ct)e^{-|x-ct|}, \quad m_{21} = 2c_{21}\delta(x-ct). \end{aligned} \tag{3.2}$$

Integrating the equation (1.5) against an arbitrary test function $\phi(x,t)$ with compact support, then moving the derivatives to $\phi(x,t)$, and finally substituting (3.1) and (3.2) into the resulting equations, the left hand side of the first equation in (1.5) produces

$$\begin{aligned} &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m_{11,t}\phi(x,t)dxdt \\ =& - cc_{11}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(E-E_{xx})_x\phi(x,t)dxdt \\ =& cc_{11}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(E\phi_x(x,t)-E\phi_{xxx}(x,t))dxdt \\ =& cc_{11}\int_{-\infty}^{+\infty}[\int_{-\infty}^{ct}e^{x-ct}\phi_x(x,t)dx+\int_{ct}^{+\infty}e^{-(x-ct)}\phi_x(x,t)dx \\ &-\int_{-\infty}^{ct}e^{x-ct}\phi_{xxx}(x,t)dx-\int_{ct}^{+\infty}e^{-(x-ct)}\phi_{xxx}(x,t)dx]dt \\ =& cc_{11}\int_{-\infty}^{+\infty}[\int_{-\infty}^{ct}e^{x-ct}\phi_x(x,t)dx+\int_{ct}^{+\infty}e^{-(x-ct)}\phi_x(x,t)dx \\ &+\int_{-\infty}^{ct}e^{x-ct}\phi_{xx}(x,t)dx-\int_{ct}^{+\infty}e^{-(x-ct)}\phi_{xx}(x,t)dx]dt \\ =& cc_{11}\int_{-\infty}^{+\infty}[\int_{-\infty}^{ct}e^{x-ct}\phi_x(x,t)dx+\int_{ct}^{+\infty}e^{-(x-ct)}\phi_x(x,t)dx \\ &+2\phi'(ct,t)-\int_{-\infty}^{ct}e^{x-ct}\phi_x(x,t)dx-\int_{ct}^{+\infty}e^{-(x-ct)}\phi_x(x,t)dx]dt \\ =& 2cc_{11}\int_{-\infty}^{+\infty}\phi'(ct,t)dt, \end{aligned}$$

where the notations $E = e^{-|x-ct|}$ and $\phi'(ct,t) = \phi_x(x,t)|_{x=ct}$. We split the right hand side of the first equation in (1.5) into the following four parts

$$\begin{aligned} &-\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m_{11}(u_{11}^2+u_{12}u_{21})\phi_x(x,t)dxdt \\ &+\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m_{11}(u_{11,x}^2+u_{12,x}u_{21,x})\phi_x(x,t)dxdt \\ &+\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m_{12}(u_{11,x}u_{21}-u_{11}u_{21,x})\phi(x,t)dxdt \\ &-\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m_{21}(u_{11}u_{12,x}-u_{11,x}u_{12})\phi(x,t)dxdt \\ &\triangleq I_1+I_2+I_3+I_4. \end{aligned}$$

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We compute I_1 as follows

$$
\begin{aligned}
I_1 =& -\frac{1}{2}c_{11}(c_{11}^2+c_{12}c_{21})\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(E-E_{xx})E^2\phi_x(x,t)dxdt \\
=& -\frac{1}{2}c_{11}(c_{11}^2+c_{12}c_{21})\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[E^3-(E^2E_x)_x+2EE_x^2]\phi_x(x,t)dxdt \\
=& -\frac{1}{2}c_{11}(c_{11}^2+c_{12}c_{21})\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[(E^3+2EE_x^2)\phi_x(x,t)+E^2E_x\phi_{xx}(x,t)]dxdt \\
=& -\frac{1}{2}c_{11}(c_{11}^2+c_{12}c_{21})\int_{-\infty}^{+\infty}[3\int_{-\infty}^{ct}e^{3(x-ct)}\phi_x(x,t)dx+3\int_{ct}^{+\infty}e^{-3(x-ct)}\phi_x(x,t)dx \\
&+\int_{-\infty}^{ct}e^{3(x-ct)}\phi_{xx}(x,t)dx-\int_{ct}^{+\infty}e^{-3(x-ct)}\phi_{xx}(x,t)dx]dt \\
=& -\frac{1}{2}c_{11}(c_{11}^2+c_{12}c_{21})\int_{-\infty}^{+\infty}[3\int_{-\infty}^{ct}e^{3(x-ct)}\phi_x(x,t)dx+3\int_{ct}^{+\infty}e^{-3(x-ct)}\phi_x(x,t)dx \\
&+2\phi'(ct,t)-3\int_{-\infty}^{ct}e^{3(x-ct)}\phi_x(x,t)dx-3\int_{ct}^{+\infty}e^{-3(x-ct)}\phi_x(x,t)dx]dt \\
=& -c_{11}(c_{11}^2+c_{12}c_{21})\int_{-\infty}^{+\infty}\phi'(ct,t)dt.
\end{aligned}
$$

In a similar manner, we obtain

$$
\begin{aligned}
I_2 &= \frac{1}{3}c_{11}(c_{11}^2+c_{12}c_{21})\int_{-\infty}^{+\infty}\phi'(ct,t)dt, \\
I_3 &= I_4 = 0.
\end{aligned}
$$

Thus it follows from the first equation in (1.5) that

$$
c_{11}^2+c_{12}c_{21} = -3c. \tag{3.3}
$$

By similar calculations, we find that the second and the third equations in (1.5) also give rise to (3.3). Hence the single peakon solution becomes

$$
u_{11} = c_{11}e^{-|x+\frac{c_{11}^2+c_{12}c_{21}}{3}t|}, \quad u_{12} = c_{12}e^{-|x+\frac{c_{11}^2+c_{12}c_{21}}{3}t|}, \quad u_{21} = c_{21}e^{-|x+\frac{c_{11}^2+c_{12}c_{21}}{3}t|}. \tag{3.4}
$$

In general, we assume that an N-peakon solution has the following form

$$
u_{11} = \sum_{j=1}^{N}p_j(t)e^{-|x-q_j(t)|}, \; u_{12} = \sum_{j=1}^{N}r_j(t)e^{-|x-q_j(t)|}, \; u_{21} = \sum_{j=1}^{N}s_j(t)e^{-|x-q_j(t)|}. \tag{3.5}
$$

In the distribution sense, we have

$$
\begin{aligned}
u_{11,x} &= -\sum_{j=1}^{N}p_j sgn(x-q_j)e^{-|x-q_j|}, \quad m_{11} = 2\sum_{j=1}^{N}p_j\delta(x-q_j), \\
u_{12,x} &= -\sum_{j=1}^{N}r_j sgn(x-q_j)e^{-|x-q_j|}, \quad m_{12} = 2\sum_{j=1}^{N}r_j\delta(x-q_j), \\
u_{21,x} &= -\sum_{j=1}^{N}s_j sgn(x-q_j)e^{-|x-q_j|}, \quad m_{21} = 2\sum_{j=1}^{N}s_j\delta(x-q_j).
\end{aligned} \tag{3.6}
$$

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Substituting (3.5) and (3.6) into (1.5) and integrating against test function with compact support, we obtain the N-peakon dynamic system as follows:

$$\begin{cases} p_{j,t} = \frac{1}{2} r_j \sum_{i,k=1}^{N} p_i s_k \left(sgn(q_j - q_k) - sgn(q_j - q_i)\right) e^{-|q_j - q_k| - |q_j - q_i|} \\ \qquad - \frac{1}{2} s_j \sum_{i,k=1}^{N} p_i r_k \left(sgn(q_j - q_i) - sgn(q_j - q_k)\right) e^{-|q_j - q_k| - |q_j - q_i|}, \\ r_{j,t} = p_j \sum_{i,k=1}^{N} p_i r_k \left(sgn(q_j - q_i) - sgn(q_j - q_k)\right) e^{-|q_j - q_k| - |q_j - q_i|} \\ \qquad + \frac{1}{2} r_j \sum_{i,k=1}^{N} r_i s_k \left(sgn(q_j - q_k) - sgn(q_j - q_i)\right) e^{-|q_j - q_k| - |q_j - q_i|}, \\ s_{j,t} = p_j \sum_{i,k=1}^{N} p_i s_k \left(sgn(q_j - q_i) - sgn(q_j - q_k)\right) e^{-|q_j - q_k| - |q_j - q_i|} \\ \qquad - \frac{1}{2} s_j \sum_{i,k=1}^{N} r_i s_k \left(sgn(q_j - q_k) - sgn(q_j - q_i)\right) e^{-|q_j - q_k| - |q_j - q_i|}, \\ q_{j,t} = \frac{1}{6}(p_j^2 + r_j s_j) - \frac{1}{2} \sum_{i,k=1}^{N} (p_i p_k + r_i s_k) \left(1 - sgn(q_j - q_i) sgn(q_j - q_k)\right) e^{-|q_j - q_i| - |q_j - q_k|}. \end{cases} \tag{3.7}$$

We still do not know whether this system is integrable for $N \geq 2$ with respect to a suitable Poisson structure.

4. Reductions and a new integrable perturbation equation

As mentioned above, system (1.5) can be reduced to the CH equation (1.1) as $u_{11} = 0,\ u_{21} = 2$ and to the cubic nonlinear CH equation (1.3) as $u_{12} = u_{21} = 0$. Now we discuss the two-component reductions of system (1.5).

Example 1. The integrable two-component system proposed in [42]

As $u_{11} = 0$, equation (1.5) is reduced to a two-component equation

$$\begin{cases} m_{12,t} = \frac{1}{2}[m_{12}(u_{12}u_{21} - u_{12,x}u_{21,x})]_x + \frac{1}{2} m_{12}(u_{12,x}u_{21} - u_{12}u_{21,x}), \\ m_{21,t} = \frac{1}{2}[m_{21}(u_{12}u_{21} - u_{12,x}u_{21,x})]_x + \frac{1}{2} m_{21}(u_{12}u_{21,x} - u_{12,x}u_{21}), \\ m_{12} = u_{12} - u_{12,xx}, \\ m_{21} = u_{21} - u_{21,xx}, \end{cases} \tag{4.1}$$

which is exactly the system we derived in [42]. For the bi-Hamiltonian structure and peakon solutions of this system, one may see [42].

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Example 2. The integrable two-component system presented in [37]

As $u_{12} = u_{21}$, equation (1.5) is reduced to a two-component equation

$$\begin{cases} m_{11,t} = \frac{1}{2}[m_{11}(u_{11}^2 + u_{12}^2 - u_{11,x}^2 - u_{12,x}^2)]_x + m_{12}(u_{11,x}u_{12} - u_{11}u_{12,x}), \\ m_{12,t} = \frac{1}{2}[m_{12}(u_{11}^2 + u_{12}^2 - u_{11,x}^2 - u_{12,x}^2)]_x + m_{11}(u_{11}u_{12,x} - u_{11,x}u_{12}), \\ m_{11} = u_{11} - u_{11,xx}, \\ m_{12} = u_{12} - u_{12,xx}, \end{cases} \tag{4.2}$$

which was proposed by Qu, Song and Yao in [37]. Here in our paper, we want to derive the peakon solutions to this system. Suppose that an N-peakon solution of (4.2) has the form

$$u_{11} = \sum_{j=1}^{N} p_j(t)e^{-|x-q_j(t)|}, \;\; u_{12} = \sum_{j=1}^{N} r_j(t)e^{-|x-q_j(t)|}. \tag{4.3}$$

From (3.7) and the reduction condition $u_{12} = u_{21}$, we immediately arrive at the N-peakon dynamic system of (4.2):

$$\begin{cases} p_{j,t} = r_j \sum_{i,k=1}^{N} p_i r_k \left(sgn(q_j - q_k) - sgn(q_j - q_i)\right) e^{-|q_j-q_k|-|q_j-q_i|}, \\ r_{j,t} = p_j \sum_{i,k=1}^{N} p_i r_k \left(sgn(q_j - q_i) - sgn(q_j - q_k)\right) e^{-|q_j-q_k|-|q_j-q_i|}, \\ q_{j,t} = \frac{1}{6}(p_j^2 + r_j^2) - \frac{1}{2}\sum_{i,k=1}^{N} (p_i p_k + r_i r_k)\left(1 - sgn(q_j - q_i)sgn(q_j - q_k)\right) e^{-|q_j-q_i|-|q_j-q_k|}. \end{cases} \tag{4.4}$$

For $N = 1$, we find that the single-peakon solution reads

$$u_{11} = c_1 e^{-|x+\frac{c_1^2+c_2^2}{3}t|}, \quad u_{12} = c_2 e^{-|x+\frac{c_1^2+c_2^2}{3}t|}, \tag{4.5}$$

where c_1 and c_2 are integration constants. For $N = 2$, we may solve (4.4) as

$$\begin{cases} q_1(t) = -\frac{1}{3}A_1^2 t + \frac{3A_1A_2\cos(A_3-A_4)}{|A_1^2-A_2^2|} sgn(t)\left(e^{-\frac{1}{3}|(A_1^2-A_2^2)t|} - 1\right), \\ q_2(t) = -\frac{1}{3}A_2^2 t + \frac{3A_1A_2\cos(A_3-A_4)}{|A_1^2-A_2^2|} sgn(t)\left(e^{-\frac{1}{3}|(A_1^2-A_2^2)t|} - 1\right), \\ p_1(t) = A_1 \sin\left(\frac{3A_1A_2\sin(A_3-A_4)}{A_1^2-A_2^2} e^{-\frac{1}{3}|(A_1^2-A_2^2)t|} + A_3\right), \\ p_2(t) = A_2 \sin\left(\frac{3A_1A_2\sin(A_3-A_4)}{A_1^2-A_2^2} e^{-\frac{1}{3}|(A_1^2-A_2^2)t|} + A_4\right), \\ r_1(t) = A_1 \cos\left(\frac{3A_1A_2\sin(A_3-A_4)}{A_1^2-A_2^2} e^{-\frac{1}{3}|(A_1^2-A_2^2)t|} + A_3\right), \\ r_2(t) = A_2 \cos\left(\frac{3A_1A_2\sin(A_3-A_4)}{A_1^2-A_2^2} e^{-\frac{1}{3}|(A_1^2-A_2^2)t|} + A_4\right), \end{cases} \tag{4.6}$$

where $A_1, \cdots, A_4$ are integration constants. In particular, letting $A_1 = 1, A_2 = 2, A_3 = 0$ and $A_4 = \frac{\pi}{6}$, we obtain the two-peakon solution of (4.2)

$$\begin{cases} u_{11} = \sin(e^{-|t|})e^{-|x-q_1(t)|} + 2\sin(e^{-|t|} + \frac{\pi}{6})e^{-|x-q_2(t)|}, \\ u_{12} = \cos(e^{-|t|})e^{-|x-q_1(t)|} + 2\cos(e^{-|t|} + \frac{\pi}{6})e^{-|x-q_2(t)|}, \end{cases} \tag{4.7}$$

with

$$q_1(t) = -\frac{1}{3}t + \sqrt{3}sgn(t)\left(e^{-|t|} - 1\right), \quad q_2(t) = -\frac{4}{3}t + \sqrt{3}sgn(t)\left(e^{-|t|} - 1\right). \tag{4.8}$$

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The two-peakon collides at $t=0$, since $q_1(0)=q_2(0)=0$. See Figures 1 and 2 for the two-peakon dynamics of the potentials $u_{11}(x,t)$ and $u_{12}(x,t)$.

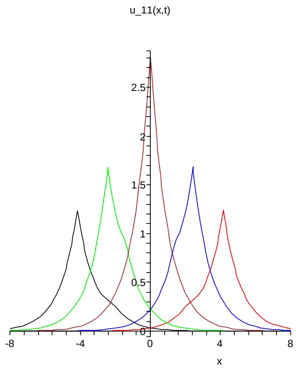

Fig. 1. The two-peakon dynamic for the potential $u_{11}(x,t)$ in (4.7). Red line: $t=-2$; Blue line: $t=-1$; Brown line: $t=0$ (collision); Green line: $t=1$; Black line: $t=2$.

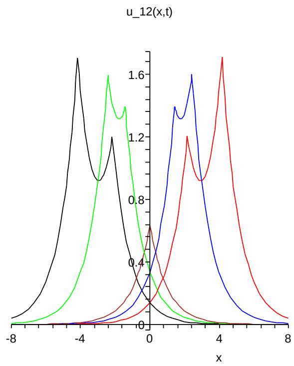

Fig. 2. The two-peakon dynamic for the potential $u_{12}(x,t)$ in (4.7). Red line: $t=-2$; Blue line: $t=-1$; Brown line: $t=0$ (collision); Green line: $t=1$; Black line: $t=2$.

Example 3. A new integrable perturbation of cubic nonlinear CH equation

As $u_{12}=0$, equation (1.5) is cast into

$$\begin{cases} m_{11,t}=\frac{1}{2}[m_{11}(u_{11}^2-u_{11,x}^2)]_x, \\ m_{21,t}=\frac{1}{2}[m_{21}(u_{11}^2-u_{11,x}^2)]_x+m_{11}(u_{11}u_{21,x}-u_{11,x}u_{21}), \\ m_{11}=u_{11}-u_{11,xx}, \\ m_{21}=u_{21}-u_{21,xx}. \end{cases} \tag{4.9}$$

This equation is different from the standard perturbation equation of the cubic nonlinear CH system

$$\begin{cases} m_{11,t}=\frac{1}{2}[m_{11}(u_{11}^2-u_{11,x}^2)]_x, \\ m_{21,t}=\frac{1}{2}[m_{21}(u_{11}^2-u_{11,x}^2)]_x+[m_{11}(u_{11}u_{21}-u_{11,x}u_{21,x})]_x, \\ m_{11}=u_{11}-u_{11,xx}, \\ m_{21}=u_{21}-u_{21,xx}, \end{cases} \tag{4.10}$$

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which can be constructed by a small disturbance (in the sense of [19, 30]) of the cubic nonlinear CH equation (for detail, see Remark 1 at the end of this example). Thus (4.9) is a new integrable perturbation of the cubic nonlinear CH equation. It is noticed that by direct calculations one may see that the second potential u_{21} of the standard perturbation equation (4.10) does not admit peakon solution in the form of $u_{21} = p(t)e^{-|x-q(t)|}$. However, we find that our new perturbation equation (4.9) admits peakon solutions. In fact, suppose that an N-peakon solution of (4.9) has the form

$$u_{11} = \sum_{j=1}^{N} p_j(t)e^{-|x-q_j(t)|}, \ \ u_{21} = \sum_{j=1}^{N} s_j(t)e^{-|x-q_j(t)|}, \tag{4.11}$$

we obtain the N-peakon dynamic system of (4.9) as follows:

$$\begin{cases} p_{j,t} =0, \\ q_{j,t} =\frac{1}{6}p_j^2 - \frac{1}{2}\sum_{i,k=1}^{N} p_i p_k \left(1 - sgn(q_j - q_i)sgn(q_j - q_k)\right) e^{-|q_j-q_i|-|q_j-q_k|}, \\ s_{j,t} =p_j \sum_{i,k=1}^{N} p_i s_k \left(sgn(q_j - q_i) - sgn(q_j - q_k)\right) e^{-|q_j-q_k|-|q_j-q_i|}. \end{cases} \tag{4.12}$$

For $N = 1$, we find that the single-peakon solution takes the form

$$u_{11} = \sqrt{-3c}e^{-|x-ct|}, \quad u_{21} = c_{21}e^{-|x-ct|}, \tag{4.13}$$

where c_{21} is an arbitrary constant.

For $N = 2$, (4.12) becomes

$$\begin{cases} p_{1,t} =p_{2,t} = 0, \\ q_{1,t} =-\frac{1}{3}p_1^2 - p_1 p_2 e^{-|q_1-q_2|}, \\ q_{2,t} =-\frac{1}{3}p_2^2 - p_1 p_2 e^{-|q_1-q_2|}, \\ s_{1,t} =p_1(p_2 s_1 - p_1 s_2)sgn(q_1 - q_2)e^{-|q_1-q_2|}, \\ s_{2,t} =p_2(p_2 s_1 - p_1 s_2)sgn(q_1 - q_2)e^{-|q_1-q_2|}. \end{cases} \tag{4.14}$$

From the first equation of (4.14), we obtain

$$p_1 = A_1, \ \ p_2 = A_2, \tag{4.15}$$

where A_1 and A_2 are integration constants. Let us set $0 < A_1 < A_2$. Then from (4.14), we arrive at

$$\begin{cases} q_1(t) = -\frac{1}{3}A_1^2 t + \frac{3A_1A_2}{A_2^2-A_1^2} sgn(t)\left(e^{-\frac{1}{3}(A_2^2-A_1^2)|t|} - 1\right), \\ q_2(t) = -\frac{1}{3}A_2^2 t + \frac{3A_1A_2}{A_2^2-A_1^2} sgn(t)\left(e^{-\frac{1}{3}(A_2^2-A_1^2)|t|} - 1\right), \\ s_1(t) = \frac{3A_1A_3}{A_1^2-A_2^2} e^{-\frac{1}{3}(A_2^2-A_1^2)|t|} + A_4, \\ s_2(t) = \frac{1}{A_1}(A_2 s_1 - A_3), \end{cases} \tag{4.16}$$

where A_3 and A_4 are integration constants. In particular, taking $A_1 = -A_3 = 1$, $A_2 = 2$ and $A_4 = 0$, we obtain the two-peakon solution of (4.9)

$$u_{11} = e^{-|x-q_1(t)|} + 2e^{-|x-q_2(t)|}, \ \ u_{21} = e^{-|t|}e^{-|x-q_1(t)|} + (2e^{-|t|} + 1)e^{-|x-q_2(t)|}, \tag{4.17}$$

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with

$$q_1(t) = -\frac{1}{3}t + 2sgn(t)\left(e^{-|t|} - 1\right), \quad q_2(t) = -\frac{4}{3}t + 2sgn(t)\left(e^{-|t|} - 1\right). \tag{4.18}$$

This two-peakon collides at the moment of $t = 0$, since $q_1(0) = q_2(0) = 0$. See Figures 3 and 4 for the two-peakon dynamics of the potentials $u_{11}(x,t)$ and $u_{21}(x,t)$.

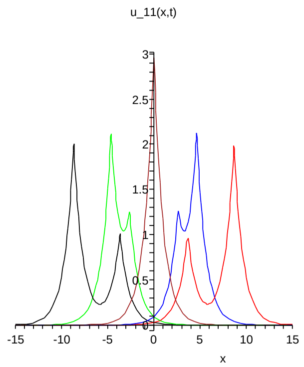

Fig. 3. The two-peakon dynamic for the potential $u_{11}(x,t)$ in (4.17). Red line: $t = -5$; Blue line: $t = -2$; Brown line: $t = 0$ (collision); Green line: $t = 2$; Black line: $t = 5$.

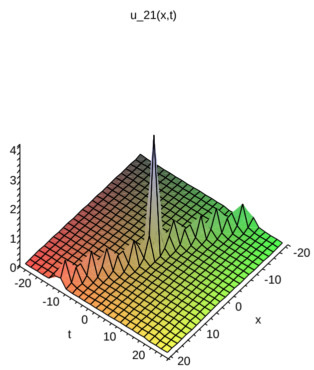

Fig. 4. 3-dimensional graph for the two-peakon dynamic of the potential $u_{21}(x,t)$ in (4.17).

Remark 1. There have been a lot of studies investigating the perturbation equations constructed by a small disturbance of the soliton equations, such as the KdV equation, the MKdV equation, and the AKNS equation (see for example Refs. [19,30,39]). However, discussions about the perturbation equations of the CH type equations are rare [27]. This is mainly due to the CH type equations being of non-evolutionary type. To the best of our knowledge, perturbation equations of the cubic nonlinear CH system have not been proposed in the literature yet. So, we provide a brief derivation in our paper. Following the paper [30], we make a perturbation expansion

$$u = \sum_{j=0}^{N} \eta_j \varepsilon^j, \quad N \geq 1. \tag{4.19}$$

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Replacing u with (4.19) in the cubic nonlinear CH equation (1.3), and equating the coefficients of powers of ε (up to ε^N), we obtain the nth perturbation system of the CH equation with variables η_0, $\eta_1, \cdots, \eta_N$:

$$\begin{cases} m_{0,t} = \dfrac{1}{2}[m_0(\eta_0^2 - \eta_{0,x}^2)]_x, \\ m_{1,t} = [m_0(\eta_0\eta_1 - \eta_{0,x}\eta_{1,x})]_x + \dfrac{1}{2}[m_1(\eta_0^2 - \eta_{0,x}^2)]_x, \\ \quad\vdots \\ m_{j,t} = \dfrac{1}{2}\left[\sum\limits_{k+l=j,\ k,l\geq 0} m_k \sum\limits_{i+h=l,\ i,h\geq 0} (\eta_i\eta_h - \eta_{i,x}\eta_{h,x})\right]_x, \\ \quad\vdots \\ m_{N,t} = \dfrac{1}{2}\left[\sum\limits_{k+l=N,\ k,l\geq 0} m_k \sum\limits_{i+h=l,\ i,h\geq 0} (\eta_i\eta_h - \eta_{i,x}\eta_{h,x})\right]_x, \\ m_0 = \eta_0 - \eta_{0,xx},\ m_1 = \eta_1 - \eta_{1,xx},\ \cdots,\ m_N = \eta_N - \eta_{N,xx}. \end{cases} \tag{4.20}$$

For example, as $N = 1$, we arrive at the first-order perturbation of cubic nonlinear CH equation

$$\begin{cases} m_{0,t} = \frac{1}{2}[m_0(\eta_0^2 - \eta_{0,x}^2)]_x, \\ m_{1,t} = [m_0(\eta_0\eta_1 - \eta_{0,x}\eta_{1,x})]_x + \frac{1}{2}[m_1(\eta_0^2 - \eta_{0,x}^2)]_x, \\ m_0 = \eta_0 - \eta_{0,xx}, \\ m_1 = \eta_1 - \eta_{1,xx}, \end{cases} \tag{4.21}$$

which is nothing but equation (4.10). With suitable adaptations of the techniques used in [30], we may derive the Lax representation, bi-Hamiltonian structure and recursion operator for the resulting perturbation system (4.20). We neglect the details of these results here, since this topic is out of the scope of the present paper. But we would like to stress that, as mentioned before, this perturbation equation does not admit a peakon solution.

5. Conclusions and discussions

We have presented an integrable 3CH peakon system with cubic nonlinearity. The Lax representation, Hamiltonian structure and infinitely many conservation laws of this system are investigated. We also discuss the reductions of this system. In particular, by a reduction we found a new integrable perturbation equation of the cubic nonlinear CH system. In contrast with the standard perturbation of the cubic nonlinear CH equation, this new integrable perturbation of the cubic nonlinear CH equation admits peakon solutions.

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