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A new two-component integrable system with peakon solutions

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A new two-component system with cubic nonlinearity and linear dispersion:

$$m_t = bu_x + \frac{1}{2}[m(uv - u_x v_x)]_x - \frac{1}{2}m(uv_x - u_x v),$$

$$n_t = bv_x + \frac{1}{2}[n(uv - u_x v_x)]_x + \frac{1}{2}n(uv_x - u_x v),$$

$$m = u - u_{xx}, \quad n = v - v_{xx},$$

where b is an arbitrary real constant, is proposed in this paper. This system is shown integrable with its Lax pair, bi-Hamiltonian structure and infinitely many conservation laws. Geometrically, this system describes a non-trivial one-parameter family of pseudo-spherical surfaces. In the case $b=0$, the peaked soliton (peakon) and multi-peakon solutions to this two-component system are derived. In particular, the two-peakon dynamical system is explicitly solved and their interactions are investigated in details. Moreover, a new integrable cubic nonlinear equation with linear dispersion

$$m_t = bu_x + \frac{1}{2}[m(|u|^2 - |u_x|^2)]_x - \frac{1}{2}m(uu_x^* - u_x u^*),$$

$$m = u - u_{xx},$$

is obtained by imposing the complex conjugate reduction $v = u^*$ to the two-component system. The complex-valued N -peakon solution and kink wave solution to this complex equation are also derived.

1. Introduction

In recent years, the Camassa–Holm (CH) equation [1]

$$m_t - bu_x + 2mu_x + m_x u = 0, \quad m = u - u_{xx}, \quad (1.1)$$

where b is an arbitrary constant, derived by Camassa & Holm [1] as a shallow water wave model, has attracted

much attention in the theory of soliton and integrable system. As an integrable equation it was implied in the work of Fuchssteiner & Fokas [2] on hereditary symmetries as a very special case. Since the work of Camassa & Holm [1], more diverse studies on this equation have been remarkably developed [3–14]. The most interesting feature of the CH equation (1.1) is that it admits peaked soliton (peakon) solutions in the case $b=0$ [1,3]. A peakon is a weak solution in some Sobolev space with corner at its crest. The stability and interaction of peakons were discussed in several references [15–19]. Moreover, in [20], the author discussed the potential applications of the CH equation to tsunami dynamics.

In addition to the CH equation, other integrable models with peakon solutions have been found [21–30]. Among these models, there are two integrable peakon equations with cubic nonlinearity, which are

$$m_t = bu_x + [m(u^2 - u_x^2)]_x, \quad m = u - u_{xx}, \quad (1.2)$$

and

$$m_t = u^2 m_x + 3uu_x m, \quad m = u - u_{xx}. \quad (1.3)$$

Equation (1.2) was proposed independently by Fokas [5], Fuchssteiner [6], Olver & Rosenau [4] and Qiao [26–28]. Equation (1.2) is the first cubic nonlinear integrable system possessing peakon solutions. Recently, the peakon stability of equation (1.2) with $b=0$ was worked out by Gui *et al.* [31]. In 2009, Novikov [30] derived another cubic equation, which is equation (1.3), from the symmetry approach, and Hone & Wang [29] gave its Lax pair, bi-Hamiltonian structure and peakon solutions. Very recently [32], we derived the Lax pair, bi-Hamiltonian structure, peakons, weak kinks, kink-peakon interactional and smooth soliton solutions for the following integrable equation with both quadratic and cubic nonlinearity [5,6]:

$$m_t = bu_x + \frac{1}{2}k_1[m(u^2 - u_x^2)]_x + \frac{1}{2}k_2(2mu_x + m_x u), \quad m = u - u_{xx}, \quad (1.4)$$

where b , k_1 and k_2 are three arbitrary constants. By some appropriate rescaling, equation (1.4) was implied in the papers of Fokas & Fuchssteiner [5,6], where it was derived from the two-dimensional hydrodynamical equations for surface waves. Equation (1.4) can also be derived by applying the tri-Hamiltonian duality to the bi-Hamiltonian Gardner equation [4].

The above shown equations are one-component integrable peakon models. It is very interesting for us to study multi-component integrable generalizations of peakon equations. For example, in [4,33–37], the authors proposed two-component generalizations of the CH equation (1.1) with $b=0$, and in [38,39], the authors presented two-component extensions of the cubic nonlinear equation (1.3) and equation (1.2) with $b=0$.

In this paper, we propose the following two-component system with cubic nonlinearity and linear dispersion

$$\left. \begin{aligned} m_t &= bu_x + \frac{1}{2}[m(uv - u_x v_x)]_x - \frac{1}{2}m(uv_x - u_x v), \\ n_t &= bv_x + \frac{1}{2}[n(uv - u_x v_x)]_x + \frac{1}{2}n(uv_x - u_x v), \\ m &= u - u_{xx}, \quad n = v - v_{xx}, \end{aligned} \right\} \quad (1.5)$$

where b is an arbitrary real constant. This system is reduced to the CH equation (1.1) as $v = -2$, to the cubic CH equation (1.2) as $v = 2u$, and to the generalized CH equation (1.4) as $v = k_1 u + k_2$. Moreover, by imposing the complex conjugate reduction $v = u^*$, equation (1.5) is reduced to a new integrable equation with cubic nonlinearity and linear dispersion

$$m_t = bu_x + \frac{1}{2}[m(|u|^2 - |u_x|^2)]_x - \frac{1}{2}m(uu_x^* - u_x u^*), \quad m = u - u_{xx}, \quad (1.6)$$

where the symbol $*$ denotes the complex conjugate of a potential. The above reductions of the two-component system (1.5) look very like the ones of AKNS system, which can be reduced to the KdV equation, the mKdV equation, the Gardner equation and the nonlinear Schrödinger equation. We prove the integrability of system (1.5) by providing its Lax pair, bi-Hamiltonian structure and infinitely many conservation laws. Geometrically system (1.5) describes pseudo-spherical surfaces and thus it is also integrable in the sense of geometry. In the case $b=0$ (dispersionless

case), we show that this system admits the single-peakon of travelling wave solution as well as multi-peakon solutions. In particular, the two-peakon dynamic system is explicitly solved and their interactions are investigated in details. Moreover, we propose the complex-valued N -peakon solution and kink wave solution to the cubic nonlinear complex equation (1.6). To the best of our knowledge, equation (1.6) is the first model admitting complex peakon solution and kink solution.

The whole paper is organized as follows. In §2, the Lax pair, bi-Hamiltonian structure as well as infinitely many conservation laws of equation (1.5) are presented. In §3, the geometric integrability of equation (1.5) are studied. In §4, the single-peakon, multi-peakon and two-peakon dynamics are discussed. Section 5 shows that equation (1.6) admits the complex-valued peakon solution and kink wave solution. Some conclusions and open problems are described in §6.

2. Lax pair, bi-Hamiltonian structure and conservation laws

Let us consider a pair of linear spectral problems

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad U = \frac{1}{2} \begin{pmatrix} -\alpha & \lambda m \\ -\lambda n & \alpha \end{pmatrix} \quad (2.1)$$

and

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = V \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad V = -\frac{1}{2} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & -V_{11} \end{pmatrix}, \quad (2.2)$$

where λ is a spectral parameter, $m = u - u_{xx}$, $n = v - v_{xx}$, $\alpha = \sqrt{1 - \lambda^2 b}$, b is an arbitrary constant and

$$\left. \begin{aligned} V_{11} &= \lambda^{-2} \alpha + \frac{\alpha}{2}(uv - u_x v_x) + \frac{1}{2}(u v_x - u_x v), \\ V_{12} &= -\lambda^{-1}(u - \alpha u_x) - \frac{1}{2} \lambda m (uv - u_x v_x) \\ V_{21} &= \lambda^{-1}(v + \alpha v_x) + \frac{1}{2} \lambda n (uv - u_x v_x). \end{aligned} \right\} \quad (2.3)$$

and

The compatibility condition of (2.1) and (2.2) generates

$$U_t - V_x + [U, V] = 0. \quad (2.4)$$

Substituting the expressions of U and V given by (2.1) and (2.2) into (2.4), we find that (2.4) is nothing but equation (1.5). Hence, (2.1) and (2.2) exactly give the Lax pair of (1.5).

Let

$$\left. \begin{aligned} K &= \begin{pmatrix} 0 & \partial^2 - 1 \\ 1 - \partial^2 & 0 \end{pmatrix} \\ \text{and} \quad J &= \begin{pmatrix} \partial m \partial^{-1} m \partial - m \partial^{-1} m & \partial m \partial^{-1} n \partial + m \partial^{-1} n + 2b \partial \\ \partial n \partial^{-1} m \partial + n \partial^{-1} m + 2b \partial & \partial n \partial^{-1} n \partial - n \partial^{-1} n \end{pmatrix}. \end{aligned} \right\} \quad (2.5)$$

Lemma 2.1. J and K are a pair of Hamiltonian operators.

Proof. It is obvious that K is Hamiltonian, since it is a skew-symmetric operator with constant-coefficient. It is easy to check if J is skew-symmetric. We need to prove that J satisfies the Jacobi identity

$$\langle \zeta, J'[J\eta]\theta \rangle + \langle \eta, J'[J\theta]\zeta \rangle + \langle \theta, J'[J\zeta]\eta \rangle = 0, \quad (2.6)$$

where $\zeta = (\zeta_1, \zeta_2)^T$, $\eta = (\eta_1, \eta_2)^T$ and $\theta = (\theta_1, \theta_2)^T$ are arbitrary testing functions, and the prime-sign means the Gâteaux derivative of an operator F on q in the direction σ defined as [6]

$$F'[\sigma] = F'(q)[\sigma] = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} F(q + \epsilon \sigma). \quad (2.7)$$

For brevity, we introduce the notations

$$\left. \begin{aligned} \tilde{A} &= \partial^{-1}(m\zeta_{1,x} + n\zeta_{2,x}), & \tilde{B} &= \partial^{-1}(m\eta_{1,x} + n\eta_{2,x}), & \tilde{C} &= \partial^{-1}(m\theta_{1,x} + n\theta_{2,x}) \\ \text{and} & & A &= \partial^{-1}(m\zeta_1 - n\zeta_2), & B &= \partial^{-1}(m\eta_1 - n\eta_2), & C &= \partial^{-1}(m\theta_1 - n\theta_2). \end{aligned} \right\} \quad (2.8)$$

By direct calculations, we arrive at

$$\begin{aligned} \langle \zeta, J'[J\eta]\theta \rangle &= \int_{-\infty}^{+\infty} [(\theta_{1,x}m_x + \theta_{2,x}n_x)\tilde{B}\tilde{A} - (\zeta_{1,x}m_x + \zeta_{2,x}n_x)\tilde{B}\tilde{C} + \tilde{C}_x\tilde{B}_x\tilde{A} - \tilde{A}_x\tilde{B}_x\tilde{C}] dx \\ &+ \int_{-\infty}^{+\infty} [(\zeta_{1,x}m - \zeta_{2,x}n)(\tilde{B}\tilde{C} + C\tilde{B}) - (\theta_{1,x}m - \theta_{2,x}n)(\tilde{B}\tilde{A} + A\tilde{B})] dx \\ &+ \int_{-\infty}^{+\infty} [(\zeta_1m + \zeta_2n)BC - (\theta_1m + \theta_2n)BA] dx \\ &- 2b \int_{-\infty}^{+\infty} [(\zeta_{1,x}\eta_{2,x} + \zeta_{2,x}\eta_{1,x})\tilde{C} - (\eta_{2,x}\theta_{1,x} + \eta_{1,x}\theta_{2,x})\tilde{A}] dx \\ &+ 2b \int_{-\infty}^{+\infty} [(\zeta_2\eta_{1,x} - \zeta_1\eta_{2,x})C - (\eta_{1,x}\theta_2 - \eta_{2,x}\theta_1)A] dx. \end{aligned} \quad (2.9)$$

Based on (2.9), we may verify (2.6) directly. This completes the proof of lemma 2.1. ■

Lemma 2.2. *The following relation holds*

$$\langle \zeta, J'[K\eta]\theta \rangle + \langle \eta, J'[K\theta]\zeta \rangle + \langle \theta, J'[K\zeta]\eta \rangle + \langle \zeta, K'[J\eta]\theta \rangle + \langle \eta, K'[J\theta]\zeta \rangle + \langle \theta, K'[J\zeta]\eta \rangle = 0. \quad (2.10)$$

Proof. Direct calculations yield that

$$\begin{aligned} \langle \zeta, J'[K\eta]\theta \rangle &= - \int_{-\infty}^{+\infty} [(\zeta_{2,x}\eta_1 - \zeta_{1,x}\eta_2)\tilde{C} - (\theta_{2,x}\eta_1 - \theta_{1,x}\eta_2)\tilde{A}] dx \\ &- \int_{-\infty}^{+\infty} [(\zeta_{1,x}\eta_{2,xx} - \zeta_{2,x}\eta_{1,xx})\tilde{C} - (\eta_{2,xx}\theta_{1,x} - \eta_{1,xx}\theta_{2,x})\tilde{A}] dx \\ &+ \int_{-\infty}^{+\infty} [(\zeta_1\eta_2 + \zeta_2\eta_1)C - (\eta_1\theta_2 + \eta_2\theta_1)A] dx \\ &- \int_{-\infty}^{+\infty} [(\zeta_1\eta_{2,xx} + \zeta_2\eta_{1,xx})C - (\eta_{1,xx}\theta_2 + \eta_{2,xx}\theta_1)A] dx. \end{aligned} \quad (2.11)$$

Formula (2.10) may be verified based on (2.11). The proof of lemma 2.2 is finished. ■

From lemmas 2.1 and 2.2, we immediately obtain

Proposition 2.3. *J and K are compatible Hamiltonian operators.*

Furthermore, we have

Proposition 2.4. *Equation (1.5) can be rewritten in the following bi-Hamiltonian form*

$$(m_t, n_t)^T = J \left(\frac{\delta H_1}{\delta m}, \frac{\delta H_1}{\delta n} \right)^T = K \left(\frac{\delta H_2}{\delta m}, \frac{\delta H_2}{\delta n} \right)^T, \quad (2.12)$$

where J and K are given by (2.5), and

$$\left. \begin{aligned} H_1 &= \frac{1}{2} \int_{-\infty}^{+\infty} (uv + u_x v_x) dx \\ \text{and} & & H_2 &= \frac{1}{4} \int_{-\infty}^{+\infty} [(u^2 v_x + u_x^2 v_x - 2uu_x v)n + 2b(uv_x - u_x v)] dx. \end{aligned} \right\} \quad (2.13)$$

Next, we construct conservation laws of equation (1.5). Let $\varphi = \phi_2/\phi_1$, where ϕ_1 and ϕ_2 are determined through equations (2.1) and (2.2). From (2.1), one can easily verify that φ satisfies the

Riccati equation

$$\varphi_x = -\frac{1}{2}\lambda m\varphi^2 + \alpha\varphi - \frac{1}{2}\lambda n. \quad (2.14)$$

Equations (2.1) and (2.2) give rise to

$$(\ln \phi_1)_x = -\frac{\alpha}{2} + \frac{1}{2}\lambda m\varphi \quad \text{and} \quad (\ln \phi_1)_t = -\frac{1}{2}(V_{11} + V_{12}\varphi), \quad (2.15)$$

which yields conservation law of equation (1.5):

$$\rho_t = F_x, \quad (2.16)$$

where

$$\left. \begin{aligned} \rho &= m\varphi \\ \text{and} \quad F &= \lambda^{-2}(u - \alpha u_x)\varphi - \frac{1}{2}\lambda^{-1}(\alpha uv - \alpha u_x v_x + uv_x - u_x v) + \frac{1}{2}m(uv - u_x v_x)\varphi. \end{aligned} \right\} \quad (2.17)$$

Usually, ρ and F are called a conserved density and an associated flux, respectively. In the case $b=0$, we are able to derive the explicit forms of conservation densities by expanding φ in powers of λ in two ways. The first one is to expand φ in terms of negative powers of λ as

$$\varphi = \sum_{j=0}^{\infty} \varphi_j \lambda^{-j}. \quad (2.18)$$

Substituting (2.18) into (2.14) and equating the coefficients of powers of λ , we arrive at

$$\varphi_0 = \sqrt{-\frac{n}{m}} \quad \text{and} \quad \varphi_1 = \frac{mn_x - m_x n - 2mn}{2m^2 n}, \quad (2.19)$$

and the recursion relation for φ_j :

$$\varphi_{j+1} = \frac{1}{m\varphi_0} \left[\varphi_j - \varphi_{j,x} - \frac{1}{2}m \sum_{i+k=j+1, i,k \geq 1} \varphi_i \varphi_k \right], \quad j \geq 1. \quad (2.20)$$

Inserting (2.18), (2.19) and (2.20) into (2.17), we obtain the following infinitely many conserved densities and the associated fluxes of equation (1.5):

$$\left. \begin{aligned} \rho_0 &= \sqrt{-mn}, \quad F_0 = \frac{1}{2}\sqrt{-mn}(uv - u_x v_x), \\ \rho_1 &= \frac{mn_x - m_x n - 2mn}{2mn}, \quad F_1 = -\frac{1}{2}(uv - u_x v_x + uv_x - u_x v) + \frac{1}{2}\rho_1(uv - u_x v_x) \end{aligned} \right\} \quad (2.21)$$

and $\rho_j = m\varphi_j, \quad F_j = (u - u_x)\varphi_{j-2} + \frac{1}{2}\rho_j(uv - u_x v_x), \quad j \geq 2,$

where φ_j is given by (2.19) and (2.20).

The second expansion of φ is in the positive powers of λ ,

$$\varphi = \sum_{j=0}^{\infty} \varphi_j \lambda^j. \quad (2.22)$$

Substituting (2.22) into (2.14) and comparing powers of λ produce

$$\left. \begin{aligned} \varphi_{2j} &= 0, \quad j \geq 0 \\ \text{and} \quad \varphi_1 &= \frac{1}{2}(v + v_x), \quad \varphi_{2j+1} - \varphi_{2j+1,x} = \frac{1}{2}m \sum_{i+k=2j, 0 \leq i,k \leq 2j} \varphi_i \varphi_k, \quad j \geq 1. \end{aligned} \right\} \quad (2.23)$$

By inserting (2.22) and (2.23) into (2.17), we arrive at

$$\rho_{2j} = 0, \quad A_{2j} = 0, \quad j \geq 0, \quad (2.24)$$

and

$$\left. \begin{aligned} \rho_1 &= \frac{1}{2}m(v + v_x), \quad A_1 = (u - u_x)\varphi_3 + \frac{1}{4}m(uv - u_x v_x)(v + v_x) \\ \text{and} \quad \rho_{2j+1} &= m\varphi_{2j+1}, \quad A_{2j+1} = (u - u_x)\varphi_{2j+3} + \frac{1}{2}m(uv - u_x v_x)\varphi_{2j+1}, \quad j \geq 1, \end{aligned} \right\} \quad (2.25)$$

where the odd-index φ_{2j+1} is defined by the recursion relation

$$\varphi_{2j+1} = \frac{1}{2}(1 - \partial_x)^{-1} \left(m \sum_{i+k=2j, 0 \leq i, k \leq 2j} \varphi_i \varphi_k \right), \quad j \geq 1. \quad (2.26)$$

Formula (2.24) means that the even-index conserved densities and associated fluxes are trivial. Formulas (2.25) and (2.26) show that the non-trivial high-order odd-index conserved densities may involve in non-local expressions in u and v .

Remark 2.5. Here, we have derived two sequences of infinitely many conserved densities and the associated fluxes for equation (1.5). The conserved densities in the sequence (2.21) become singular when the denominators have zero points. The conserved densities in the sequence (2.25) have no singularity, but they might involve in non-local expressions.

3. Geometric integrability

Based on the work of Chern & Tenenblat [40] and the subsequent works [41,42], a differential equation for a real-valued function $u(x, t)$ is said to describe pseudo-spherical surfaces if it is the necessary and sufficient condition for the existence of smooth functions f_{ij} , $i = 1, 2, 3$, $j = 1, 2$, depending on x , t , u and its derivatives, such that the one-forms $\omega_i = f_{i1} dx + f_{i2} dt$ satisfy the structure equations of a surface of constant Gaussian curvature equal to -1 with metric $\omega_1^2 + \omega_2^2$ and connection one-form ω_3 , namely

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3 \quad \text{and} \quad d\omega_3 = \omega_1 \wedge \omega_2. \quad (3.1)$$

Let us consider

$$\left. \begin{aligned} f_{11} &= -\frac{1}{2}\lambda[e^{(\alpha-\lambda)x}m - e^{(\lambda-\alpha)x}n], \\ f_{12} &= \frac{1}{2}\lambda^{-1}[e^{(\lambda-\alpha)x}(v + \alpha v_x) - e^{(\alpha-\lambda)x}(u - \alpha u_x)] + \frac{1}{4}\lambda[e^{(\lambda-\alpha)x}n - e^{(\alpha-\lambda)x}m](uv - u_x v_x), \\ f_{21} &= \lambda, \\ f_{22} &= \lambda^{-2}\alpha + \frac{\alpha}{2}(uv - u_x v_x) + \frac{1}{2}(uv_x - u_x v), \\ f_{31} &= -\frac{1}{2}\lambda[e^{(\alpha-\lambda)x}m + e^{(\lambda-\alpha)x}n] \\ \text{and} \quad f_{32} &= -\frac{1}{2}\lambda^{-1}[e^{(\lambda-\alpha)x}(v + \alpha v_x) + e^{(\alpha-\lambda)x}(u - \alpha u_x)] - \frac{1}{4}\lambda[e^{(\lambda-\alpha)x}n + e^{(\alpha-\lambda)x}m](uv - u_x v_x), \end{aligned} \right\} \quad (3.2)$$

and introduce the following three one-forms

$$\omega_1 = f_{11} dx + f_{12} dt, \quad \omega_2 = f_{21} dx + f_{22} dt \quad \text{and} \quad \omega_3 = f_{31} dx + f_{32} dt. \quad (3.3)$$

Through a direct computation, we find that the structure equations (3.1) hold whenever $u(x, t)$ and $v(x, t)$ are solutions of system (1.5). Thus, we have

Theorem 3.1. *System (1.5) describes pseudo-spherical surfaces.*

Recall that a differential equation is geometrically integrable if it describes a non-trivial one-parameter family of pseudo-spherical surfaces. It follows that

Corollary 3.2. *System (1.5) is geometrically integrable.*

According to [40–43], we have the following fact

Proposition 3.3. A geometrically integrable equation with associated one-forms ω_i , $i = 1, 2, 3$, is the integrability condition of a one-parameter family of $sl(2, R)$ -valued linear problem

$$d\Phi = \Omega\Phi, \quad (3.4)$$

where Ω is the matrix-valued one-form

$$\Omega = X dx + T dt = \frac{1}{2} \begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix}. \quad (3.5)$$

Therefore, the one-forms (3.3) and (3.4) yield an $sl(2, R)$ -valued linear problem $\Phi_x = X\Phi$ and $\Phi_t = T\Phi$, whose integrability condition is the two-component system (1.5). The expression (3.5) implies that the matrices X and T are

$$\left. \begin{aligned} X &= \frac{1}{2} \begin{pmatrix} \lambda & \lambda e^{(\lambda-\alpha)x} n \\ -\lambda e^{(\alpha-\lambda)x} m & -\lambda \end{pmatrix} \\ \text{and } T &= \frac{1}{2} \begin{pmatrix} \lambda^{-2}\alpha + \frac{\alpha}{2}(uv - u_x v_x) + \frac{1}{2}(uv_x - u_x v) & \left[\lambda^{-1}(v + \alpha v_x) + \frac{\lambda}{2}n(uv - u_x v_x) \right] e^{(\lambda-\alpha)x} \\ -\left[\lambda^{-1}(u - \alpha u_x) + \frac{\lambda}{2}m(uv - u_x v_x) \right] e^{(\alpha-\lambda)x} & -\lambda^{-2}\alpha - \frac{\alpha}{2}(uv - u_x v_x) - \frac{1}{2}(uv_x - u_x v) \end{pmatrix} \end{aligned} \right\} \quad (3.6)$$

4. Peakon solutions to system (1.5) in the case $b = 0$

In this section, we shall derive the peakon solutions to the two-component system (1.5) with $b = 0$ in two situations. The first situation is the peakon solutions with the same peakon position. The second situation is the peakon solutions with different peakon positions, which is studied by Cotter *et al.* [37] for a cross-coupled CH equation.

(a) Peakon solutions to the two-component system (1.5) with the same peakon position

Let us suppose that a single peakon solution of (1.5) with $b = 0$ is of the following form

$$u = c_1 e^{-|x-ct|} \quad \text{and} \quad v = c_2 e^{-|x-ct|}, \quad (4.1)$$

where the two constants c_1 and c_2 are to be determined. With the help of distribution theory, we are able to write out u_x , m and v_x , n as follows:

$$\left. \begin{aligned} u_x &= -c_1 \operatorname{sgn}(x - ct) e^{-|x-ct|}, & m &= 2c_1 \delta(x - ct) \\ \text{and } v_x &= -c_2 \operatorname{sgn}(x - ct) e^{-|x-ct|}, & n &= 2c_2 \delta(x - ct). \end{aligned} \right\} \quad (4.2)$$

Substituting (4.1) and (4.2) into (1.5) with $b = 0$ and integrating in the distribution sense, one can readily see that c_1 and c_2 should satisfy

$$c_1 c_2 = -3c. \quad (4.3)$$

In particular, for $c_1 = c_2$, we recover the single peakon solution $u = \pm \sqrt{-3c} e^{-|x-ct|}$ of the cubic CH equation (1.2) with $b = 0$ [31,32].

Let us now assume a two-peakon solution as follows:

$$u = p_1(t) e^{-|x-q_1(t)|} + p_2(t) e^{-|x-q_2(t)|} \quad \text{and} \quad v = r_1(t) e^{-|x-q_1(t)|} + r_2(t) e^{-|x-q_2(t)|}. \quad (4.4)$$

In the sense of distribution, we have

$$\left. \begin{aligned} u_x &= -p_1 \operatorname{sgn}(x - q_1) e^{-|x-q_1|} - p_2 \operatorname{sgn}(x - q_2) e^{-|x-q_2|}, \\ m &= 2p_1 \delta(x - q_1) + 2p_2 \delta(x - q_2), \\ v_x &= -r_1 \operatorname{sgn}(x - q_1) e^{-|x-q_1|} - r_2 \operatorname{sgn}(x - q_2) e^{-|x-q_2|} \\ \text{and } n &= 2r_1 \delta(x - q_1) + 2r_2 \delta(x - q_2). \end{aligned} \right\} \quad (4.5)$$

Substituting (4.4) and (4.5) into (1.5) with $b = 0$ and integrating through test functions yield the following dynamic system:

$$\left. \begin{aligned} p_{1,t} &= \frac{1}{2}p_1(p_1r_2 - p_2r_1) \operatorname{sgn}(q_1 - q_2)e^{-|q_1 - q_2|}, \\ p_{2,t} &= \frac{1}{2}p_2(p_2r_1 - p_1r_2) \operatorname{sgn}(q_2 - q_1)e^{-|q_2 - q_1|}, \\ q_{1,t} &= -\frac{1}{3}p_1r_1 - \frac{1}{2}(p_1r_2 + p_2r_1)e^{-|q_1 - q_2|}, \\ q_{2,t} &= -\frac{1}{3}p_2r_2 - \frac{1}{2}(p_1r_2 + p_2r_1)e^{-|q_2 - q_1|}, \\ r_{1,t} &= -\frac{1}{2}r_1(p_1r_2 - p_2r_1) \operatorname{sgn}(q_1 - q_2)e^{-|q_1 - q_2|} \\ r_{2,t} &= -\frac{1}{2}r_2(p_2r_1 - p_1r_2) \operatorname{sgn}(q_2 - q_1)e^{-|q_2 - q_1|}. \end{aligned} \right\} \quad (4.6)$$

and

Guided by the above equations, we may conclude the following relations:

$$p_1 = Dp_2, \quad p_1r_1 = A_1 \quad \text{and} \quad p_2r_2 = A_2, \quad (4.7)$$

where D , A_1 and A_2 are three arbitrary integration constants.

If $A_1 = A_2$, we arrive at the following solution of (4.6):

$$\left. \begin{aligned} p_1(t) &= Be^{(1/2D)(D^2A_1 - A_1) \operatorname{sgn}(C_1)e^{-|C_1|}t}, \quad p_2(t) = \frac{p_1}{D}, \quad r_1(t) = \frac{A_1}{p_1}, \quad r_2(t) = \frac{A_1}{p_2}, \\ \text{and} \quad q_1(t) &= -\left[\frac{1}{3}A_1 + \frac{1}{2D}(D^2A_1 + A_1)e^{-|C_1|} \right]t + \frac{1}{2}C_1, \quad q_2(t) = q_1(t) - C_1, \end{aligned} \right\} \quad (4.8)$$

where B and C_1 are two arbitrary non-zero constants. In this case, the collision between two peakons will never happen since $q_2(t) = q_1(t) - C_1$. For example, as $A_1 = B = D = 1$, $C_1 = 2$, (4.8) is reduced to

$$p_1(t) = p_2(t) = r_1(t) = r_2(t) = 1,$$

$$q_1(t) = -\left(\frac{1}{3} + e^{-2}\right)t + 1$$

and

$$q_2(t) = -\left(\frac{1}{3} + e^{-2}\right)t - 1.$$

Thus, the associated solution of (1.5) with $b = 0$ becomes

$$u(x, t) = v(x, t) = e^{-|x + (1/3 + e^{-2})t - 1|} + e^{-|x + (1/3 + e^{-2})t + 1|}. \quad (4.9)$$

This wave has two peaks, and looks like a M-shape soliton. See figure 1 for this M-shape two-peakon solution. As $A_1 = -B = -D = 1$, $C_1 = 2$, the associated solution of (1.5) with $b = 0$ becomes

$$u(x, t) = v(x, t) = -e^{-|x + (1/3 - e^{-2})t - 1|} + e^{-|x + (1/3 - e^{-2})t + 1|}, \quad (4.10)$$

which has one peak and one trough and looks like N-shape soliton solution. See figure 2 for this N-shape two-peakon solution. As $B = 2D = 1$, $A_1 = C_1 = 2$, (4.8) becomes

$$\left. \begin{aligned} p_1(t) &= \frac{1}{2}p_2(t) = e^{-(3/2)e^{-2}t}, \quad r_1(t) = 2r_2(t) = 2e^{(3/2)e^{-2}t} \\ \text{and} \quad q_1(t) &= -\left(\frac{2}{3} + \frac{5}{2}e^{-2}\right)t + 1, \quad q_2(t) = -\left(\frac{2}{3} + \frac{5}{2}e^{-2}\right)t - 1. \end{aligned} \right\} \quad (4.11)$$

and the associated solution of (1.5) with $b = 0$ becomes

$$\left. \begin{aligned} u(x, t) &= e^{-(3/2)e^{-2}t} \left(e^{-|x + (2/3 + (5/2)e^{-2})t - 1|} + 2e^{-|x + (2/3 + (5/2)e^{-2})t + 1|} \right) \\ \text{and} \quad v(x, t) &= e^{(3/2)e^{-2}t} \left(2e^{-|x + (2/3 + (5/2)e^{-2})t - 1|} + e^{-|x + (2/3 + (5/2)e^{-2})t + 1|} \right). \end{aligned} \right\} \quad (4.12)$$

From (4.11), one can easily see that the amplitudes $p_1(t)$ and $p_2(t)$ of potential $u(x, t)$ are two monotonically decreasing functions of t , while the amplitudes $r_1(t)$ and $r_2(t)$ of potential $v(x, t)$ are two monotonically increasing functions of t . Figures 3 and 4 show the profiles of the potentials $u(x, t)$ and $v(x, t)$.

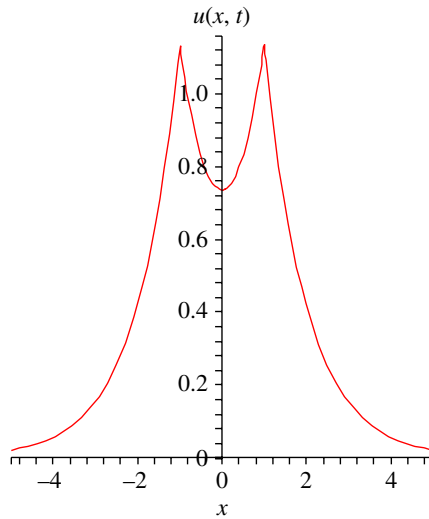


Figure 1. The M-shape two-peakon solution $u(x, t)$ in (4.9) at the moment of $t = 0$. (Online version in colour.)

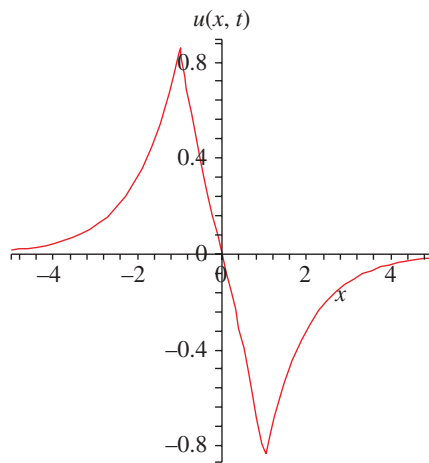


Figure 2. The N-shape peak-trough solution $u(x, t)$ in (4.10) at the moment of $t = 0$. (Online version in colour.)

If $A_1 \neq A_2$, we may obtain the following solution of (4.6):

$$\left. \begin{aligned}
 p_1(t) &= B e^{(3A_2 D^2 - A_1)/2D(A_1 - A_2)t} e^{-(1/3)|(A_1 - A_2)t|}, & p_2(t) &= \frac{p_1}{D}, \\
 r_1(t) &= \frac{A_1}{p_1}, & r_2(t) &= \frac{A_2}{p_2}, \\
 q_1(t) &= -\frac{1}{3} A_1 t + \frac{3(A_2 D^2 + A_1)}{2D(A_1 - A_2)} \operatorname{sgn}[(A_1 - A_2)t] (e^{-(1/3)|(A_1 - A_2)t|} - 1) \\
 \text{and} & & q_2(t) &= -\frac{1}{3} A_2 t + \frac{3(A_2 D^2 + A_1)}{2D(A_1 - A_2)} \operatorname{sgn}[(A_1 - A_2)t] (e^{-(1/3)|(A_1 - A_2)t|} - 1),
 \end{aligned} \right\} \quad (4.13)$$

where B is an arbitrary integration constant. Let us study the following special cases of this solution.

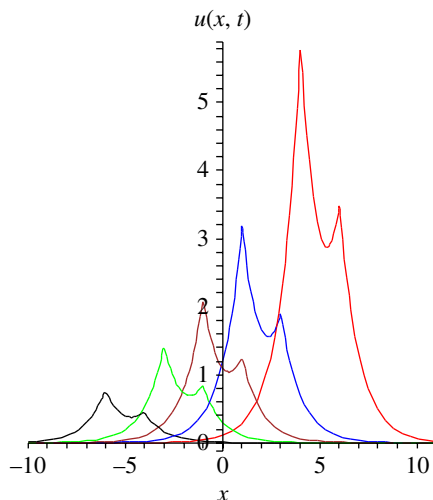


Figure 3. The two-peak solution $u(x, t)$ in (4.12). Red line, $t = -5$; blue line, $t = -2$; brown line, $t = 0$; green line, $t = 2$ and black line, $t = 5$.

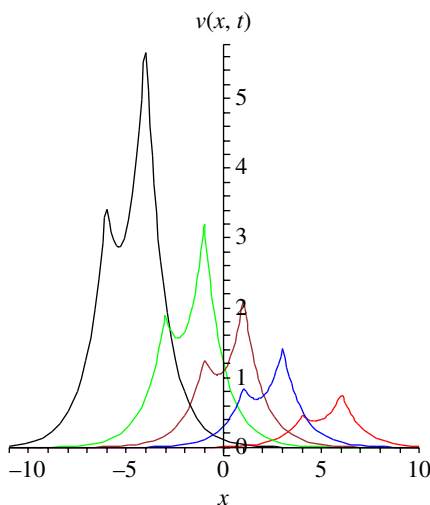


Figure 4. The two-peak solution $v(x, t)$ in (4.12). Red line: $t = -5$; blue line, $t = -2$; brown line, $t = 0$; green line, $t = 2$ and black line, $t = 5$.

Example 4.1. Let $A_1 = 1$, $A_2 = 4$, $B = 1$ and $D = \frac{1}{2}$, then

$$\left. \begin{aligned} p_1(t) &= r_1(t) = 1, & p_2 &= r_2(t) = 2, \\ q_1(t) &= -\frac{1}{3}t + 2 \operatorname{sgn}(t)(e^{-|t|} - 1) \\ q_2(t) &= -\frac{4}{3}t + 2 \operatorname{sgn}(t)(e^{-|t|} - 1). \end{aligned} \right\} \quad (4.14)$$

and

The associated two-peak solution of (1.5) becomes

$$u(x, t) = v(x, t) = e^{-|x+(1/3)t-2 \operatorname{sgn}(t)(e^{-|t|}-1)|} + 2e^{-|x+(4/3)t-2 \operatorname{sgn}(t)(e^{-|t|}-1)|}. \quad (4.15)$$

As $t < 0$ and t is going to 0, the tall peakon with the amplitude 2 chases after the short peakon with the amplitude 1. The two-peakon collides at time $t = 0$. After the collision ($t > 0$), the peaks

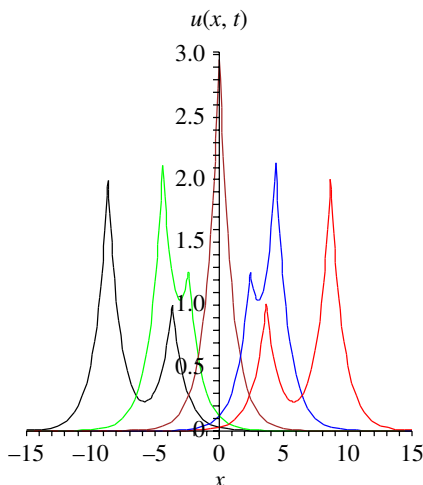


Figure 5. The two-peakon solution $u(x, t)$ in (4.15). Red line, $t = -5$; blue line, $t = -2$; brown line, $t = 0$ (collision); green line, $t = 2$ and black line, $t = 5$.

separate (the tall peakon surpasses the short one) and develop on their own way. See figure 5 for the detailed development of this kind of two-peakon.

Example 4.2. Let $A_1 = 1$, $A_2 = 4$, $B = 1$ and $D = 1$, then we have

$$\left. \begin{aligned} p_1(t) &= p_2(t) = e^{-(3/2)e^{-|t|}}, \\ r_1(t) &= e^{(3/2)e^{-|t|}}, \quad r_2(t) = 4e^{(3/2)e^{-|t|}}, \\ q_1(t) &= -\frac{1}{3}t + \frac{5}{2} \operatorname{sgn}(t)(e^{-|t|} - 1) \\ q_2(t) &= -\frac{4}{3}t + \frac{5}{2} \operatorname{sgn}(t)(e^{-|t|} - 1). \end{aligned} \right\} \quad (4.16)$$

and

The associated two-peakon solution of (1.5) becomes

$$\left. \begin{aligned} u(x, t) &= e^{-(3/2)e^{-|t|}} \left(e^{-|x+(1/3)t-(5/2) \operatorname{sgn}(t)(e^{-|t|}-1)|} + e^{-|x+(4/3)t-(5/2) \operatorname{sgn}(t)(e^{-|t|}-1)|} \right) \\ \text{and } v(x, t) &= e^{(3/2)e^{-|t|}} \left(e^{-|x+(1/3)t-(5/2) \operatorname{sgn}(t)(e^{-|t|}-1)|} + 4e^{-|x+(4/3)t-(5/2) \operatorname{sgn}(t)(e^{-|t|}-1)|} \right). \end{aligned} \right\} \quad (4.17)$$

For the potential $u(x, t)$, the two-peakon solution possesses the same amplitude $e^{-(3/2)e^{-|t|}}$, which reaches the minimum value at $t = 0$. Figure 6 shows the profile of the two-peakon dynamics for the potential $u(x, t)$. For the potential $v(x, t)$, the two-peakon solution with the amplitudes $e^{(3/2)e^{-|t|}}$ and $4e^{(3/2)e^{-|t|}}$ collides at $t = 0$. At this moment, the amplitudes attain the maximum value and the two-peakon overlaps into one peakon $5e^{(3/2)e^{-|x|}}$, which is much higher than other moments. See figures 7 and 8 for the two- and three-dimensional graphs of the two-peakon dynamics for the potential $v(x, t)$.

Example 4.3. Let $A_1 = 1$, $A_2 = 4$, $B = 1$ and $D = -1$, then we have

$$\left. \begin{aligned} p_1(t) &= -p_2(t) = e^{(3/2)e^{-|t|}}, \\ r_1(t) &= e^{-(3/2)e^{-|t|}}, \quad r_2(t) = -4e^{-(3/2)e^{-|t|}}, \\ q_1(t) &= -\frac{1}{3}t - \frac{5}{2} \operatorname{sgn}(t)(e^{-|t|} - 1) \\ q_2(t) &= -\frac{4}{3}t - \frac{5}{2} \operatorname{sgn}(t)(e^{-|t|} - 1). \end{aligned} \right\} \quad (4.18)$$

and

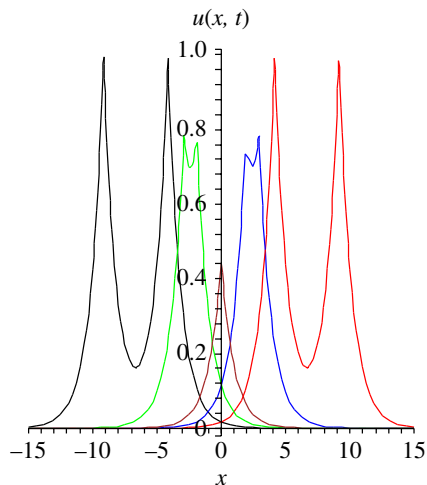


Figure 6. The two-peak solution $u(x, t)$ in (4.17). Red line, $t = -5$; blue line, $t = -1$; brown line, $t = 0$ (collision); green line, $t = 1$ and black line, $t = 5$.

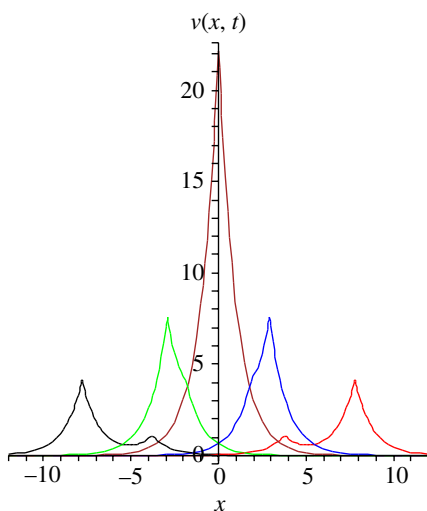


Figure 7. The two-peak solution $v(x, t)$ in (4.17). Red line, $t = -4$; blue line, $t = -1$; brown line, $t = 0$ (collision); green line, $t = 1$ and black line, $t = 4$.

The associated two-peak solution of (1.5) becomes

$$\left. \begin{aligned} u(x, t) &= e^{(3/2)e^{-|t|}} \left(e^{-|x+(1/3)t+(5/2)\operatorname{sgn}(t)(e^{-|t|}-1)|} - e^{-|x+(4/3)t+(5/2)\operatorname{sgn}(t)(e^{-|t|}-1)|} \right) \\ \text{and } v(x, t) &= e^{-(3/2)e^{-|t|}} \left(e^{-|x+(1/3)t+(5/2)\operatorname{sgn}(t)(e^{-|t|}-1)|} - 4e^{-|x+(4/3)t+(5/2)\operatorname{sgn}(t)(e^{-|t|}-1)|} \right). \end{aligned} \right\} \quad (4.19)$$

For the potential $u(x, t)$, the peakon–antipeakon collides and vanishes at $t = 0$. After the collision, the peakon and antipeakon re-emerge and separate. For the potential $v(x, t)$, the peakon and trough overlap at $t = 0$, and then they separate. Figures 9 and 10 show the peakon–antipeakon dynamics for the potentials $u(x, t)$ and $v(x, t)$.

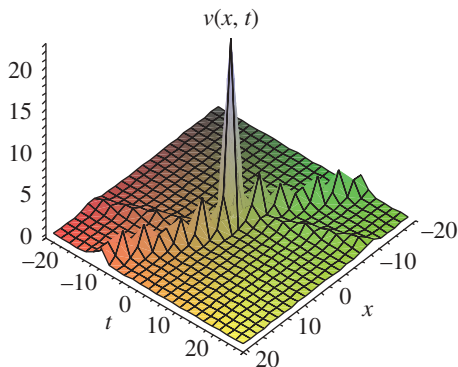


Figure 8. Three-dimensional graph for the two-peakon solution $v(x, t)$ in (4.17).

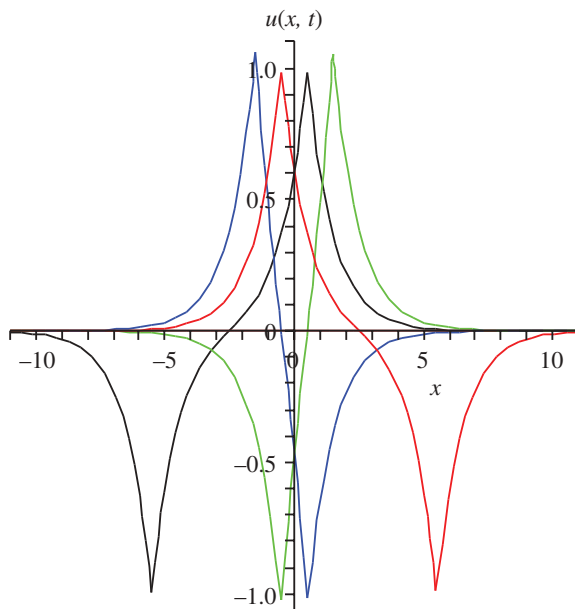


Figure 9. Peakon–antipeakon solution $u(x, t)$ in (4.19). Red line, $t = -6$; blue line, $t = -2$; at $t = 0$ they collide and vanish; green line, $t = 2$ and black line, $t = 6$.

In general, we suppose an N -peakon solution has the following form

$$u(x, t) = \sum_{j=1}^N p_j(t) e^{-|x - q_j(t)|} \quad \text{and} \quad v(x, t) = \sum_{j=1}^N r_j(t) e^{-|x - q_j(t)|}. \tag{4.20}$$

Substituting (4.20) into (1.5) with $b = 0$ and integrating through test functions, we obtain the N -peakon dynamic system as follows:

$$\left. \begin{aligned} p_{j,t} &= \frac{1}{2} p_j \sum_{i,k=1}^N p_i r_k (\operatorname{sgn}(q_j - q_k) - \operatorname{sgn}(q_j - q_i)) e^{-|q_j - q_k| - |q_j - q_i|}, \\ q_{j,t} &= \frac{1}{6} p_j r_j - \frac{1}{2} \sum_{i,k=1}^N p_i r_k (1 - \operatorname{sgn}(q_j - q_i) \operatorname{sgn}(q_j - q_k)) e^{-|q_j - q_i| - |q_j - q_k|} \end{aligned} \right\} \tag{4.21}$$

and

$$r_{j,t} = -\frac{1}{2} r_j \sum_{i,k=1}^N p_i r_k (\operatorname{sgn}(q_j - q_k) - \operatorname{sgn}(q_j - q_i)) e^{-|q_j - q_k| - |q_j - q_i|}.$$

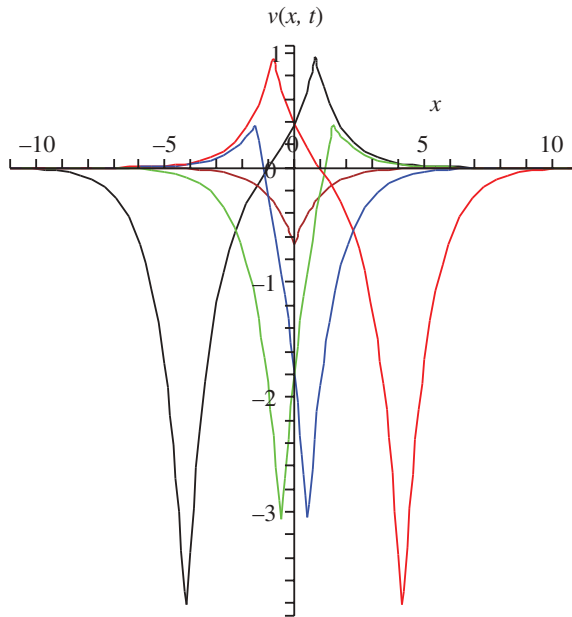


Figure 10. Peakon–antipeakon solution $v(x, t)$ in (4.19). Red line, $t = -5$; blue line, $t = -2$; brown line, $t = 0$ (collision); green line, $t = 2$ and black line, $t = 5$.

(b) Peakon solutions to the two-component system (1.5) with different peakon positions

In this section, we discuss the N -peakon solutions with different peakon positions based on the work in [37]. Let us assume that the N -peakon solutions of the two potentials u and v with different peakon positions are given in the form

$$u(x, t) = \sum_{j=1}^N p_j(t) e^{-|x - q_j(t)|} \quad \text{and} \quad v(x, t) = \sum_{j=1}^N r_j(t) e^{-|x - s_j(t)|}, \quad (4.22)$$

where $q_i(t) \neq s_j(t)$, $1 \leq i, j \leq N$. With the help of delta functions, we have

$$m(x, t) = 2 \sum_{j=1}^N p_j(t) \delta(x - q_j(t)) \quad \text{and} \quad n(x, t) = 2 \sum_{j=1}^N r_j(t) \delta(x - s_j(t)). \quad (4.23)$$

Substituting (4.22) and (4.23) into (1.5) with $b = 0$ and integrating through test functions, we arrive at the following system regarding p_j , q_j , r_j and s_j :

$$\left. \begin{aligned} p_{j,t} &= \frac{1}{2} p_j \sum_{i,k=1}^N p_i r_k (\operatorname{sgn}(q_j - s_k) - \operatorname{sgn}(q_j - q_i)) e^{-|q_j - s_k| - |q_j - q_i|}, \\ r_{j,t} &= -\frac{1}{2} r_j \sum_{i,k=1}^N p_i r_k (\operatorname{sgn}(s_j - s_k) - \operatorname{sgn}(s_j - q_i)) e^{-|s_j - s_k| - |s_j - q_i|}, \\ q_{j,t} &= -\frac{1}{2} \sum_{i,k=1}^N p_i r_k (1 - \operatorname{sgn}(q_j - q_i) \operatorname{sgn}(q_j - s_k)) e^{-|q_j - q_i| - |q_j - s_k|} \\ \text{and} \quad s_{j,t} &= -\frac{1}{2} \sum_{i,k=1}^N p_i r_k (1 - \operatorname{sgn}(s_j - q_i) \operatorname{sgn}(s_j - s_k)) e^{-|s_j - q_i| - |s_j - s_k|}. \end{aligned} \right\} \quad (4.24)$$

Different from the N -peakon dynamic system of the coupled CH equation proposed in [37], our system (4.24) can not directly be rewritten in the standard form of a canonical Hamiltonian system. It is interesting to study whether (4.24) is able to be rewritten as an integrable Hamiltonian system by introducing a Poisson bracket. We will investigate the related topics in the near future.

For $N = 1$, (4.24) is reduced to

$$\left. \begin{aligned} p_{1,t} &= \frac{1}{2} p_1^2 r_1 \operatorname{sgn}(q_1 - s_1) e^{-|q_1 - s_1|}, \\ r_{1,t} &= \frac{1}{2} p_1 r_1^2 \operatorname{sgn}(s_1 - q_1) e^{-|s_1 - q_1|}, \\ q_{1,t} &= -\frac{1}{2} p_1 r_1 e^{-|q_1 - s_1|} \\ s_{1,t} &= -\frac{1}{2} p_1 r_1 e^{-|s_1 - q_1|}. \end{aligned} \right\} \quad (4.25)$$

and

From the last two equations of (4.25), we obtain

$$s_1 = q_1 + A_1, \quad (4.26)$$

where $A_1 \neq 0$ is an integration constant. Without loss of generality, we suppose $A_1 > 0$. Substituting (4.26) into (4.25) leads to

$$\left. \begin{aligned} p_1 &= A_3 e^{-(1/2)e^{-A_1} A_2 t}, \\ r_1 &= \frac{A_2}{A_3} e^{(1/2)e^{-A_1} A_2 t}, \\ q_1 &= -\frac{1}{2} e^{-A_1} A_2 t \\ s_1 &= -\frac{1}{2} e^{-A_1} A_2 t + A_1, \end{aligned} \right\} \quad (4.27)$$

and

where A_2 and A_3 are integration constants. In particular, we take $A_1 = \ln 2$ and $A_2 = A_3 = 1$, then the single-peakon solutions with different peakon positions become

$$u = e^{-(1/4)t} e^{-|x+(1/4)t|} \quad \text{and} \quad v = e^{(1/4)t} e^{-|x+(1/4)t - \ln 2|}. \quad (4.28)$$

See figure 11 for the profile of this single-peakon solution at $t = 0$. We have not yet explicitly solved (4.24) with $N = 2$. This is due to the complexity of (4.24) with $N = 2$, which is a coupled ordinary differential equation with eight components.

5. Solutions to the integrable system (1.6)

As mentioned above, system (1.5) is cast into the integrable cubic nonlinear equation (1.6) under the complex conjugate reduction $v = u^*$. Thus, equation (1.6) possesses the following Lax pair

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad U = \frac{1}{2} \begin{pmatrix} -\alpha & \lambda m \\ -\lambda m^* & \alpha \end{pmatrix} \quad (5.1)$$

and

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = V \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad V = -\frac{1}{2} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & -V_{11} \end{pmatrix}, \quad (5.2)$$

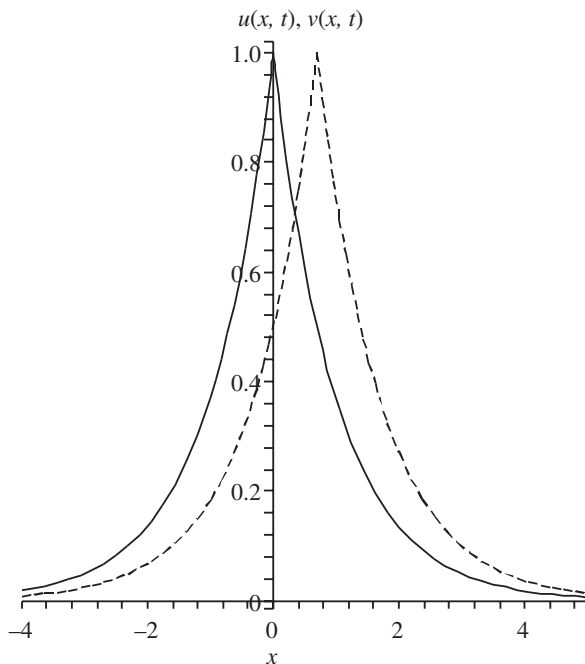


Figure 11. The single-peakon solution (4.28) at $t = 0$. Solid line, $u(x, 0)$ and dashed line, $v(x, 0)$.

with $\alpha = \sqrt{1 - \lambda^2 b}$, and

$$\left. \begin{aligned} V_{11} &= \lambda^{-2} \alpha + \frac{\alpha}{2} (|u|^2 - |u_x|^2) + \frac{1}{2} (uu_x^* - u^* u_x), \\ V_{12} &= -\lambda^{-1} (u - \alpha u_x) - \frac{1}{2} \lambda m (|u|^2 - |u_x|^2) \\ \text{and} \quad V_{21} &= \lambda^{-1} (v + \alpha v_x) + \frac{1}{2} \lambda n (|u|^2 - |u_x|^2). \end{aligned} \right\} \quad (5.3)$$

Next, we show that the dispersionless version of equation (1.6) with $b = 0$ admits the complex-valued N -peakon solution, while the dispersion version of equation (1.6) with $b \neq 0$ allows the complex-valued kink wave solution.

(a) Complex-valued peakon solution of (1.6) with $b = 0$

Let us assume that a complex-valued N -peakon solution of (1.6) with $b = 0$ has the following form

$$u = \sum_{j=1}^N (p_j(t) + \sqrt{-1} r_j(t)) e^{-|x - q_j(t)|}, \quad (5.4)$$

where $p_j(t)$, $r_j(t)$ and $q_j(t)$ are real-valued functions. Substituting (5.4) into (1.6) with $b = 0$ and integrating through real-valued test functions, and separating the real part and imaginary part, we finally obtain that $p_j(t)$, $r_j(t)$ and $q_j(t)$ evolve according to the dynamical system

$$\left. \begin{aligned} p_{j,t} &= r_j \sum_{l,k=1}^N p_l r_k (\text{sgn}(q_j - q_k) - \text{sgn}(q_j - q_l)) e^{-|q_j - q_k| - |q_j - q_l|}, \\ r_{j,t} &= p_j \sum_{l,k=1}^N p_l r_k (\text{sgn}(q_j - q_l) - \text{sgn}(q_j - q_k)) e^{-|q_j - q_k| - |q_j - q_l|} \\ \text{and} \quad q_{j,t} &= \frac{1}{6} (p_j^2 + r_j^2) - \frac{1}{2} \sum_{l,k=1}^N (p_l p_k + r_l r_k) (1 - \text{sgn}(q_j - q_l) \text{sgn}(q_j - q_k)) e^{-|q_j - q_l| - |q_j - q_k|}. \end{aligned} \right\} \quad (5.5)$$

For $N = 1$, (5.5) becomes

$$p_{1,t} = 0, \quad r_{1,t} = 0 \quad \text{and} \quad q_{1,t} = -\frac{1}{3}(p_1^2 + r_1^2), \quad (5.6)$$

which gives

$$p_1 = c_1, \quad r_1 = c_2 \quad \text{and} \quad q_1 = -\frac{1}{3}(c_1^2 + c_2^2)t, \quad (5.7)$$

where c_1 and c_2 are real-valued integration constants. Thus, we arrive at the single-peakon solution

$$u = (c_1 + \sqrt{-1}c_2)e^{-|x+(c_1^2+c_2^2)/3t|} = ce^{-|x+(1/3)|c|^2t|}, \quad (5.8)$$

where $c = c_1 + \sqrt{-1}c_2$ and $|c|$ is the modulus of c .

For $N = 2$, we may solve (5.5) as

$$\left. \begin{aligned} q_1(t) &= -\frac{1}{3}A_1^2t + \Gamma_1(t), \\ q_2(t) &= -\frac{1}{3}A_2^2t + \Gamma_1(t), \\ p_1(t) &= A_1 \sin(\Gamma_2(t) + A_3), \\ p_2(t) &= A_2 \sin(\Gamma_2(t) + A_4), \\ r_1(t) &= A_1 \cos(\Gamma_2(t) + A_3), \\ r_2(t) &= A_2 \cos(\Gamma_2(t) + A_4), \end{aligned} \right\} \quad (5.9)$$

and

where

$$\left. \begin{aligned} \Gamma_1(t) &= \frac{3A_1A_2 \cos(A_3 - A_4)}{|A_1^2 - A_2^2|} \operatorname{sgn}(t)(e^{-(1/3)(A_1^2 - A_2^2)t} - 1) \\ \Gamma_2(t) &= \frac{3A_1A_2 \sin(A_3 - A_4)}{A_1^2 - A_2^2} e^{-(1/3)(A_1^2 - A_2^2)t}, \end{aligned} \right\} \quad (5.10)$$

and

A_1, \dots, A_4 are real-valued integration constants. Hence, the two-peakon solution reads

$$\begin{aligned} u &= A_1 \sqrt{-1} e^{-\sqrt{-1}(\Gamma_2(t) + A_3)} e^{-|x+(1/3)A_1^2t - \Gamma_1(t)|} \\ &\quad + A_2 \sqrt{-1} e^{-\sqrt{-1}(\Gamma_2(t) + A_4)} e^{-|x+(1/3)A_2^2t - \Gamma_1(t)|}, \end{aligned} \quad (5.11)$$

where the Euler formula $e^{\sqrt{-1}x} = \cos x + \sqrt{-1} \sin x$ is employed.

(b) Complex-valued kink solution of (1.6) with $b \neq 0$

We suppose that a complex-valued kink wave solution of equation (1.6) with $b \neq 0$ has the form

$$u = (C_1 + \sqrt{-1}C_2) \operatorname{sgn}(x - ct)(e^{-|x-ct|} - 1), \quad (5.12)$$

where the real constant c is the wave speed, and C_1 and C_2 are two real constants to be determined. Substituting (5.12) into equation (1.6) with $b \neq 0$ and integrating through real-valued test functions, and separating its real part and imaginary part, we finally arrive at

$$c = -\frac{1}{2}b, \quad C_1^2 + C_2^2 = -b. \quad (5.13)$$

Formula (5.13) implies that the wave speed is exactly $-\frac{1}{2}b$, where $b < 0$ is the coefficient of the linear dispersive term. Thus, the complex-valued weak kink solution becomes

$$u = C \operatorname{sgn}(x + \frac{1}{2}bt)(e^{-|x+(1/2)bt|} - 1), \quad (5.14)$$

where $C = C_1 + \sqrt{-1}C_2$ and $|C|^2 = -b$, $b < 0$. We remark that in (5.14) only the constant C is complex, the variables x and t are real-valued variables.

6. Conclusion and discussions

In our paper, we propose a new integrable two-component system with cubic nonlinearity and linear dispersion. The system is shown to possess Lax pair, bi-Hamiltonian structure and infinitely many conservation laws. Geometrically, this system describes a non-trivial one-parameter family of pseudo-spherical surfaces. In the dispersionless case, we show the system admits N -peakon solution and explicitly solve the system for the single-peakon and the two-peakon dynamical system. Moreover, we propose a scalar integrable complex cubic nonlinear equation and find the complex-valued N -peakon solution and kink wave solution to the integrable complex equation.

In [39], the authors introduced an integrable two-component extension of the dispersionless version of cubic nonlinear equation (1.2) (or the FORQ equation called in some literature)

$$\left. \begin{aligned} m_t &= -[m(uv - u_x v_x + u_x v - uv_x)]_x, \\ n_t &= -[n(uv - u_x v_x + u_x v - uv_x)]_x \end{aligned} \right\} \quad (6.1)$$

and

$$m = u - u_{xx}, \quad n = v - v_{xx}.$$

We remark that the dispersionless version of our two-component system (1.5) with $b=0$ is not equivalent to system (6.1). System (1.5) in our paper is able to be reduced to the CH equation, but system (6.1) is not, which apparently implies that these two equations are not equivalent. In fact, both system (1.5) with $b=0$ and system (6.1) belong to a more general negative flow in a hierarchy. For the details of this topic, one may see our very recent paper [44].

It is an interesting task to study whether there are other new features in the structure of solutions for our two-component system, and particularly for our complex equation with a linear dispersive term. Also other topics, such as smooth soliton solutions [45], cuspons, peakon stability and algebra-geometric solutions, remain open for our system (1.5) and (1.6).

Data accessibility. There are no primary data in this article.

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References

1. Camassa R, Holm DD. 1993 An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* **71**, 1661–1664. (doi:10.1103/PhysRevLett.71.1661)
2. Fuchssteiner B, Fokas AS. 1981 Symplectic structures their Bäcklund transformation and hereditary symmetries. *Phys. D* **4**, 47–66. (doi:10.1016/0167-2789(81)90004-X)
3. Camassa R, Holm DD, Hyman JM. 1994 A new integrable shallow water equation. *Adv. Appl. Mech.* **31**, 1–33. (doi:10.1016/S0065-2156(08)70254-0)
4. Olver PJ, Rosenau P. 1996 Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support. *Phys. Rev. E* **53**, 1900–1906. (doi:10.1103/PhysRevE.53.1900)
5. Fokas AS. 1995 On a class of physically important integrable equations. *Phys. D* **87**, 145–150. (doi:10.1016/0167-2789(95)00133-O)
6. Fuchssteiner B. 1996 Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation. *Phys. D* **95**, 229–243. (doi:10.1016/0167-2789(96)00048-6)
7. Fokas AS, Liu QM. 1996 Asymptotic integrability of water waves. *Phys. Rev. Lett.* **77**, 2347–2351. (doi:10.1103/PhysRevLett.77.2347)

8. Beals R, Sattinger D, Szmigielski J. 1999 Multipeakons and a theorem of Stieltjes. *Inverse Probl.* **15**, L1–L4. (doi:10.1088/0266-5611/15/1/001)
9. Constantin A. 1998 On the inverse spectral problem for the Camassa-Holm equation. *J. Funct. Anal.* **155**, 352–363. (doi:10.1006/jfan.1997.3231)
10. Dullin HR, Gottwald GA, Holm DD. 2001 An integrable shallow water equation with linear and nonlinear dispersion. *Phys. Rev. Lett.* **87**, 194501. (doi:10.1103/PhysRevLett.87.194501)
11. Gesztesy F, Holden H. 2003 Algebro-geometric solutions of the Camassa-Holm hierarchy. *Rev. Mat. Iberoamericana* **19**, 73–142. (doi:10.4171/RMI/339)
12. Qiao ZJ. 2003 Camassa-Holm hierarchy related N-dimensional integrable systems and algebro-geometric solution on a symplectic submanifold. *Commun. Math. Phys.* **239**, 309–341. (doi:10.1007/s00220-003-0880-y)
13. Lorenzoni P, Pedroni M. 2004 On the bi-Hamiltonian structures of the Camassa-Holm and Harry Dym equations. *Int. Math. Res. Not.* **75**, 4019–4029. (doi:10.1155/S1073792804142554)
14. Constantin A, Gerdjikov VS, Ivanov RI. 2006 Inverse scattering transform for the Camassa-Holm equation. *Inverse Probl.* **22**, 2197–2207. (doi:10.1088/0266-5611/22/6/017)
15. Constantin A, Strauss WA. 2000 Stability of peakons. *Comm. Pure Appl. Math.* **53**, 603–610. (doi:10.1002/(SICI)1097-0312(200005)53:5<603::AID-CPA3>3.0.CO;2-L)
16. Constantin A, Strauss WA. 2002 Stability of the Camassa-Holm solitons. *J. Nonlinear Sci.* **12**, 415–422. (doi:10.1007/s00332-002-0517-x)
17. Beals R, Sattinger D, Szmigielski J. 2000 Multipeakons and the classical moment problem. *Adv. Math.* **154**, 229–257. (doi:10.1006/aima.1999.1883)
18. Alber MS, Camassa R, Fedorov YN, Holm DD, Marsden JE. 2001 The complex geometry of weak piecewise smooth solutions of integrable nonlinear PDE's of shallow water and dym type. *Commun. Math. Phys.* **221**, 197–227. (doi:10.1007/PL00005573)
19. Johnson RS. 2003 On solutions of the Camassa-Holm equation. *Proc. R. Soc. Lond. A* **459**, 1687–1708. (doi:10.1098/rspa.2002.1078)
20. Lakshmanan M. 2007 Integrable nonlinear wave equations and possible connections to tsunami dynamics. In *Tsunami and nonlinear waves*, (ed. A Kundu), pp. 31–49. Berlin, Germany: Springer.
21. Degasperis A, Procesi M. 1999 *Asymptotic integrability symmetry and perturbation theory* (eds A Degasperis, G Gaeta), pp. 23–37. Singapore: World Scientific.
22. Degasperis A, Holm DD, Hone ANW. 2002 A new integrable equation with peakon solitons. *Theor. Math. Phys.* **133**, 1463–1474. (doi:10.1023/A:1021186408422)
23. Hone ANW, Wang JP. 2003 Prolongation algebras and Hamiltonian operators for peakon equations. *Inverse Probl.* **19**, 129–145. (doi:10.1088/0266-5611/19/1/307)
24. Lundmark H, Szmigielski J. 2003 Multi-peakon solutions of the Degasperis-Procesi equation. *Inverse Probl.* **19**, 1241–1245. (doi:10.1088/0266-5611/19/6/001)
25. Lundmark H. 2007 Formation and dynamics of shock waves in the Degasperis-Procesi equation. *J. Nonlinear Sci.* **17**, 169–198. (doi:10.1007/s00332-006-0803-3)
26. Qiao ZJ. 2006 A new integrable equation with cuspons and W/M-shape-peaks solitons. *J. Math. Phys.* **47**, 112701. (doi:10.1063/1.2365758)
27. Qiao ZJ. 2007 New integrable hierarchy its parametric solutions, cuspons, one-peak solutions, and M/W-shape peak solitons. *J. Math. Phys.* **48**, 082701. (doi:10.1063/1.2759830)
28. Qiao ZJ, Li XQ. 2011 An integrable equation with nonsmooth solitons. *Theor. Math. Phys.* **167**, 584–589. (doi:10.1007/s11232-011-0044-8)
29. Hone ANW, Wang JP. 2008 Integrable peakon equations with cubic nonlinearity. *J. Phys. A* **41**, 372002. (doi:10.1088/1751-8113/41/37/372002)
30. Novikov V. 2009 Generalizations of the Camassa-Holm equation. *J. Phys. A* **42**, 342002. (doi:10.1088/1751-8113/42/34/342002)
31. Gui GL, Liu Y, Olver PJ, Qu CZ 2013 Wave-breaking and peakons for a modified Camassa-Holm equation. *Commun. Math. Phys.* **319**, 731–759. (doi:10.1007/s00220-012-1566-0)
32. Qiao ZJ, Xia BQ, Li JB. Integrable system with peakon, weak kink, and kink-peakon interactional solutions. (<http://arxiv.org/abs/1205.2028v2>)
33. Chen M, Liu SQ, Zhang Y. 2006 A two-component generalization of the Camassa-Holm equation and its solutions. *Lett. Math. Phys.* **75**, 1–15. (doi:10.1007/s11005-005-0041-7)
34. Liu SQ, Zhang Y. 2005 Deformations of semisimple bihamiltonian structures of hydrodynamic type. *J. Geom. Phys.* **54**, 427–453. (doi:10.1016/j.geomphys.2004.11.003)
35. Falqui G. 2006 On a Camassa-Holm type equation with two dependent variables. *J. Phys. A* **39**, 327–342. (doi:10.1088/0305-4470/39/2/004)

36. Holm DD, Ivanov RI. 2010 Multi-component generalizations of the CH equation: geometrical aspects, peakons and numerical examples. *J. Phys A* **43**, 492001. (doi:10.1088/1751-8113/43/49/492001)
37. Cotter CJ, Holm DD, Ivanov RI, Percival JR. 2011 Waltzing peakons and compacton pairs in a cross-coupled Camassa-Holm equation. *J. Phys A* **44**, 265205. (doi:10.1088/1751-8113/44/26/265205)
38. Geng XG, Xue B. 2009 An extension of integrable peakon equations with cubic nonlinearity. *Nonlinearity* **22**, 1847–1856. (doi:10.1088/0951-7715/22/8/004)
39. Song JF, Qu CZ, Qiao ZJ. 2011 A new integrable two-component system with cubic nonlinearity. *J. Math. Phys.* **52**, 013503. (doi:10.1063/1.3530865)
40. Chern SS, Tenenblat K. 1986 Pseudo-spherical surfaces and evolution equations. *Stud. Appl. Math.* **74**, 55–83.
41. Reyes EG. 1998 Pseudo-spherical surfaces and integrability of evolution equations. *J. Diff. Equ.* **147**, 195–230. (doi:10.1006/jdeq.1998.3430)
42. Reyes EG. 2002 Geometric integrability of the Camassa-Holm equation. *Lett. Math. Phys.* **59**, 117–131. (doi:10.1023/A:1014933316169)
43. Sasaki R. 1979 Soliton equations and pseudospherical surfaces. *Nucl. Phys.* **13**, 343–357. (doi:10.1016/0550-3213(79)90517-0)
44. Xia BQ, Qiao ZJ, Zhou RG. A synthetical integrable two-component model with peakon solutions. (<http://arxiv.org/abs/1301.3216>)
45. Matsuno Y. 2013 Bäcklund transformation and smooth multisoliton solutions for a modified Camassa-Holm equation with cubic nonlinearity. *J. Math. Phys.* **54**, 051504. (doi:10.1063/1.4807417)