

A Synthetical Two-Component Model with Peakon Solutions

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A generalized two-component model with peakon solutions is proposed in this paper. It allows an arbitrary function to be involved in as well as including some existing integrable peakon equations as special reductions. The generalized two-component system is shown to possess Lax pair and infinitely many conservation laws. Bi-Hamiltonian structures and peakon interactions are discussed in detail for typical representative equations of the generalized system. In particular, a new type of N -peakon solution, which is not in the traveling wave type, is obtained from the generalized system.

1. Introduction

In recent years, the Camassa–Holm (CH) equation [1]

$$m_t = 2mu_x + m_xu, \quad m = u - u_{xx} + k, \quad (1)$$

where (k is an arbitrary constant) derived by Camassa and Holm [1] as a shallow water wave model, has attracted much attention and various studies. The CH equation admits Lax representation [1], bi-Hamiltonian structure [2, 3], and is integrable by the inverse scattering transformation [4]. Also it possesses multiple peaked soliton solutions [1, 5] and algebrogeometric solutions [6, 7]. The most interesting feature of the CH equation is that it admits peaked soliton (peakon) solutions in the case $k = 0$ [1, 5]. A peakon is a weak solution in some Sobolev space with corner at its crest. The stability and analysis study of peakons were discussed in several references [8–12].

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The interesting characteristics of the CH equation stimulated more people to search new integrable models which admit peakon solutions. Among them, for example, there are:

- (1) The Degasperis–Procesi (DP) equation [13]

$$m_t = 3mu_x + m_xu, \quad m = u - u_{xx}, \quad (2)$$

which was shown integrable with Lax pair and bi-Hamiltonian structure in [14], multipeakon dynamics in [15], and shocks formation in [16];

- (2) The cubic nonlinear peakon equation—the Fokas–Olver–Rosenau–Qiao (FORQ) equation [3, 17, 18]

$$m_t = [m(u^2 - u_x^2)]_x, \quad m = u - u_{xx}, \quad (3)$$

which was shown to have Lax pair and bi-Hamiltonian structure in [19], and peakon solutions in [20];

- (3) The Novikov’s cubic nonlinear equation [21]

$$m_t = u^2m_x + 3uu_xm, \quad m = u - u_{xx}, \quad (4)$$

which was shown integrable with Lax pair, bi-Hamiltonian structure, and conservation laws in [22]; and

- (4) The generalized CH equation with both quadratic and cubic nonlinearity [17, 18, 23]

$$m_t = \frac{1}{2}k_1 [m(u^2 - u_x^2)]_x + \frac{1}{2}k_2(2mu_x + m_xu), \quad m = u - u_{xx}, \quad (5)$$

where k_1 and k_2 are two arbitrary constants. Equation (5) was proven to possess Lax pair, conservation laws, and peakon solutions in [23].

Equation (5) is actually a linear combination of CH equation (1) and cubic nonlinear equation (3). This structure is very similar to the Gardner equation, known as a linear combination of KdV and mKdV equations. Thus, equation (5) is the dual system of the Gardner equation from the viewpoint of tri-Hamiltonian duality [3, 18]. In the literature [17], a more generalized version of Equation (5) was derived by Fokas from the two-dimensional hydrodynamical equations for surface waves. We also notice that by some appropriate rescaling or gauge transformations Equation (5) is equivalent to Equation (3).

All equations shown above are scalar integrable peakon models. Another important task is to find integrable multicomponent peakon systems to enrich the theory of soliton and integrable systems. For example, the integrable two-component CH equations are proposed in [3, 24–26]. The integrable two-component forms of the cubic peakon systems (3) and (4) are presented in [27–29].

In addition to the integrable generalizations of peakon equations, there are also some works appearing to study the nonintegrable generalizations of

peakon equations. The most well-known example is the so-called b -family equation by Holm and Staley [30,31]

$$m_t = bmu_x + m_x u, \quad m = u - u_{xx}, \quad (6)$$

where b is an arbitrary constant. The case of $b = 2$ is exactly the CH equation, while the case of $b = 3$ recovers the DP equation. According to various tests for integrability, it is known that the cases of $b = 2$ and $b = 3$ are the only integrable equations within this family [32–35]. However, for any b , all those equations admit peakon solutions [31]. Holm and Staley also studied peakon dynamics of (6) for different values of b and discussed how they behave with changing b [31]. In the literature [36], Popowicz proposed a two-component system, which can be considered as a coupling between the CH equation and the DP equation. Later, Hone and Irle showed that the two-component Popowicz system is nonintegrable, but admits single-peakon solution as well as multipeakon solutions [37].

In this paper, we propose the following generalized version of the two-component peakon system:

$$\begin{cases} m_t = (mH)_x + mH - \frac{1}{2}m(u - u_x)(v + v_x), \\ n_t = (nH)_x - nH + \frac{1}{2}n(u - u_x)(v + v_x), \\ m = u - u_{xx}, \\ n = v - v_{xx}, \end{cases} \quad (7)$$

where H is an arbitrary function of u , v , and their derivatives. As $v = 2$ and $H = u$, Equation (7) is reduced to the CH equation (1). As $v = 2u$ and $H = (u^2 - u_x^2)$, Equation (7) is reduced to the FORQ equation (3). As $v = k_1 u + k_2$ and $H = \frac{1}{2}[k_1(u^2 - u_x^2) + k_2 u]$, Equation (7) is cast into the generalized CH equation (5). Thus, Equation (7) is a kind of the two-component generalization of Equations (1), (3), and (5). We show that the generalized system (7) possesses an $sl(2)$ -valued Lax pair and infinitely many conservation laws. Since the arbitrary function H is involved in (7), we do not expect all those equations have bi-Hamiltonian structures in general. Nevertheless, we demonstrate that for some special choices of H we may find the corresponding bi-Hamiltonian structures. Such a system is interesting, because we may obtain quite a large number of integrable peakon equations by choosing different H . We take some examples to discuss in detail the bi-Hamiltonian structures and the peakon interactions for some equations in the family (7). From the equations in the family (7), we obtain a new type of N -peakon solution which is not presented in the traveling wave type.

The whole paper is organized as follows. Section 2 provides the Lax pair and conservation laws for the system (7). Section 3 studies the bi-Hamiltonian structures and the multipeakon solutions of some two-component equations in the family (7). Section 4 supplies a proof for the bi-Hamiltonian property in

each example discussed in Section 3. Some conclusions and open problems are addressed in Section 5.

2. Lax pair and conservation laws

Let us consider a pair of 2×2 matrix spectral problems of the following type:

$$\phi_x = U\phi, \quad U = \frac{1}{2} \begin{pmatrix} -1 & \lambda m \\ -\lambda n & 1 \end{pmatrix}, \quad (8)$$

$$\phi_t = V\phi, \quad V = -\frac{1}{2} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & -V_{11} \end{pmatrix}, \quad (9)$$

where λ is a spectral parameter, $\phi = (\phi_1, \phi_2)^T$, $m = u - u_{xx}$, $n = v - v_{xx}$, and

$$\begin{aligned} V_{11} &= \lambda^{-2} + \frac{1}{2}(u - u_x)(v + v_x), \\ V_{12} &= -\lambda^{-1}(u - u_x) - \lambda m H, \\ V_{21} &= \lambda^{-1}(v + v_x) + \lambda n H, \end{aligned} \quad (10)$$

and H is an arbitrary function of u , v , and their derivatives.

It is easy to see that the compatibility condition of (8) and (9) reads

$$U_t - V_x + [U, V] = 0. \quad (11)$$

Substituting the expressions of (8) and (9) into (11), we immediately find that (11) is nothing but Equation (7). Thus, (8) and (9) compose of a Lax pair of Equation (7).

Remark 1. Our generalized system with an arbitrary function H involved does admit an $sl(2)$ -valued Lax representation. System (7) is produced by the compatibility condition (11) of the spectral problems (8) and (9) where such an arbitrary function is included in V part. The arbitrary function H is able to appear because the Lax equation (11) is an overdetermined system by choosing the appropriate V (dependent on λ) to match U .

Next, let us construct the conservation laws for system (7) by using spectral problems (8) and (9). Let $\omega = \frac{\phi_2}{\phi_1}$, then it follows from (8) that ω satisfies the Riccati equation

$$\omega_x = -\frac{1}{2}\lambda m \omega^2 + \omega - \frac{1}{2}\lambda n. \quad (12)$$

Based on (8) and (9), we obtain

$$\begin{aligned} (\ln \phi_1)_x &= -\frac{1}{2} + \frac{1}{2}\lambda m\omega, \\ (\ln \phi_1)_t &= -\frac{1}{2} [\lambda^{-2} - \lambda^{-1}(u - u_x)\omega + \frac{1}{2}(u - u_x)(v + v_x) - \lambda m H\omega], \end{aligned} \quad (13)$$

which generates the following conservation law of Equation (7):

$$\rho_t = A_x, \quad (14)$$

where

$$\begin{aligned} \rho &= m\omega, \\ A &= -\frac{1}{2}\lambda^{-1}(u - u_x)(v + v_x) + \lambda^{-2}(u - u_x)\omega + mH\omega. \end{aligned} \quad (15)$$

Usually ρ and A are called a conserved density and an associated flux, respectively.

We are able to derive the explicit forms of conservation densities by expanding ω in powers of λ in two ways. The first one is to expand ω in terms of negative powers of λ as

$$\omega = \sum_{j=0}^{\infty} \omega_j \lambda^{-j}. \quad (16)$$

By substituting (16) into (12) and equating the coefficients of powers of λ , we arrive at

$$\begin{aligned} \omega_0 &= \sqrt{-\frac{n}{m}}, \quad \omega_1 = \frac{mn_x - m_x n - 2mn}{2m^2 n}, \\ \omega_{j+1} &= \frac{1}{m\omega_0} \left[\omega_j - \omega_{j,x} - \frac{1}{2}m \sum_{i+k=j+1, 1 \leq i, k \leq j} \omega_i \omega_k \right], \quad j \geq 1. \end{aligned} \quad (17)$$

Inserting (16) and (17) into (15), we obtain the following infinitely many conserved densities and the associated fluxes

$$\begin{aligned} \rho_0 &= \sqrt{-mn}, \quad A_0 = H\sqrt{-mn}, \\ \rho_1 &= \frac{mn_x - m_x n - 2mn}{2mn}, \\ A_1 &= -\frac{1}{2}(u - u_x)(v + v_x) + \frac{(mn_x - m_x n - 2mn)H}{2mn}, \\ \rho_j &= m\omega_j, \quad A_j = (u - u_x)\omega_{j-2} + mH\omega_j, \quad j \geq 2, \end{aligned} \quad (18)$$

where ω_j is given by (17).

The second expansion of ω is in the positive powers of λ as

$$\omega = \sum_{j=0}^{\infty} \omega_j \lambda^j. \quad (19)$$

Substituting (19) into (12) and comparing powers of λ lead to

$$\omega_{2j} = 0, \quad j \geq 0, \quad (20)$$

$$\omega_1 = \frac{1}{2}(v + v_x), \quad \omega_{2j+1} - \omega_{2j+1,x} = \frac{1}{2}m \sum_{i+k=2j, 0 \leq i, k \leq 2j} \omega_i \omega_k, \quad j \geq 1. \quad (21)$$

From formula (20), we know

$$\rho_{2j} = 0, \quad A_{2j} = 0, \quad j \geq 0, \quad (22)$$

which means the even-index conserved densities and associated fluxes are trivial. From formula (21), we arrive at the odd-index conserved densities and associated fluxes

$$\begin{aligned} \rho_1 &= \frac{1}{2}m(v + v_x), \quad A_1 = (u - u_x)\omega_3 + \frac{1}{2}m(v + v_x)H, \\ \rho_{2j+1} &= m\omega_{2j+1}, \quad A_{2j+1} = (u - u_x)\omega_{2j+3} + mH\omega_{2j+1}, \quad j \geq 1, \end{aligned} \quad (23)$$

where the odd-index ω_{2j+1} is defined by the recursion relation

$$\omega_{2j+1} = \frac{1}{2}(1 - \partial_x)^{-1} \left(m \sum_{i+k=2j, 0 \leq i, k \leq 2j} \omega_i \omega_k \right), \quad j \geq 1. \quad (24)$$

We should remark that the relation (24) shows the nontrivial high-order conserved densities in the sequence (23) may involve in nonlocal expressions in u and v . However, the conserved densities in the sequence (18) are local ones.

Remark 2. The expressions (18) and (23) show that all members in our generalized system possess the same conserved quantities but different conserved fluxes. This is because the conserved quantities are derived from the Riccati equation (12) that only depends on the spatial part of the Lax representation which keeps the same for all members in the family; while the conserved fluxes rely on the temporal part of the Lax representation which changes for different members.

3. Two-component peakon systems

The two-component system (7) is of great interest because different choices of H lead to different peakon equations. Let us discuss some special cases in the following examples.

EXAMPLE 1. *A new integrable system with a new type of peakon solutions.*

Taking $H = 0$ in Equation (7) gives rise to the following integrable two-component model:

$$\begin{cases} m_t = -\frac{1}{2}m(u - u_x)(v + v_x), \\ n_t = \frac{1}{2}n(u - u_x)(v + v_x), \\ m = u - u_{xx}, n = v - v_{xx}. \end{cases} \quad (25)$$

This model can be rewritten as the following bi-Hamiltonian form

$$(m_t, n_t)^T = J \left(\frac{\delta H_2}{\delta m}, \frac{\delta H_2}{\delta n} \right)^T = K \left(\frac{\delta H_1}{\delta m}, \frac{\delta H_1}{\delta n} \right)^T, \quad (26)$$

where

$$J = \begin{pmatrix} 0 & -\partial - 1 \\ -\partial + 1 & 0 \end{pmatrix}, K = \begin{pmatrix} -m\partial^{-1}m & m\partial^{-1}n \\ n\partial^{-1}m & -n\partial^{-1}n \end{pmatrix}, \quad (27)$$

$$H_1 = \frac{1}{2} \int_{-\infty}^{+\infty} m(v + v_x)dx, H_2 = \frac{1}{4} \int_{-\infty}^{+\infty} (u - u_x)^2(v + v_x)ndx. \quad (28)$$

In Section 4, we will provide a detailed proof for the compatibility of the Hamiltonian pairs J and K in this example and the next three examples.

Let us assume that (25) has the following one-peakon solution

$$u = p_1(t)e^{-|x-q_1(t)|}, \quad v = r_1(t)e^{-|x-q_1(t)|}, \quad (29)$$

where $p_1(t)$, $r_1(t)$, and $q_1(t)$ are functions of t needed to be determined. Substituting (29) into (25) and integrating against the test function with support around the peak, we obtain

$$p_{1,t} = -\frac{1}{3}p_1^2r_1, \quad r_{1,t} = \frac{1}{3}p_1r_1^2, \quad q_{1,t} = 0, \quad (30)$$

which yields

$$p_1(t) = A_2e^{-\frac{1}{3}A_1t}, \quad r_1(t) = \frac{A_1}{A_2}e^{\frac{1}{3}A_1t}, \quad q_1(t) = A_3, \quad (31)$$

where A_1 , A_2 , and A_3 are integration constants. Thus, we obtain the peakon solutions as follows:

$$u(x, t) = A_2e^{-\frac{1}{3}A_1t}e^{-|x-A_3|}, \quad v(x, t) = \frac{A_1}{A_2}e^{\frac{1}{3}A_1t}e^{-|x-A_3|}. \quad (32)$$

This pair of single-peakon solutions is not presented in the traveling wave type, because the peakon position $q_1(t) = A_3$ is stationary. To the best of our knowledge, almost all integrable peakon models have single peakons which are of traveling wave type. So, we find a new integrable peakon system (25) whose

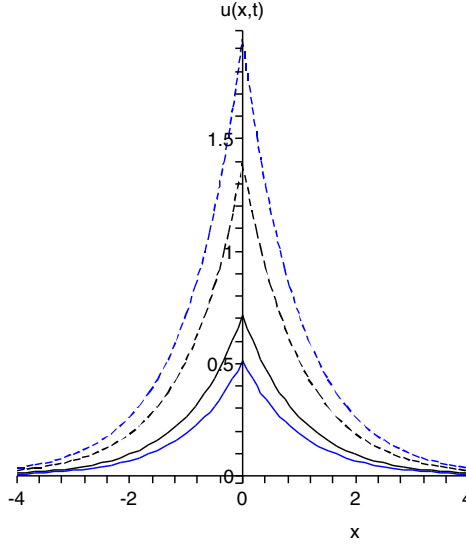


Figure 1. The single-peakon solution given by (32) with $A_1 = A_2 = 1$ and $A_3 = 0$. Solid line: $u(x, t)$; Dashed line: $v(x, t)$; Black: $t = 1$; Blue: $t = 2$.

peakon solution is not in traveling wave type. See Figure 1 for the profile of the new single-peakon solution. We remark that the amplitudes of the peakons of Equation (25) grow/decay exponentially with time. Recently, Lundmark and Szmigielski [38] found that the Geng-Xue two-component system [29] has a similar type of peakons (with amplitudes exponentially growing/decaying with time).

Let us suppose the N -peakon solution in the form of

$$u(x, t) = \sum_{j=1}^N p_j(t) e^{-|x-q_j(t)|}, \quad v(x, t) = \sum_{j=1}^N r_j(t) e^{-|x-q_j(t)|}. \quad (33)$$

By substituting (33) into (25) and integrating against test functions, we obtain the N -peakon dynamic system of (25)

$$\begin{cases} q_{j,t} = 0, \\ p_{j,t} = \frac{1}{6} p_j^2 r_j \\ \quad + \frac{1}{2} p_j \sum_{i,k=1}^N p_i r_k (\operatorname{sgn}(q_j - q_i) + 1) \\ \quad (\operatorname{sgn}(q_j - q_k) - 1) e^{-|q_j - q_i| - |q_j - q_k|}, \\ r_{j,t} = -\frac{1}{6} p_j r_j^2 \\ \quad - \frac{1}{2} r_j \sum_{i,k=1}^N p_i r_k (\operatorname{sgn}(q_j - q_i) + 1) (\operatorname{sgn}(q_j - q_k) - 1) e^{-|q_j - q_i| - |q_j - q_k|}. \end{cases} \quad (34)$$

In the above formula, $q_{j,t} = 0$ implies that the peak position does not change along with the time t .

For $N = 2$, solving (34) leads to

$$\begin{cases} q_1(t) = A_4, & q_2(t) = A_5, \\ r_1(t) = \frac{A_1}{p_1}, & r_2(t) = \frac{A_2}{p_2}, \\ p_1(t) = A_6 \Gamma_1, & p_2(t) = \frac{p_1}{A_3} e^{\frac{1}{3}(A_1 - A_2)t}, \end{cases} \quad (35)$$

where A_1, A_2, \dots, A_6 are integration constants, and

$$\Gamma_1 = e^{-\frac{1}{3}A_1 t - \frac{e^{-|A_4 - A_5|}}{2} \left[\frac{3A_1(1 + \text{sgn}(A_4 - A_5))}{(A_1 - A_2)A_3} e^{\frac{1}{3}(A_1 - A_2)t} - \frac{3A_2 A_3(1 - \text{sgn}(A_4 - A_5))}{A_1 - A_2} e^{-\frac{1}{3}(A_1 - A_2)t} \right]}.$$

If $A_4 = A_5$, it is reduced to the one-peakon solution. If $A_4 \neq A_5$, this two-peakon solution will never collide because $q_1(t) \neq q_2(t)$ for any t . In particular, for $A_1 = A_3 = A_4 = -A_5 = A_6 = 1$ and $A_2 = 4$, the two-peakon becomes

$$\begin{cases} u(x, t) = e^{-\frac{1}{3}t + e^{-t-2}} e^{-|x-1|} + e^{-\frac{4}{3}t + e^{-t-2}} e^{-|x+1|}, \\ v(x, t) = e^{\frac{1}{3}t - e^{-t-2}} e^{-|x-1|} + 4e^{\frac{4}{3}t - e^{-t-2}} e^{-|x+1|}. \end{cases} \quad (36)$$

See Figure 2 for the profile of the two-peakon dynamics for the potentials $u(x, t)$ and $v(x, t)$.

Remark 3. We point out that from (25) one may conclude $(mn)_t = 0$ and thus $mn = f(x)$, where $f(x)$ is a free function of x . It then follows that $v = (1 - \partial^2)^{-1} \left(\frac{f(x)}{m} \right)$. This means we can remove the component v in Equation (25) and thus write (25) in the form of a single field equation. However, the resulting single field equation involves in nonlocal expressions and a free function $f(x)$. Guided by this, we still write the equation associated with the case of $H = 0$ in the form of (25).

EXAMPLE 2. *The integrable two-component system proposed in [28].*

By choosing $H = \frac{1}{2}(uv - u_x v_x)$, we obtain

$$\begin{cases} m_t = \frac{1}{2}[m(uv - u_x v_x)]_x - \frac{1}{2}m(uv_x - u_x v), \\ n_t = \frac{1}{2}[n(uv - u_x v_x)]_x + \frac{1}{2}n(uv_x - u_x v), \\ m = u - u_{xx}, n = v - v_{xx}, \end{cases} \quad (37)$$

which is exactly the dispersionless version of the system we derived in [28]. This system possesses the bi-Hamiltonian form

$$(m_t, n_t)^T = J \left(\frac{\delta H_2}{\delta m}, \frac{\delta H_2}{\delta n} \right)^T = K \left(\frac{\delta H_1}{\delta m}, \frac{\delta H_1}{\delta n} \right)^T, \quad (38)$$

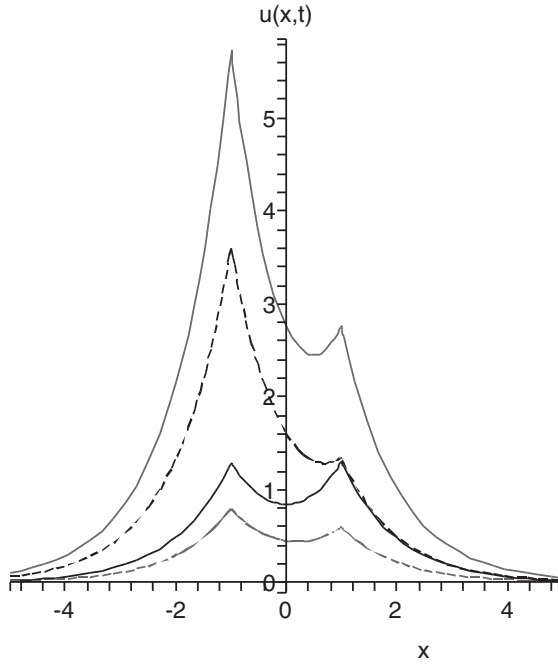


Figure 2. The two-peakon solution given by (36). Solid line: $u(x, t)$; Dashed line: $v(x, t)$; Black: $t = 0$; Blue: $t = -1$.

where

$$J = \begin{pmatrix} 0 & \partial^2 - 1 \\ 1 - \partial^2 & 0 \end{pmatrix}, \quad (39)$$

$$K = \begin{pmatrix} \partial m \partial^{-1} m \partial - m \partial^{-1} m & \partial m \partial^{-1} n \partial + m \partial^{-1} n \\ \partial n \partial^{-1} m \partial + n \partial^{-1} m & \partial n \partial^{-1} n \partial - n \partial^{-1} n \end{pmatrix},$$

$$H_1 = \frac{1}{2} \int_{-\infty}^{+\infty} (uv + u_x v_x) dx, \quad H_2 = \frac{1}{4} \int_{-\infty}^{+\infty} (u^2 v_x + u_x^2 v_x - 2uu_x v) ndx. \quad (40)$$

In [28], we have derived the one-peakon of (37)

$$u(x, t) = c_1 e^{-|x + \frac{1}{3}c_1 c_2 t|}, \quad v(x, t) = c_2 e^{-|x + \frac{1}{3}c_1 c_2 t|}, \quad (41)$$

where c_1 and c_2 are two arbitrary integration constants. We also investigated the N -peakon dynamical system. In particular, the two-peakon solution was given explicitly and the collisions are discussed (for details, see [28]).

EXAMPLE 3. *A new integrable two-component peakon system with the same bi-Hamiltonian operators as (37) but different Hamiltonian functions.*

Taking $H = \frac{1}{2}(uv_x - u_x v)$, we arrive at

$$\begin{cases} m_t = \frac{1}{2} [m(uv_x - u_x v)]_x - \frac{1}{2} m (uv - u_x v_x), \\ n_t = \frac{1}{2} [n(uv_x - u_x v)]_x + \frac{1}{2} n (uv - u_x v_x), \\ m = u - u_{xx}, n = v - v_{xx}. \end{cases} \quad (42)$$

This system can be rewritten as the following bi-Hamiltonian form:

$$(m_t, n_t)^T = J \left(\frac{\delta H_2}{\delta m}, \frac{\delta H_2}{\delta n} \right)^T = K \left(\frac{\delta H_1}{\delta m}, \frac{\delta H_1}{\delta n} \right)^T, \quad (43)$$

where J, K are given by (39), and

$$H_1 = \frac{1}{2} \int_{-\infty}^{+\infty} (uv_x + u_x v_{xx}) dx, H_2 = \frac{1}{4} \int_{-\infty}^{+\infty} (u^2 v + u_x^2 v - 2uu_x v_x) ndx. \quad (44)$$

From (38) and (43), we find that Equations (37) and (42) share the same bi-Hamiltonian operators but with different Hamiltonian functions. We will comment this at the end of this example (see Remark 4).

Let us study the peakon solutions of this example. By direct calculations, we find that the one-peakon solution of (42) takes the form as

$$u(x, t) = c_2 e^{-\frac{1}{3}c_1 t} e^{-|x-c_3|}, \quad v(x, t) = \frac{c_1}{c_2} e^{\frac{1}{3}c_1 t} e^{-|x-c_3|}, \quad (45)$$

where c_1, c_2 , and c_3 are three integration constants. In general, we suppose the N -peakon solution of (42) in the form of (33). Then, we obtain the N -peakon dynamical system of (42)

$$\begin{cases} p_{j,t} = \frac{1}{6} p_j^2 r_j \\ \quad + \frac{1}{2} p_j \sum_{i,k=1}^N p_i r_k (\text{sgn}(q_j - q_i) \text{sgn}(q_j - q_k) - 1) e^{-|q_j - q_i| - |q_j - q_k|}, \\ r_{j,t} = -\frac{1}{6} p_j r_j^2 \\ \quad - \frac{1}{2} r_j \sum_{i,k=1}^N p_i r_k (\text{sgn}(q_j - q_i) \text{sgn}(q_j - q_k) - 1) e^{-|q_j - q_i| - |q_j - q_k|}, \\ q_{j,t} = \frac{1}{2} \sum_{i,k=1}^N p_i r_k (\text{sgn}(q_j - q_k) - \text{sgn}(q_j - q_i)) e^{-|q_j - q_i| - |q_j - q_k|}. \end{cases} \quad (46)$$

For $N = 2$, the two-peakon dynamical system reads as

$$\begin{cases} p_{1,t} = -\frac{1}{3}p_1^2r_1 - \frac{1}{2}p_1(p_1r_2 + p_2r_1)e^{-|q_1-q_2|}, \\ p_{2,t} = -\frac{1}{3}p_2^2r_2 - \frac{1}{2}p_2(p_1r_2 + p_2r_1)e^{-|q_1-q_2|}, \\ r_{1,t} = \frac{1}{3}p_1r_1^2 + \frac{1}{2}r_1(p_1r_2 + p_2r_1)e^{-|q_1-q_2|}, \\ r_{2,t} = \frac{1}{3}p_2r_2^2 + \frac{1}{2}r_2(p_1r_2 + p_2r_1)e^{-|q_1-q_2|}, \\ q_{1,t} = \frac{1}{2}(p_1r_2 - p_2r_1)\operatorname{sgn}(q_1 - q_2)e^{-|q_1-q_2|}, \\ q_{2,t} = q_{1,t}. \end{cases} \quad (47)$$

From the first four equations of (47), we may conclude $p_1(t)r_1(t) = A_1$ and $p_2(t)r_2(t) = A_2$, where A_1 and A_2 are two integration constants. From the last two equations of (47), we know $q_2(t) = q_1(t) - B_1$ where B_1 is a nonzero constant, which indicates that the two-peakon will never collide. For $A_1 = A_2$, we have

$$\begin{cases} p_1(t) = De^{[-\frac{1}{3}A_1 - \frac{1}{2}(A_1C_1 + \frac{A_1}{C_1})e^{-|B_1|}]t}, \\ p_2(t) = \frac{p_1(t)}{C_1}, r_1(t) = \frac{A_1}{p_1(t)}, r_2 = \frac{A_1C_1}{p_1(t)}, \\ q_1(t) = \frac{1}{2}\left[\left(A_1C_1 - \frac{A_1}{C_1}\right)\operatorname{sgn}(B_1)e^{-|B_1|}\right]t + \frac{B_1}{2}, \\ q_2(t) = q_1(t) - B_1, \end{cases} \quad (48)$$

where B_1, C_1 , and D are three integration constants. For example, choosing $C_1 = D = 1$, $B_1 = 2$, $A_1 = 3$, we have $p_2(t) = p_1(t) = e^{-(3e^{-2}+1)t}$, $r_2(t) = r_1(t) = 3e^{(3e^{-2}+1)t}$. Thus, the two-peakon solution accordingly reads as

$$\begin{cases} u(x, t) = e^{-(3e^{-2}+1)t} (e^{-|x-1|} + e^{-|x+1|}), \\ v(x, t) = 3e^{(3e^{-2}+1)t} (e^{-|x-1|} + e^{-|x+1|}), \end{cases} \quad (49)$$

which are apparently M-shape peakon solutions with two peaks (see Figure 3 for details). If choosing $C_1 = B_1 = 2$, $D = 1$, $A_1 = 3$, then we have the following two-peakon solution

$$\begin{cases} u(x, t) = \frac{1}{2}e^{-(\frac{15}{4}e^{-2}+1)t} \left(2e^{-|x-\frac{9}{4}e^{-2}t-1|} + e^{-|x-\frac{9}{4}e^{-2}t+1|}\right), \\ v(x, t) = 3e^{(\frac{15}{4}e^{-2}+1)t} \left(e^{-|x-\frac{9}{4}e^{-2}t-1|} + 2e^{-|x-\frac{9}{4}e^{-2}t+1|}\right). \end{cases} \quad (50)$$

Figure 4 shows the profile of this two-peakon solution.

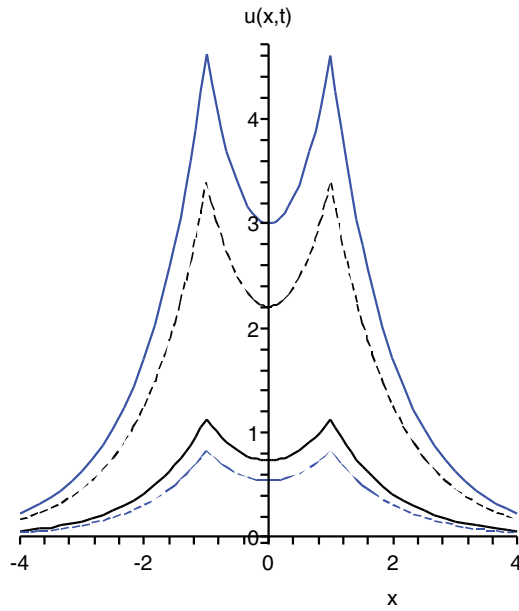


Figure 3. The M-shape peakon solution given by (49). Solid line: $u(x, t)$; Dashed line: $v(x, t)$; Black: $t = 0$; Blue: $t = -1$.

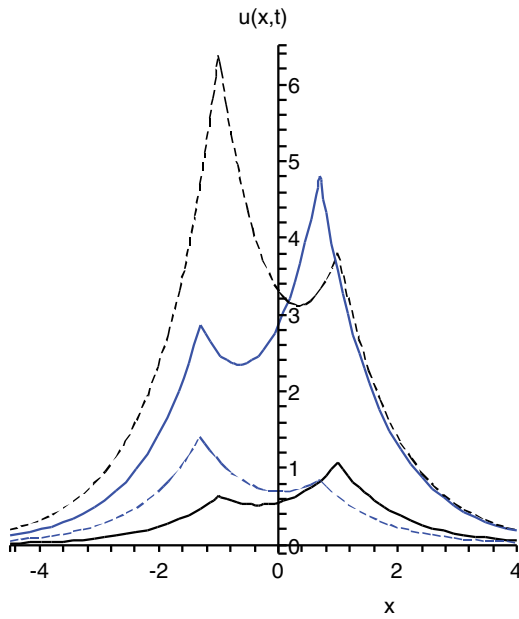


Figure 4. The two-peakon solution given by (50). Solid line: $u(x, t)$; Dashed line: $v(x, t)$; Black: $t = 0$; Blue: $t = -1$.

For $A_1 \neq A_2$, we obtain the following solution of (47):

$$\begin{cases} p_1(t) = B_3 e^{-\frac{1}{3}A_1 t - \frac{3e^{-|B_1|}}{2(A_1 - A_2)} \left(\frac{A_1}{B_2}\right)^{\frac{1}{3}(A_1 - A_2)t} - A_2 B_2 e^{-\frac{1}{3}(A_1 - A_2)t}}, \\ p_2(t) = \frac{p_1}{B_2} e^{\frac{1}{3}(A_1 - A_2)t}, \\ r_1(t) = \frac{A_1}{p_1}, r_2 = \frac{A_2}{p_2}, \\ q_1(t) = -\frac{3\text{sgn}(B_1)e^{-|B_1|}}{2(A_1 - A_2)} \left[A_2 B_2 e^{-\frac{1}{3}(A_1 - A_2)t} + \frac{A_1}{B_2} e^{\frac{1}{3}(A_1 - A_2)t} \right] + B_4, \\ q_2(t) = q_1 - B_1, \end{cases} \quad (51)$$

where A_1, A_2, B_1, B_2, B_3 , and B_4 are six integration constants. Let us consider a special case of choosing $A_1 = B_1 = B_2 = B_3 = 1, A_2 = 4, B_4 = 0$. Then, we have

$$\begin{cases} p_1 = e^{-\frac{1}{3}t + \frac{1}{2}e^{-t-1} - 2e^{t-1}}, \\ p_2 = e^{-\frac{4}{3}t + \frac{1}{2}e^{-t-1} - 2e^{t-1}}, \\ r_1 = e^{\frac{1}{3}t - \frac{1}{2}e^{-t-1} + 2e^{t-1}}, \\ r_2 = 4e^{\frac{4}{3}t - \frac{1}{2}e^{-t-1} + 2e^{t-1}}, \\ q_1 = \frac{1}{2}e^{-t-1} + 2e^{t-1}, \\ q_2 = q_1 - 1. \end{cases} \quad (52)$$

Figure 5 shows the dynamics of this two-peakon for the potentials $u(x, t)$ and $v(x, t)$ determined by (52).

Remark 4. It has been shown that Equations (37) and (42) share the same bi-Hamiltonian operators (but with different Hamiltonian functions). In fact, the bi-Hamiltonian operators (39) generate two hierarchies of equations. To see this, we define Lenard sequence recursively by

$$Jb_{-k} = Kb_{-k+1}, \quad Jb_0 = 0, \quad k = 1, 2, \dots,$$

and the soliton hierarchy by

$$m_{t_{-n}} = Kb_{-n}, \quad n = 1, 2, \dots \quad (53)$$

Let us take an initial value $b_0 = (0, 0)^T$. Then from $Jb_{-1} = Kb_0$, we may reach $b_{-1} = \frac{1}{2}(v, u)^T$ or $b_{-1} = \frac{1}{2}(v_x, -u_x)^T$. For $b_{-1} = \frac{1}{2}(v, u)^T$, the first member $m_{t_{-1}} = Kb_{-1}$ in the hierarchy (53) is just Equation (37). While for $b_{-1} = \frac{1}{2}(v_x, -u_x)^T$, the first member $m_{t_{-1}} = Kb_{-1}$ is nothing but Equation (42).

Remark 5. Although Equations (37) and (42) share the same bi-Hamiltonian operators, their peakon dynamics are very different. In the single-peakon case, the peakon solution of (37) is in the type of traveling wave (see (41)), while the peakon solution of (42) is not, since the peak point does not change along with the time t (see (45)). In the two-peakon case, the collision of the two-peakon

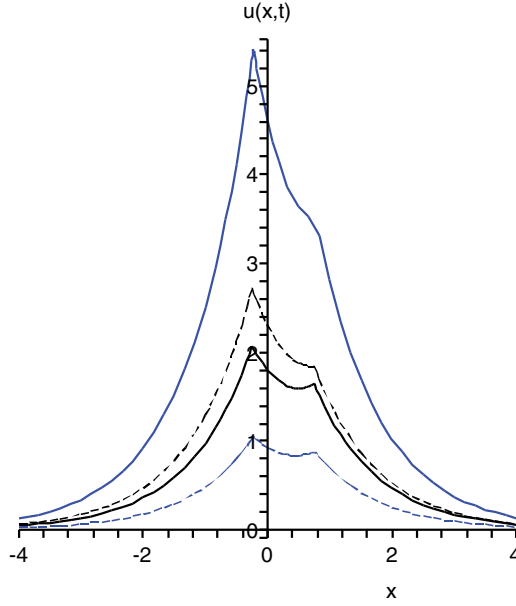


Figure 5. The two-peakon solution determined by (52). Solid line: $u(x, t)$; Dashed line: $v(x, t)$; Black: $t = -0.5$; Blue: $t = -1$.

of Equation (37) is discussed in [28], while the two-peakon of Equation (42) never collides since their positions are satisfied with $q_2(t) = q_1(t) - B_1$, where B_1 is a nonzero constant.

EXAMPLE 4. *The two-component integrable system proposed by Song et al. [27].*

Choosing $H = \frac{1}{2}(u - u_x)(v + v_x)$ casts Equation (7) into

$$\begin{cases} m_t = \frac{1}{2} [m(u - u_x)(v + v_x)]_x, \\ n_t = \frac{1}{2} [n(u - u_x)(v + v_x)]_x, \\ m = u - u_{xx}, n = v - v_{xx}, \end{cases} \quad (54)$$

which is exactly the equation derived by Song et al. [27]. This system possesses a bi-Hamiltonian structure [39]:

$$(m_t, n_t)^T = J \begin{pmatrix} \delta H_2 \\ \delta H_2 \end{pmatrix}^T = K \begin{pmatrix} \delta H_1 \\ \delta H_1 \end{pmatrix}^T, \quad (55)$$

where

$$J = \begin{pmatrix} 0 & \partial^2 + \partial \\ -\partial^2 + \partial & 0 \end{pmatrix}, K = \begin{pmatrix} \partial m \partial^{-1} m \partial & \partial m \partial^{-1} n \partial \\ \partial n \partial^{-1} m \partial & \partial n \partial^{-1} n \partial \end{pmatrix}, \quad (56)$$

$$H_1 = \frac{1}{2} \int_{-\infty}^{+\infty} m(v + v_x) dx, H_2 = \frac{1}{4} \int_{-\infty}^{+\infty} (u - u_x)^2 (v + v_x) n dx. \quad (57)$$

In the following, we want to derive the peakon solutions and discuss the peakon interactions for this system. It is easy to check that the one-peakon solution of (54) takes the same form as (41). In general, by direct calculations, we can obtain the N -peakon dynamical system of (54) as follows:

$$\begin{cases} p_{j,t} = 0, \\ r_{j,t} = 0, \\ q_{j,t} = \frac{1}{6}p_j r_j + \frac{1}{2} \sum_{i,k=1}^N p_i r_k (\operatorname{sgn}(q_j - q_i) + 1) \\ \quad (\operatorname{sgn}(q_j - q_k) - 1) e^{-|q_j - q_i| - |q_j - q_k|}. \end{cases} \quad (58)$$

If $N = 2$, then the two-peakon system reads as

$$\begin{cases} p_{1,t} = p_{2,t} = r_{1,t} = r_{2,t} = 0, \\ q_{1,t} = -\frac{1}{3}p_1 r_1 + \frac{1}{2} [p_1 r_2 (\operatorname{sgn}(q_1 - q_2) - 1) \\ \quad - p_2 r_1 (\operatorname{sgn}(q_1 - q_2) + 1)] e^{-|q_1 - q_2|}, \\ q_{2,t} = -\frac{1}{3}p_2 r_2 + \frac{1}{2} [p_1 r_2 (\operatorname{sgn}(q_1 - q_2) - 1) \\ \quad - p_2 r_1 (\operatorname{sgn}(q_1 - q_2) + 1)] e^{-|q_1 - q_2|}. \end{cases} \quad (59)$$

From the first equation of (59), we know

$$p_1 = A_1, \quad p_2 = A_2, \quad r_1 = B_1, \quad r_2 = B_2, \quad (60)$$

where $A_1, A_2, B_1,$ and B_2 are four integration constants. If $A_1 B_1 = A_2 B_2$, then we have

$$\begin{cases} q_1(t) = \left\{ -\frac{1}{3}A_1 B_1 + \frac{1}{2}[A_1 B_2 (\operatorname{sgn}(C_1) - 1) \right. \\ \quad \left. - A_2 B_1 (\operatorname{sgn}(C_1) + 1)] e^{-|C_1|} \right\} t + \frac{C_1}{2}, \\ q_2(t) = q_1(t) - C_1. \end{cases} \quad (61)$$

If $A_1 B_1 \neq A_2 B_2$, then we arrive at

$$\begin{cases} q_1(t) = -\frac{1}{3}A_1 B_1 t + \Gamma(t), \\ q_2(t) = -\frac{1}{3}A_2 B_2 t + \Gamma(t), \end{cases} \quad (62)$$

where

$$\begin{aligned} \Gamma(t) = & \frac{3(A_1 B_2 + A_2 B_1)}{2|A_1 B_1 - A_2 B_2|} \operatorname{sgn}(t) \left(e^{-\frac{1}{3}|(A_1 B_1 - A_2 B_2)t|} - 1 \right) \\ & + \frac{3(A_1 B_2 - A_2 B_1)}{2(A_1 B_1 - A_2 B_2)} e^{-\frac{1}{3}|(A_1 B_1 - A_2 B_2)t|}. \end{aligned} \quad (63)$$

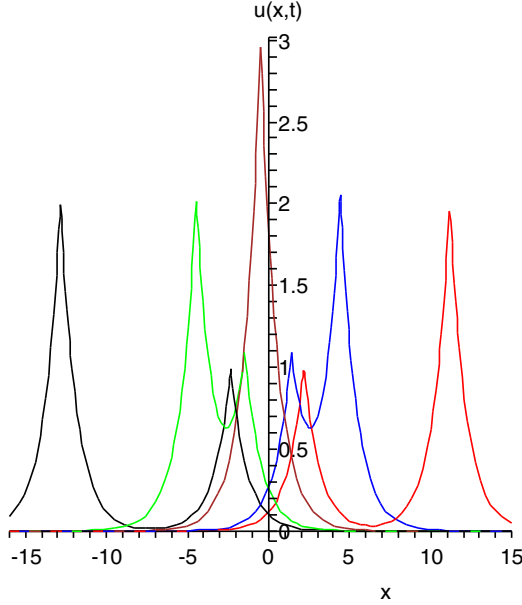


Figure 6. The two-peakon solution for the potential $u(x, t)$ given by (64). Red line: $t = -3$; Blue line: $t = -1$; Brown line: $t = 0$ (collision); Green line: $t = 1$; Black line: $t = 3.5$.

In particular, taking $A_1 = B_1 = 1$, $A_2 = 2$, and $B_2 = 5$ sends the two-peakon solution to the following form:

$$\begin{cases} u(x, t) = e^{-|x-q_1(t)|} + 2e^{-|x-q_2(t)|}, \\ v(x, t) = e^{-|x-q_1(t)|} + 5e^{-|x-q_2(t)|}, \end{cases} \quad (64)$$

where

$$\begin{cases} q_1(t) = -\frac{t}{3} + \frac{7}{6}\text{sgn}(t)(e^{-3|t|} - 1) - \frac{1}{2}e^{-3|t|}, \\ q_2(t) = -\frac{10t}{3} + \frac{7}{6}\text{sgn}(t)(e^{-3|t|} - 1) - \frac{1}{2}e^{-3|t|}. \end{cases} \quad (65)$$

For the potential $u(x, t)$, the two-peakon collides at the moment $t = 0$, since $q_1(0) = q_2(0) = 0$. For $t < 0$, the tall and fast peakon with the amplitude 2 and peak position q_2 chases after the short and slow peakon with the amplitude 1 and peak position q_1 . At the moment of $t = 0$, the two-peakon overlaps. After the collision ($t > 0$), the two-peakon separates, and the tall and fast peakon surpasses the short and slow one. Similarly, we may discuss the collision of the two-peakon for the potential $v(x, t)$. See Figures 6 and 7 for the two-peakon dynamics of the potentials $u(x, t)$ and $v(x, t)$.

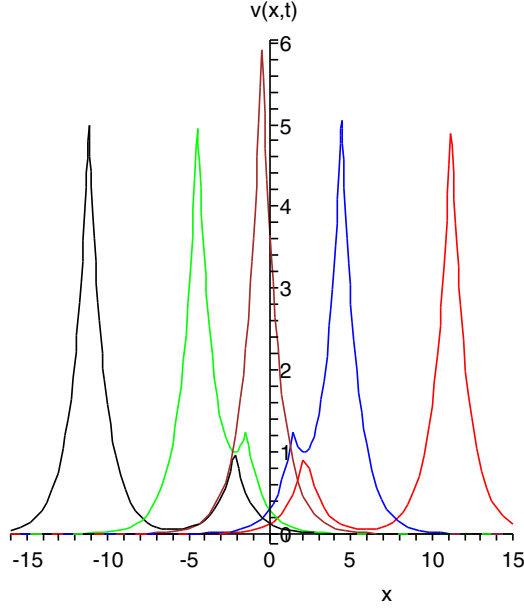


Figure 7. The two-peakon solution for the potential $v(x, t)$ given by (64). Red line: $t = -3$; Blue line: $t = -1$; Brown line: $t = 0$ (collision); Green line: $t = 1$; Black line: $t = 3$.

4. A proof for the bi-Hamiltonian property

In this section, we will supply a proof for the bi-Hamiltonian property in each example presented in the above section. Let us introduce the following basic operators:

$$J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & \partial^2 \\ -\partial^2 & 0 \end{pmatrix}, \quad (66)$$

$$K_1 = \begin{pmatrix} -m\partial^{-1}m & m\partial^{-1}n \\ n\partial^{-1}m & -n\partial^{-1}n \end{pmatrix}, \quad K_2 = \begin{pmatrix} \partial m\partial^{-1}m\partial & \partial m\partial^{-1}n\partial \\ \partial n\partial^{-1}m\partial & \partial n\partial^{-1}n\partial \end{pmatrix}. \quad (67)$$

LEMMA 1. *All the above operators are Hamiltonian operators.*

Proof: It is obvious that J_1 , J_2 , and J_3 are Hamiltonian operators since they are skew-symmetric operators with constant-coefficient. It is easy to check K_1 and K_2 are skew-symmetric. We need to prove that both K_1 and K_2 satisfy the Jacobi identities

$$\langle \alpha, K_1'[K_1\beta]\gamma \rangle + \text{cycle}(\alpha, \beta, \gamma) = 0, \quad (68)$$

$$\langle \alpha, K_2'[K_2\beta]\gamma \rangle + \text{cycle}(\alpha, \beta, \gamma) = 0, \quad (69)$$

where $\alpha = (\alpha_1, \alpha_2)^T, \beta = (\beta_1, \beta_2)^T, \gamma = (\gamma_1, \gamma_2)^T$ are arbitrary vector functions, the symbol cycle (α, β, γ) means the cyclic permutation of α, β, γ , and the prime-sign means the Gâteaux derivative of an operator F on q in the direction σ defined as [18]

$$F'[\sigma] = F'(q)[\sigma] = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} F(q + \epsilon \sigma). \quad (70)$$

Let us first prove the Jacobi identity (68). For brevity, we introduce the notations:

$$A = \partial^{-1}(m\alpha_1 - n\alpha_2), B = \partial^{-1}(m\beta_1 - n\beta_2), C = \partial^{-1}(m\gamma_1 - n\gamma_2). \quad (71)$$

Direct calculations give rise to

$$\begin{aligned} & \langle \alpha, K'_1[K_1\beta]\gamma \rangle \\ &= \int_{-\infty}^{+\infty} [(\alpha_1 m B + \alpha_2 n B)C + (\alpha_1 m - \alpha_2 n)\partial^{-1}(\gamma_1 m B + \gamma_2 n B)]dx. \end{aligned} \quad (72)$$

Integrating (72) by parts, we obtain

$$\begin{aligned} & \langle \alpha, K'_1[K_1\beta]\gamma \rangle \\ &= \int_{-\infty}^{+\infty} [(\alpha_1 m B + \alpha_2 n B)C - (\gamma_1 m B + \gamma_2 n B)\partial^{-1}(\alpha_1 m - \alpha_2 n)]dx \\ &= \int_{-\infty}^{+\infty} [(\alpha_1 m B + \alpha_2 n B)C - (\gamma_1 m B + \gamma_2 n B)A]dx \\ &= \int_{-\infty}^{+\infty} [(\alpha_1 m + \alpha_2 n)BC - (\gamma_1 m + \gamma_2 n)BA]dx. \end{aligned} \quad (73)$$

Thus

$$\begin{aligned} & \langle \alpha, K'_1[K_1\beta]\gamma \rangle + \text{cycle}(\alpha, \beta, \gamma) \\ &= \int_{-\infty}^{+\infty} [(\alpha_1 m + \alpha_2 n)BC - (\gamma_1 m + \gamma_2 n)BA]dx \\ & \quad + \int_{-\infty}^{+\infty} [(\beta_1 m + \beta_2 n)CA - (\alpha_1 m + \alpha_2 n)CB]dx \\ & \quad + \int_{-\infty}^{+\infty} [(\gamma_1 m + \gamma_2 n)AB - (\beta_1 m + \beta_2 n)AC]dx \\ &= 0. \end{aligned} \quad (74)$$

Now we turn to the proof of Jacobi identity (69). Let us set

$$\begin{aligned}\tilde{A} &= \partial^{-1}(m\alpha_{1,x} + n\alpha_{2,x}), \tilde{B} = \partial^{-1}(m\beta_{1,x} + n\beta_{2,x}), \\ \tilde{C} &= \partial^{-1}(m\gamma_{1,x} + n\gamma_{2,x}).\end{aligned}\quad (75)$$

With the similar calculations as (72) and (73), we arrive at

$$\begin{aligned}&\langle \alpha, K_2'[K_2\beta]\gamma \rangle \\ &= \int_{-\infty}^{+\infty} [(\gamma_{1,x}m_x + \gamma_{2,x}n_x)\tilde{B}\tilde{A} - (\alpha_{1,x}m_x + \alpha_{2,x}n_x)\tilde{B}\tilde{C} \\ &\quad + \tilde{C}_x\tilde{B}_x\tilde{A} - \tilde{A}_x\tilde{B}_x\tilde{C}]dx.\end{aligned}\quad (76)$$

Then it follows that

$$\begin{aligned}&\langle \alpha, K_2'[K_2\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) \\ &= \int_{-\infty}^{+\infty} [(\gamma_{1,x}m_x + \gamma_{2,x}n_x)\tilde{B}\tilde{A} - (\alpha_{1,x}m_x + \alpha_{2,x}n_x)\tilde{B}\tilde{C} \\ &\quad + \tilde{C}_x\tilde{B}_x\tilde{A} - \tilde{A}_x\tilde{B}_x\tilde{C}]dx \\ &\quad + \int_{-\infty}^{+\infty} [(\alpha_{1,x}m_x + \alpha_{2,x}n_x)\tilde{C}\tilde{B} - (\beta_{1,x}m_x + \beta_{2,x}n_x)\tilde{C}\tilde{A} \\ &\quad + \tilde{A}_x\tilde{C}_x\tilde{B} - \tilde{B}_x\tilde{C}_x\tilde{A}]dx \\ &\quad + \int_{-\infty}^{+\infty} [(\beta_{1,x}m_x + \beta_{2,x}n_x)\tilde{A}\tilde{C} - (\gamma_{1,x}m_x + \gamma_{2,x}n_x)\tilde{A}\tilde{B} \\ &\quad + \tilde{B}_x\tilde{A}_x\tilde{C} - \tilde{C}_x\tilde{A}_x\tilde{B}]dx \\ &= 0.\end{aligned}\quad (77)$$

The proof of Lemma 1 is finished.

LEMMA 2. *The following relations hold*

$$\langle \alpha, J_1'[J_2\beta]\gamma \rangle + \langle \alpha, J_2'[J_1\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) = 0, \quad (78)$$

$$\langle \alpha, J_1'[J_3\beta]\gamma \rangle + \langle \alpha, J_3'[J_1\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) = 0, \quad (79)$$

$$\langle \alpha, J_2'[J_3\beta]\gamma \rangle + \langle \alpha, J_3'[J_2\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) = 0, \quad (80)$$

$$\langle \alpha, J_1'[K_1\beta]\gamma \rangle + \langle \alpha, K_1'[J_1\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) = 0, \quad (81)$$

$$\langle \alpha, J_2'[K_1\beta]\gamma \rangle + \langle \alpha, K_1'[J_2\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) = 0, \quad (82)$$

$$\langle \alpha, J_2'[K_2\beta]\gamma \rangle + \langle \alpha, K_2'[J_2\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) = 0, \quad (83)$$

$$\langle \alpha, J_3'[K_2\beta]\gamma \rangle + \langle \alpha, K_2'[J_3\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) = 0, \quad (84)$$

$$\langle \alpha, K_1'[K_2\beta]\gamma \rangle + \langle \alpha, K_2'[K_1\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) = 0. \quad (85)$$

Proof: It is clearly formulas (78)–(80) hold since J_1 , J_2 , and J_3 are constant-coefficient operators. For (81), we have $\langle \alpha, J_1'[K_1\beta]\gamma \rangle = 0$, and

$$\langle \alpha, K_1'[J_1\beta]\gamma \rangle = \int_{-\infty}^{+\infty} [(\alpha_1\beta_2 + \alpha_2\beta_1)C - (\beta_1\gamma_2 + \beta_2\gamma_1)A]dx.$$

Thus the left-hand side (LHS) of (81) becomes

$$\begin{aligned} & \int_{-\infty}^{+\infty} [(\alpha_1\beta_2 + \alpha_2\beta_1)C - (\beta_1\gamma_2 + \beta_2\gamma_1)A]dx \\ & + \int_{-\infty}^{+\infty} [(\beta_1\gamma_2 + \beta_2\gamma_1)A - (\gamma_1\alpha_2 + \gamma_2\alpha_1)B]dx \\ & + \int_{-\infty}^{+\infty} [(\gamma_1\alpha_2 + \gamma_2\alpha_1)B - (\alpha_1\beta_2 + \alpha_2\beta_1)C]dx \\ & = 0. \end{aligned}$$

Similarly, by direct calculations, the LHS of (82) becomes

$$\begin{aligned} & \int_{-\infty}^{+\infty} [(\beta_{1,x}\alpha_2 - \beta_{2,x}\alpha_1)C - (\beta_{1,x}\gamma_2 - \beta_{2,x}\gamma_1)A]dx \\ & + \int_{-\infty}^{+\infty} [(\gamma_{1,x}\beta_2 - \gamma_{2,x}\beta_1)A - (\gamma_{1,x}\alpha_2 - \gamma_{2,x}\alpha_1)B]dx \\ & + \int_{-\infty}^{+\infty} [(\alpha_{1,x}\gamma_2 - \alpha_{2,x}\gamma_1)B - (\alpha_{1,x}\beta_2 - \alpha_{2,x}\beta_1)C]dx \\ & = \int_{-\infty}^{+\infty} [(\beta_1\alpha_2 - \beta_2\alpha_1)_x C - (\beta_1\gamma_2 - \beta_2\gamma_1)_x A - (\gamma_1\alpha_2 - \gamma_2\alpha_1)_x B]dx \\ & = - \int_{-\infty}^{+\infty} [(\beta_1\alpha_2 - \beta_2\alpha_1)C_x - (\beta_1\gamma_2 - \beta_2\gamma_1)A_x - (\gamma_1\alpha_2 - \gamma_2\alpha_1)B_x]dx \\ & = - \int_{-\infty}^{+\infty} [(\beta_1\alpha_2 - \beta_2\alpha_1)(m\gamma_1 - n\gamma_2) - (\beta_1\gamma_2 - \beta_2\gamma_1)(m\alpha_1 - n\alpha_2) \\ & \quad - (\gamma_1\alpha_2 - \gamma_2\alpha_1)(m\beta_1 - n\beta_2)]dx \\ & = 0. \end{aligned}$$

The LHS of (83) is equal to

$$\begin{aligned}
& - \int_{-\infty}^{+\infty} [(\alpha_{1,x}\beta_{2,x} + \alpha_{2,x}\beta_{1,x})\tilde{C} - (\beta_{2,x}\gamma_{1,x} + \beta_{1,x}\gamma_{2,x})\tilde{A}]dx \\
& - \int_{-\infty}^{+\infty} [(\beta_{1,x}\gamma_{2,x} + \beta_{2,x}\gamma_{1,x})\tilde{A} - (\gamma_{2,x}\alpha_{1,x} + \gamma_{1,x}\alpha_{2,x})\tilde{B}]dx \\
& - \int_{-\infty}^{+\infty} [(\gamma_{1,x}\alpha_{2,x} + \gamma_{2,x}\alpha_{1,x})\tilde{B} - (\alpha_{2,x}\beta_{1,x} + \alpha_{1,x}\beta_{2,x})\tilde{C}]dx \\
& = 0.
\end{aligned}$$

The LHS of (84) is equal to

$$\begin{aligned}
& - \int_{-\infty}^{+\infty} [(\alpha_{1,x}\beta_{2,xx} - \alpha_{2,x}\beta_{1,xx})\tilde{C} - (\beta_{2,xx}\gamma_{1,x} - \beta_{1,xx}\gamma_{2,x})\tilde{A}]dx \\
& - \int_{-\infty}^{+\infty} [(\beta_{1,x}\gamma_{2,xx} - \beta_{2,x}\gamma_{1,xx})\tilde{A} - (\gamma_{2,xx}\alpha_{1,x} - \gamma_{1,xx}\alpha_{2,x})\tilde{B}]dx \\
& - \int_{-\infty}^{+\infty} [(\gamma_{1,x}\alpha_{2,xx} - \gamma_{2,x}\alpha_{1,xx})\tilde{B} - (\alpha_{2,xx}\beta_{1,x} - \alpha_{1,xx}\beta_{2,x})\tilde{C}]dx \\
& = - \int_{-\infty}^{+\infty} [(\beta_{1,x}\gamma_{2,x} - \beta_{2,x}\gamma_{1,x})_x \tilde{A} + (\gamma_{1,x}\alpha_{2,x} - \gamma_{2,x}\alpha_{1,x})_x \tilde{B} \\
& \quad + (\alpha_{1,x}\beta_{2,x} - \alpha_{2,x}\beta_{1,x})_x \tilde{C}]dx \\
& = \int_{-\infty}^{+\infty} [(\beta_{1,x}\gamma_{2,x} - \beta_{2,x}\gamma_{1,x})\tilde{A}_x + (\gamma_{1,x}\alpha_{2,x} - \gamma_{2,x}\alpha_{1,x})\tilde{B}_x \\
& \quad + (\alpha_{1,x}\beta_{2,x} - \alpha_{2,x}\beta_{1,x})\tilde{C}_x]dx \\
& = \int_{-\infty}^{+\infty} [(\beta_{1,x}\gamma_{2,x} - \beta_{2,x}\gamma_{1,x})(m\alpha_{1,x} + n\alpha_{2,x}) \\
& \quad + (\gamma_{1,x}\alpha_{2,x} - \gamma_{2,x}\alpha_{1,x})(m\beta_{1,x} + n\beta_{2,x}) \\
& \quad + (\alpha_{1,x}\beta_{2,x} - \alpha_{2,x}\beta_{1,x})(m\gamma_{1,x} + n\gamma_{2,x})]dx \\
& = 0.
\end{aligned}$$

For the LHS of (85), we have

$$\begin{aligned}
\langle \alpha, K'_1[K_2\beta]\gamma \rangle &= \int_{-\infty}^{+\infty} [(m\alpha_{1,x} - n\alpha_{2,x})C\tilde{B} - (m\gamma_{1,x} - n\gamma_{2,x})A\tilde{B}]dx, \\
\langle \alpha, K'_2[K_1\beta]\gamma \rangle &= \int_{-\infty}^{+\infty} [(m\alpha_{1,x} - n\alpha_{2,x})B\tilde{C} - (m\gamma_{1,x} - n\gamma_{2,x})B\tilde{A}]dx.
\end{aligned}$$

Hence, the LHS of (85) reads as

$$\begin{aligned}
& \int_{-\infty}^{+\infty} [(m\alpha_{1,x} - n\alpha_{2,x})(C\tilde{B} + B\tilde{C}) - (m\gamma_{1,x} - n\gamma_{2,x})(A\tilde{B} + B\tilde{A})]dx \\
& + \int_{-\infty}^{+\infty} [(m\beta_{1,x} - n\beta_{2,x})(A\tilde{C} + C\tilde{A}) - (m\alpha_{1,x} - n\alpha_{2,x})(B\tilde{C} + C\tilde{B})]dx \\
& + \int_{-\infty}^{+\infty} [(m\gamma_{1,x} - n\gamma_{2,x})(B\tilde{A} + A\tilde{B}) - (m\beta_{1,x} - n\beta_{2,x})(C\tilde{A} + A\tilde{C})]dx \\
& = 0.
\end{aligned}$$

This completes the proof of Lemma 2.

Lemma 2 implies that $J_1 + J_2$, $J_1 + J_3$, $J_1 + K_1$, $J_2 + J_3$, $J_2 + K_1$, $J_2 + K_2$, $J_3 + K_2$ and $K_1 + K_2$ are Hamiltonian operators. However, we should notice that $J_1 + K_2$ and $J_3 + K_1$ are not Hamiltonian operators. In fact, we have

LEMMA 3. *The following two relations hold:*

$$\begin{aligned}
& \langle \alpha, J'_1[K_2\beta]\gamma \rangle + \langle \alpha, K'_2[J_1\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) \\
& = \int_{-\infty}^{+\infty} [(\beta_1\gamma_2 - \beta_2\gamma_1)_x\tilde{A} + (\alpha_2\gamma_1 - \alpha_1\gamma_2)_x\tilde{B} + (\alpha_1\beta_2 - \alpha_2\beta_1)_x\tilde{C}]dx, \\
& \langle \alpha, J'_3[K_1\beta]\gamma \rangle + \langle \alpha, K'_1[J_3\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) \\
& = \int_{-\infty}^{+\infty} [(\beta_{1,x}\gamma_2 - \beta_1\gamma_{2,x} + \beta_{2,x}\gamma_1 - \beta_2\gamma_{1,x})_xA \\
& + (\gamma_{1,x}\alpha_2 - \gamma_1\alpha_{2,x} + \gamma_{2,x}\alpha_1 - \gamma_2\alpha_{1,x})_xB \\
& + (\alpha_{1,x}\beta_2 - \alpha_1\beta_{2,x} + \alpha_{2,x}\beta_1 - \alpha_2\beta_{1,x})_xC]dx.
\end{aligned} \tag{86}$$

Proof: Direct calculations yield that

$$\begin{aligned}
& \langle \alpha, J'_1[K_2\beta]\gamma \rangle + \langle \alpha, K'_2[J_1\beta]\gamma \rangle + cycle(\alpha, \beta, \gamma) \\
& = - \int_{-\infty}^{+\infty} [(\beta_1\alpha_{2,x} - \beta_2\alpha_{1,x})\tilde{C} - (\beta_1\gamma_{2,x} - \beta_2\gamma_{1,x})\tilde{A}]dx \\
& - \int_{-\infty}^{+\infty} [(\gamma_1\beta_{2,x} - \gamma_2\beta_{1,x})\tilde{A} - (\gamma_1\alpha_{2,x} - \gamma_2\alpha_{1,x})\tilde{B}]dx \\
& - \int_{-\infty}^{+\infty} [(\alpha_1\gamma_{2,x} - \alpha_2\gamma_{1,x})\tilde{B} - (\alpha_1\beta_{2,x} - \alpha_2\beta_{1,x})\tilde{A}\tilde{C}]dx \\
& = \int_{-\infty}^{+\infty} [(\beta_1\gamma_2 - \beta_2\gamma_1)_x\tilde{6}\tilde{A} + (\alpha_2\gamma_1 - \alpha_1\gamma_2)_x\tilde{B} + (\alpha_1\beta_2 - \alpha_2\beta_1)_x\tilde{4}\tilde{C}]dx,
\end{aligned}$$

and

$$\begin{aligned}
& \langle \alpha, J_3'[K_1\beta]\gamma \rangle + \langle \alpha, K_1'[J_3\beta]\gamma \rangle + \text{cycle}(\alpha, \beta, \gamma) \\
&= - \int_{-\infty}^{+\infty} [(\alpha_1\beta_{2,xx} + \alpha_2\beta_{1,xx})C - (\beta_{1,xx}\gamma_2 + \beta_{2,xx}\gamma_1)A]dx \\
&\quad - \int_{-\infty}^{+\infty} [(\beta_1\gamma_{2,xx} + \beta_2\gamma_{1,xx})A - (\gamma_{1,xx}\alpha_2 + \gamma_{2,xx}\alpha_1)B]dx \\
&\quad - \int_{-\infty}^{+\infty} [(\gamma_1\alpha_{2,xx} + \gamma_2\alpha_{1,xx})B - (\alpha_{1,xx}\beta_2 + \alpha_{2,xx}\beta_1)C]dx \\
&= \int_{-\infty}^{+\infty} [(\beta_{1,x}\gamma_2 - \beta_1\gamma_{2,x} + \beta_{2,x}\gamma_1 - \beta_2\gamma_{1,x})_x A \\
&\quad + (\gamma_{1,x}\alpha_2 - \gamma_1\alpha_{2,x} + \gamma_{2,x}\alpha_1 - \gamma_2\alpha_{1,x})_x B \\
&\quad + (\alpha_{1,x}\beta_2 - \alpha_1\beta_{2,x} + \alpha_{2,x}\beta_1 - \alpha_2\beta_{1,x})_x C]dx.
\end{aligned}$$

This finishes the proof of Lemma 3.

LEMMA 4. *The following Jacobi identity holds:*

$$\begin{aligned}
& \langle \alpha, J_1'[K_2\beta]\gamma \rangle + \langle \alpha, K_2'[J_1\beta]\gamma \rangle + \langle \alpha, J_3'[K_1\beta]\gamma \rangle + \langle \alpha, K_1'[J_3\beta]\gamma \rangle \\
& + \text{cycle}(\alpha, \beta, \gamma) = 0.
\end{aligned} \tag{87}$$

Proof: By virtue of Lemma 3 and integration by parts, we arrive at

$$\begin{aligned}
& \langle \alpha, J_1'[K_2\beta]\gamma \rangle + \langle \alpha, K_2'[J_1\beta]\gamma \rangle + \langle \alpha, J_3'[K_1\beta]\gamma \rangle + \langle \alpha, K_1'[J_3\beta]\gamma \rangle + \text{cycle}(\alpha, \beta, \gamma) \\
&= \int_{-\infty}^{+\infty} [(\beta_1\gamma_2 - \beta_2\gamma_1)_x \tilde{A} + (\alpha_2\gamma_1 - \alpha_1\gamma_2)_x \tilde{B} + (\alpha_1\beta_2 - \alpha_2\beta_1)_x \tilde{C} \\
&\quad + (\beta_{1,x}\gamma_2 - \beta_1\gamma_{2,x} + \beta_{2,x}\gamma_1 - \beta_2\gamma_{1,x})_x A \\
&\quad + (\gamma_{1,x}\alpha_2 - \gamma_1\alpha_{2,x} + \gamma_{2,x}\alpha_1 - \gamma_2\alpha_{1,x})_x B \\
&\quad + (\alpha_{1,x}\beta_2 - \alpha_1\beta_{2,x} + \alpha_{2,x}\beta_1 - \alpha_2\beta_{1,x})_x C]dx \\
&= - \int_{-\infty}^{+\infty} [(\beta_1\gamma_2 - \beta_2\gamma_1)\tilde{A}_x + (\alpha_2\gamma_1 - \alpha_1\gamma_2)\tilde{B}_x + (\alpha_1\beta_2 - \alpha_2\beta_1)\tilde{C}_x \\
&\quad + (\beta_{1,x}\gamma_2 - \beta_1\gamma_{2,x} + \beta_{2,x}\gamma_1 - \beta_2\gamma_{1,x})A_x \\
&\quad + (\gamma_{1,x}\alpha_2 - \gamma_1\alpha_{2,x} + \gamma_{2,x}\alpha_1 - \gamma_2\alpha_{1,x})B_x \\
&\quad + (\alpha_{1,x}\beta_2 - \alpha_1\beta_{2,x} + \alpha_{2,x}\beta_1 - \alpha_2\beta_{1,x})C_x]dx \\
&= - \int_{-\infty}^{+\infty} [(\beta_1\gamma_2 - \beta_2\gamma_1)(m\alpha_{1,x} + n\alpha_{2,x}) + (\alpha_2\gamma_1 - \alpha_1\gamma_2)(m\beta_{1,x} + n\beta_{2,x}) \\
&\quad + (\alpha_1\beta_2 - \alpha_2\beta_1)(m\gamma_{1,x} + n\gamma_{2,x}) \\
&\quad + (\beta_{1,x}\gamma_2 - \beta_1\gamma_{2,x} + \beta_{2,x}\gamma_1 - \beta_2\gamma_{1,x})(m\alpha_1 - n\alpha_2) \\
&\quad + (\gamma_{1,x}\alpha_2 - \gamma_1\alpha_{2,x} + \gamma_{2,x}\alpha_1 - \gamma_2\alpha_{1,x})(m\beta_1 - n\beta_2) \\
&\quad + (\alpha_{1,x}\beta_2 - \alpha_1\beta_{2,x} + \alpha_{2,x}\beta_1 - \alpha_2\beta_{1,x})(m\gamma_1 - n\gamma_2)]dx \\
&= 0.
\end{aligned}$$

The proof of Lemma 4 is finished.

Based on the above lemmas, we finally obtain:

PROPOSITION 1. *Let c_j , $1 \leq j \leq 5$, be arbitrary constants. For any $c_1c_5 = c_3c_4$, we can conclude that $J = c_1J_1 + c_2J_2 + c_3J_3 + c_4K_1 + c_5K_2$ is a Hamiltonian operator.*

Proof: We need to verify the Jacobi identity

$$\langle \alpha, J'[J\beta]\gamma \rangle + \text{cycle}(\alpha, \beta, \gamma) = 0.$$

In fact, we have

$$\begin{aligned} & \langle \alpha, J'[J\beta]\gamma \rangle + \text{cycle}(\alpha, \beta, \gamma) \\ &= c_1^2 \langle \alpha, J_1'[J_1\beta]\gamma \rangle + c_2^2 \langle \alpha, J_2'[J_2\beta]\gamma \rangle + c_3^2 \langle \alpha, J_3'[J_3\beta]\gamma \rangle \\ & \quad + c_4^2 \langle \alpha, K_1'[K_1\beta]\gamma \rangle + c_5^2 \langle \alpha, K_2'[K_2\beta]\gamma \rangle \\ & \quad + c_1c_2(\langle \alpha, J_1'[J_2\beta]\gamma \rangle + \langle \alpha, J_2'[J_1\beta]\gamma \rangle) \\ & \quad + c_1c_3(\langle \alpha, J_1'[J_3\beta]\gamma \rangle + \langle \alpha, J_3'[J_1\beta]\gamma \rangle) \\ & \quad + c_1c_4(\langle \alpha, J_1'[K_1\beta]\gamma \rangle + \langle \alpha, K_1'[J_1\beta]\gamma \rangle) \\ & \quad + c_1c_5(\langle \alpha, J_1'[K_2\beta]\gamma \rangle + \langle \alpha, K_2'[J_1\beta]\gamma \rangle) \\ & \quad + c_2c_3(\langle \alpha, J_2'[J_3\beta]\gamma \rangle + \langle \alpha, J_3'[J_2\beta]\gamma \rangle) \\ & \quad + c_2c_4(\langle \alpha, J_2'[K_1\beta]\gamma \rangle + \langle \alpha, K_1'[J_2\beta]\gamma \rangle) \\ & \quad + c_2c_5(\langle \alpha, J_2'[K_2\beta]\gamma \rangle + \langle \alpha, K_2'[J_2\beta]\gamma \rangle) \\ & \quad + c_3c_4(\langle \alpha, J_3'[K_1\beta]\gamma \rangle + \langle \alpha, K_1'[J_3\beta]\gamma \rangle) \\ & \quad + c_3c_5(\langle \alpha, J_3'[K_2\beta]\gamma \rangle + \langle \alpha, K_2'[J_3\beta]\gamma \rangle) \\ & \quad + c_4c_5(\langle \alpha, K_1'[K_2\beta]\gamma \rangle + \langle \alpha, K_2'[K_1\beta]\gamma \rangle) \\ & \quad + \text{cycle}(\alpha, \beta, \gamma) \\ &= c_1c_5(\langle \alpha, J_1'[K_2\beta]\gamma \rangle + \langle \alpha, K_2'[J_1\beta]\gamma \rangle) \\ & \quad + c_3c_4(\langle \alpha, J_3'[K_1\beta]\gamma \rangle + \langle \alpha, K_1'[J_3\beta]\gamma \rangle) \\ & \quad + \text{cycle}(\alpha, \beta, \gamma) \\ &= 0, \end{aligned}$$

where the last identity holds because of $c_1c_5 = c_3c_4$ and Lemma 4. This completes the proof of the proposition. \blacksquare

Recall that a pair of Hamiltonian operators J and K is called compatible, if $J + K$ is Hamiltonian. From the above proposition, we immediately arrive at:

COROLLARY 1. *The case of $c_1 = -c_2 = c_4 = 1$ and $c_3 = c_5 = 0$ leads to the compatibility of the Hamiltonian operators (27).*

COROLLARY 2. *The case of $c_1 = c_3 = c_4 = c_5 = 1$ and $c_2 = 0$ leads to the compatibility of the Hamiltonian operators (39).*

COROLLARY 3. *The case of $c_2 = c_3 = c_5 = 1$ and $c_1 = c_4 = 0$ leads to the compatibility of the Hamiltonian operators (56).*

Remark 6. The Hamiltonian pair J and K in each example in Section 3 is a special case of the generalized form $J = c_1J_1 + c_2J_2 + c_3J_3$ and $K = c_4K_1 + c_5K_2$, where $c_1c_5 = c_3c_4$. The compatibility of such a Hamiltonian pair is guaranteed by Proposition 1.

5. Conclusions and discussions

In the paper, from the spectral problems (8) and (9), we propose a generalized two-component model (7) which allows for an arbitrary function H to be involved in. We may generate many integrable peakon systems with different choices of H in our model. So, our model provides a large class of peakon systems and covers almost all existing integrable peakon equations associated with 2×2 spectral problems. Because of the presence of an arbitrary function in the generalized system, we do not expect all those equations possess the bi-Hamiltonian structures in general. Nevertheless, we show that for some special choices of the function H in (7) we may find the bi-Hamiltonian structures. Moreover, from the generalized model we obtain very interesting solutions, such as new type of N -peakon solution which is not in the traveling wave type.

Different from the usual integrable soliton equations, the peakon equation involved in an arbitrary function seems to be unusual. We believe that this system deserves a further investigation. The following two problems seem to be interesting:

- Is there a gauge transformation that can remove the arbitrary function H ?
- Can the inverse scattering transforms be applied to solve our system in general?

Very recently, we know that Li et al. [40] proposed a four-component peakon equation with an arbitrary function involved in, where they cited a preprint version [41] of the present paper. We believe that both our generalized peakon system and Li–Liu–Popowicz’s system deserve a further investigation.

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