A NEW POLYNOMIAL INVOLUTIVE SYSTEM AND A CLASSICAL COMPLETELY INTEGRABLE SYSTEM*

MU WEIHUA (牟卫华) ZHANG BAOCAI (张保才)

(*Shijiazhuang Railway Institute, Shijiazhuang 050043, China*)

QIAO ZHIJUN (乔志军)

(*Liaoning University, Shenyang 110036, China*)

Abstract

Under the constrained condition induced by the eigenfunctions and the potentials, the Lax systems of nonlinear evolution equations in relation to a matrix eigenvalue problem are nonlinearized to be a completely integrable system $(R^{2N}, dp \wedge dq, H)$, while the time part of it is nonlinearized to be its N-involutive system $\{F_m\}$. The involutive solution of the compatible system (F_0), (F_m) is transformed into the solution of the m-th nonlinear evolution equation.

1. Introduction

The Liouville-Arnold theory[1] of the finite-dimensional completely integrable system is beautiful, which includes a series of examples celebrated for ingenuity and skill in the history of analytic mechanics, e.g., the Jacobi problem of geodesic flow on the ellipsoid, C. Neumann problem of oscillators constrained on the sphere, Kovalevski's top, etc.(see [2]). The number of already known finite-dimensional completely integrable systems is small, which depends on the existence of N-involutive system of Hamiltonian functions.

Flaschka[3] pointed out an important principle to produce finite-dimensional integrable systems by constraining the infinite-dimensional integrable system on the finite-dimensional invariant manifold. However, it is not easy to realize the elaborately concrete framework according to this principle. Recently, Cao[4] developed a systematic approach to get a finite-dimensional integrable system by the nonlinearization of Lax pair of solution equations under certain constraints between the potentials and the eigenfunctions.

In the present note, let's consider the following eigenvalue problem

$$y_x = \begin{pmatrix} -\lambda + v & u + v \\ u - v & \lambda - v \end{pmatrix} y, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{1.1}$$

In [5], the gauge transformation between (1.1) and AKNS eigenvalue problem is given, and the nonlinear evolution equations in relation to (1.1) are presented. In this paper, firstly,

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we are going to give the Lax pairs for evolution equation hierarchy in relation to (1.1), and then, nonlinearize the Lax systems by introducing certain constraint conditions between the potentials and the eigenfunctions and constructing a new polynomial involutive system, so as to prove that the nonlinearized form of (1.1) is a complete integrable Hamiltonian system in the Liouville sense.

2. The Lax Pair for Evolution Equation Hierarchy in Relation to (1.1)

First, we know that the eigenvalue problem (1.1) can be rewritten into

$$Ly = \lambda y, \qquad \text{where } L = \begin{pmatrix} v-\partial & u+v \\ v-u & v+\partial \end{pmatrix}, \quad \partial \triangleq \frac{\partial}{\partial x}. \tag{2.1}$$

Define the tangent mapping of differential operator L:

$$L_*\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \triangleq \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(u+\varepsilon\xi_1,\ v+\varepsilon\xi_2) = \begin{pmatrix} \xi_2 & \xi_1+\xi_2 \\ \xi_2-\xi_1 & \xi_2 \end{pmatrix}, \tag{2.2}$$

and then L_* is a one-to-one mapping, i.e., $L_*\binom{\xi_1}{\xi_2} = 0$ implies $\xi_1 = 0,\ \xi_2 = 0$.

Lemma 2.1. Let $G(x) = \left(G^{(1)}(x), G^{(2)}(x)\right)^T$ be an arbitrary smooth function. Then

$$[V, L] = L_*(KG) - L_*(JG)L, \tag{2.3}$$

where

$$V = \begin{pmatrix} \frac{1}{2}G_x^{(1)} + G^{(2)}\partial & \frac{1}{2}\left(G_x^{(1)} - G_x^{(2)}\right) \\ -\frac{1}{2}\left(G_x^{(1)} + G_x^{(2)}\right) & -\frac{1}{2}G_x^{(1)} + G_x^{(2)}\partial \end{pmatrix},$$

$$J = \begin{pmatrix} -\partial & 0 \\ 0 & \partial \end{pmatrix}, \qquad K = \frac{1}{2}\begin{pmatrix} 0 & -\partial^2 + 2\partial u \\ \partial^2 + 2u\partial & 4v\partial + 2v_x \end{pmatrix}.$$

Define the Lenard sequence recursively:

$$G_{-1} = \begin{pmatrix}1\\1\end{pmatrix}, \qquad G_0 = \begin{pmatrix}-u\\ v\end{pmatrix}, \qquad KG_{j-1} = JG_j, \qquad j = 0, 1, 2, \cdots, \tag{2.4}$$

G_j is a polynomial of u, v and their derivatives[5], and is unique if its constant term is required to be zero. $X_j \triangleq JG_j$ is a vector field.

Theorem 2.2. Let G_j be a Lenard sequence. Then

$$[W_m, L] = L_*(X_m), \tag{2.5}$$

where

$$W_m = \sum_{j=0}^{m} V_{j-1}L^{m-j}, \qquad V_j = \begin{pmatrix} \frac{1}{2}G_{j,x}^{(1)} + G_j^{(2)}\partial & \frac{1}{2}\left(G_{j,x}^{(1)} - G_{j,x}^{(2)}\right) \\ -\frac{1}{2}\left(G_{j,x}^{(1)} + G_{j,x}^{(2)}\right) & -\frac{1}{2}G_{j,x}^{(1)} + G_j^{(2)}\partial \end{pmatrix}.$$

Proof. Since $[WL^s, L] = [W, L]L^s$, by Lemma 2.1 we have

$$\begin{aligned}[W_m, L] &= \sum_{j=0}^{m}[V_{j-1}, L]L^{m-j} = L_*\left[\sum_{j=0}^{m}(JG_j)L^{m-j} - (JG_{j-1})L^{m-j+1}\right] \\ &= L_*(JG_m) = L_*(X_m).\end{aligned}$$

Corollary 2.3. $U_t = X_m$ if and only if $L_{t_m} = [W_m, L]$, where $U = (u,v)^T$.

Proof.

$$L_{t_m} - [W_m, L] = L_*(U_{t_m}) - L_*(X_m) = L_*(U_{t_m} - X_m).$$

Corollary 2.4. The m-th evolution equation in relation to (1.1), $U_{t_m} = X_m$, is the consistency condition of the following Lax pair

$$Ly = \lambda y, \qquad y_{t_m} = W_m y. \tag{2.6}$$

Remark 2.5. The first few results of calculation:

$$G_0 = \begin{pmatrix} -u \\ v \end{pmatrix}, \qquad G_1 = \frac{1}{2}\begin{pmatrix} v_x - 2uv \\ -u_x + 3v^2 - u^2 \end{pmatrix}, \cdots;$$

$$X_0 = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad X_1 = \frac{1}{2}\begin{pmatrix} -v_{xx} + 2(uv)_x \\ -u_{xx} + (3v^2 - u^2)_x \end{pmatrix}, \cdots.$$

3. The Nonlinearized Form of Lax System and a Classical Integrable System

Let's consider the eigenvalue problem (1.1), and introduce the constraint conditions between the potentials and the eigenfunctions

$$\begin{cases} u = \langle y_1, y_1 \rangle - \langle y_2, y_2 \rangle, \\ v = \langle y_1 + y_2, \ y_1 + y_2 \rangle, \end{cases} \tag{3.1}$$

where $y_1 = (y_{11}, y_{12}, \cdots, y_{1N})^T$, $y_2 = (y_{21}, y_{22}, \cdots, y_{2N})^T$, $\langle \cdot, \cdot \rangle$ is the standard inner-product in $\boldsymbol{R}^N$, and $(y_{1j}, y_{2j})^T$ satisfies

$$\begin{pmatrix} y_{1j} \\ y_{2j} \end{pmatrix}_x = \begin{pmatrix} -\alpha_j + v & u + v \\ u - v & \alpha_j - v \end{pmatrix} \begin{pmatrix} y_{1j} \\ y_{2j} \end{pmatrix}, \qquad j = 1, 2, \cdots, N, \tag{3.2}$$

α_j being a real constant.

Introduce the canonical variables $q = (q_1, \cdots, q_n)^T = y_1$, $p = (p_1, \cdots, p_N)^T = y_2$, and $A = \mathrm{diag}(\alpha_1, \cdots, \alpha_n)$; and condense (3.2) into

$$\begin{cases} q_x = -Aq + vq + up + vp, \\ p_x = uq - vq + Aq - vp. \end{cases} \tag{3.3}$$

Then (3.3) can be written in the Hamiltonian canonical form under the constraint condition (3.1):

$$q_x = \frac{\partial H}{\partial p}, \qquad p_x = -\frac{\partial H}{\partial q} \tag{3.4}$$

with the Hamiltonian function

$$H = -\langle Aq, p \rangle + \big(\langle q, q \rangle + \langle q, p \rangle\big)\big(\langle p, p \rangle + \langle q, p \rangle\big). \tag{3.5}$$

Lemma 3.1. Let (q, p) satisfy (3.3), with $u = \langle q, q \rangle - \langle p, p \rangle$, and $v = \langle q + p, \ q + p \rangle$. Then there exist constants $c_1, c_2, \cdots, c_{m+1}$, such that

$$\begin{pmatrix} \langle A^m p, p \rangle - \langle A^m q, q \rangle \\ \langle A^m (q + p), q + p \rangle \end{pmatrix} = G_m + \sum_{j=1}^{m+1} c_j G_{m-j}. \tag{3.6}$$

Proof. First, we notice that

$$\begin{cases} q_{jx} = (-\alpha_j + v)q_j + (u+v)p_j, \\ p_{jx} = (u-v)q_j + (\alpha_j - v)p_j, \end{cases}$$

implies

$$K\begin{pmatrix} p_j^2 - q_j^2 \\ (q_j+p_j)^2 \end{pmatrix} = \alpha_j J\begin{pmatrix} p_j^2 - q_j^2 \\ (q_j+p_j)^2 \end{pmatrix}, \qquad j = 1, 2, \cdots, N;$$

thus

$$K\begin{pmatrix} \langle A^j p, p\rangle - \langle A^j q, q\rangle \\ \langle A^j(q+p),\, q+p\rangle \end{pmatrix} = J\begin{pmatrix} \langle A^{j+1} p, p\rangle - \langle A^{j+1} q, q\rangle \\ \langle A^{j+1}(q+p),\, q+p\rangle \end{pmatrix}.$$

Under the action of $J^{-1}K$, $G_j \longmapsto G_{j+1}$.

$$\begin{pmatrix} \langle A^j p, p\rangle - \langle A^j q, q\rangle \\ \langle A^j(q+p),\, q+p\rangle \end{pmatrix} \longmapsto \begin{pmatrix} \langle A^{j+1} p, p\rangle - \langle A^{j+1} q, q\rangle \\ \langle A^{j+1}(q+p),\, q+p\rangle \end{pmatrix}$$

with an extra constant G_{-1}. (3.6) is obtained by an action of m times upon

$$G_0 = \begin{pmatrix} \langle p, p\rangle - \langle q, q\rangle \\ \langle q+p,\, q+p\rangle \end{pmatrix}.$$

A natural problem is whether the Hamiltonian system (3.4), (3.5) can be completely integrable in Liouville sense. Next, we shall construct a polynomial involutive system, and then give a complete solution of this problem.

From (2.6), we have

$$\begin{cases} Ly_j = \alpha_j y_j, \\ y_{j,t_m} = W_m y_j, \end{cases} \qquad j = 1, 2, \cdots, N;$$

then

$$\begin{aligned} y_{1k,t_m} = &\sum_{j=0}^{m}\Big[\frac{1}{2}G^{(1)}_{j-1,x}\alpha_k^{m-j}y_{1k} \\ &+\frac{1}{2}\left(G^{(1)}_{j-1,x} - G^{(2)}_{j-1,x}\right)\alpha_k^{m-j}y_{2k} + G^{(2)}_{j-1}\alpha_k^{m-j}y_{1k,x}\Big], \end{aligned} \tag{3.7}$$

$$\begin{aligned} y_{2k,t_m} = &\sum_{j=0}^{m}\Big[-\frac{1}{2}\left(G^{(1)}_{j-1,x} + G^{(2)}_{j-1,x}\right)\alpha_k^{m-j}y_{1k} \\ &-\frac{1}{2}G^{(1)}_{j-1,x}\alpha_k^{m-j}y_{2k} + G^{(2)}_{j-1}\alpha_k^{m-j}y_{2k,x}\Big]. \end{aligned} \tag{3.7'}$$

Introduce the canonical variables

$$q = (q_1, q_2, \cdots, q_n)^T = y_1, \qquad p = (p_1, p_2, \cdots, p_n)^T = y_2;$$

then (3.7) can be condensed into

$$\begin{cases} q_{t_m} = \displaystyle\sum_{j=0}^{m}\Big[\frac{1}{2}G^{(1)}_{j-1,x}A^{m-j}q \\ \qquad\qquad +\frac{1}{2}(G^{(1)}_{j-1,x} - G^{(2)}_{j-1,x})A^{m-j}p + G^{(2)}_{j-1}A^{m-j}q_x\Big], \\ p_{t_m} = \displaystyle\sum_{j=0}^{n}\Big[-\frac{1}{2}(G^{(1)}_{j-1,x} + G^{(2)}_{j-1,x})A^{m-j}q \\ \qquad\qquad -\frac{1}{2}G^{(1)}_{j-1,x}A^{m-j}p + G^{(2)}_{j-1}A^{m-j}p_x\Big]. \end{cases} \tag{3.8}$$

In virtue of Lemma 3.1 with $c_j = 0$ $(j = 1, \cdots, m+1)$, and constraint conditions (3.1), through direct calculation, (3.8) can be written in the following Hamiltonian canonical form

$$q_{t_m} = \frac{\partial F_m}{\partial p}, \qquad p_{t_m} = -\frac{\partial F_m}{\partial q}, \tag{3.9}$$

with the Hamiltonian functions

$$\begin{aligned} F_m = &- \langle A^{m+1}q, p\rangle + \langle q, q\rangle \langle A^m p, p\rangle + \langle q, p\rangle \langle A^m p, p\rangle + \langle q, p\rangle \langle A^m q, q\rangle \\ &+ \langle q, p\rangle \langle A^m q, p\rangle + \sum_{j=1}^{m} \begin{vmatrix} \langle A^j q, q\rangle, & \langle A^j q, p\rangle \\ \langle A^{m-j} p, q\rangle, & \langle A^{m-j} p, p\rangle \end{vmatrix}, \qquad m = 0, 1, 2, \cdots. \end{aligned} \tag{3.10}$$

Let's consider the integrability of systems (3.4) and (3.5). An important fact is given in the following.

The Poisson bracket of the two functions in the symplectic space $(\mathbf{R}^{2N}, dp \wedge dq)$ is defined as:

$$(F, G) = \sum_{j=1}^{N} \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}.$$

F and G are called in involution, if $(F, G) = 0$. Define

$$\Gamma_k = \sum_{\substack{j=1 \\ j \neq k}}^{N} \frac{B_{kj}^2}{\alpha_k - \alpha_j}, \qquad \text{where } B_{kj} = p_k q_j - q_k p_j.$$

Lemma 3.2. $(\Gamma_k, \Gamma_l) = 0$, $(\langle q, q\rangle, \Gamma_l) = 0$, $(\langle p, p\rangle, \Gamma_l) = 0$, $(\Gamma_k, q_l^2) = \frac{4B_{kl}}{\alpha_k - \alpha_l} q_k q_l$, $(\Gamma_k, p_l^2) = \frac{4B_{kl}}{\alpha_k - \alpha_l} p_k p_l$.

Proof. See [2].

Lemma 3.3. $E_1, E_2, \cdots, E_N$ defined as follows constitute an N-involutive system:

$$E_k = -q_k p_k + \langle q, p\rangle \alpha_k^{-1} p_k^2 + \langle q, p\rangle \alpha_k^{-1} q_k^2 + 2\langle q, p\rangle \alpha_k^{-1} q_k p_k + \Gamma_k.$$

Proof. $(E_k, E_l) = 0$ is evidently valid for $k = l$. Suppose that $k \neq l$. Noticing $(q_k, q_l) = 0$, $(p_k, p_l) = 0$, $(q_k, p_l) = \delta_{kl} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$ and Lemma 3.1, in virtue of Leibnitz rule, we have

$$\begin{aligned} (\lambda_k E_k, \lambda_l E_l) = &4q_k^2 p_l^2 \langle q, p\rangle - 4q_l^2 p_k^2 \langle q, p\rangle + 4\langle q, p\rangle q_k p_k p_l^2 - 4q_l p_l p_k^2 \langle q, p\rangle + 4q_l p_l q_k^2 \langle q, p\rangle \\ &- 4q_k p_k q_l^2 \langle q, p\rangle + 4B_{kl} p_k p_l \langle q, p\rangle + 4B_{kl} q_k q_l \langle q, p\rangle + 4B_{kl}(q_l p_k + q_k p_l) = 0. \end{aligned}$$

So $(E_k, E_l) = 0$.

Consider a bilinear function $Q_z(\xi, \eta)$ on $\mathbf{R}^N$ and its partial fraction expansion and Laurant expansion:

$$Q(\xi, \eta) = \langle (z - A)^{-1}\xi, \ \eta\rangle = \sum_{k=1}^{N} (z - \alpha_k)^{-1} \xi_k \eta_k = \sum_{m=0}^{\infty} z^{-m-1} \langle A^m \xi, \ \eta\rangle.$$

The generating function of Γ_k is (see [2])

$$\begin{vmatrix} Q_z(q,q) & Q_z(q,p) \\ Q_z(p,q) & Q_z(p,p) \end{vmatrix} = \sum_{k=1}^{N} \frac{\Gamma_k}{z-\alpha_k}.$$

Hence, on the one hand, we have

$$\sum_{k=1}^{N} \frac{E_k}{z-\alpha_k} = \sum_{m=0}^{\infty} \frac{1}{z^{m+1}} \sum_{k=1}^{N} \alpha_k^m E_k;$$

and on the other, we have

$$\sum_{k=1}^{N} \frac{E_k}{z-\alpha_k} = \sum_{m=0}^{\infty} \frac{1}{z^{m+1}} \Bigg[-\langle A^m q, p\rangle + \langle q, p\rangle\langle A^{m-1}p, p\rangle + \langle q, p\rangle\langle A^{m-1}q, q\rangle$$
$$+ 2\langle q, p\rangle\langle A^{m-1}q, p\rangle + \sum_{j=1}^{m} \begin{vmatrix} \langle A^{j-1}q, q\rangle & \langle A^{j-1}q, p\rangle \\ \langle A^{m-j}p, q\rangle & \langle A^{m-j}p, p\rangle \end{vmatrix} \Bigg].$$

So we obtain

Theorem 3.4. The functions defined as follows are in involution in pairs, $(F_k, F_l) = 0$,

$$F_0 = -\langle Aq, p\rangle + \big(\langle q,q\rangle + \langle q,p\rangle\big)\big(\langle p,p\rangle + \langle q,p\rangle\big), \tag{3.11}$$

$$F_m = -\langle A^{m+1}q, p\rangle + \langle q,q\rangle\langle A^m p, p\rangle + \langle q,p\rangle\langle A^m p, p\rangle + \langle q,p\rangle\langle A^m q, p\rangle$$
$$+ \langle q,p\rangle\langle A^m q, q\rangle + \sum_{j=1}^{m} \begin{vmatrix} \langle A^j q, q\rangle & \langle A^j q, p\rangle \\ \langle A^{m-j}p, q\rangle & \langle A^{m-j}p, p\rangle \end{vmatrix}. \tag{3.12}$$

Moreover, $F_{m-1} = \sum_{k=1}^{N} \alpha_k^m E_k,\ m = 1, 2, \cdots$.

Proof.

$$F_{m-1} = -\langle A^m q, p\rangle + \langle q, p\rangle\langle A^{m-1}p, p\rangle + \langle q, p\rangle\langle A^{m-1}q, q\rangle + 2\langle q, p\rangle\langle A^{m-1}q, p\rangle$$
$$+ \sum_{j=1}^{m} \begin{vmatrix} \langle A^{j-1}q, q\rangle & \langle A^{j-1}p, q\rangle \\ \langle A^{m-j}p, q\rangle & \langle A^{m-j}p, p\rangle \end{vmatrix};$$

thus $F_{m-1} = \sum_{k=1}^{N} \alpha_k^m E_k$, and the involutivity of $\{E_k\}$ implies the involutivity of $\{F_m\}$.

Note that the Hamiltonian function H of (3.5) is F_0; therefore, $\{F_m\}_{m=0}^{\infty}$ is a series of involutive integrals of motion of (3.5). On the other hand, because $\alpha_i \neq \alpha_j$ when $i \neq j$, the Vandermoude determinant of $\alpha_1, \alpha_2, \cdots, \alpha_N$ is not zero. Thus it's not difficult to see that there is a region $\Omega \subseteq \mathbb{R}^{2N}$ on which the N 1-forms $dF_0, dF_1, \cdots, dF_{N-1}$ are everywhere linearly independent. Based upon these facts we obtain the following main theorem.

Theorem 3.5. The finite-dimensional Hamiltonian system (3.4) possesses a series of involutive integrals of motion $\{F_m\}_{m=0}^{\infty}$ of which $F_0, F_1, \cdots, F_{N-1}$ are linearly independent (strictly speaking, on some region $\Omega \subseteq \mathbb{R}^{2N}$), thus being completely integrable.

4. The Involutive Representation of Solutions of Evolution Equation Hierarchy in Relation to (1.1)

Consider the canonical system of the F_m-flow:

$$(F_m): \qquad \frac{\partial}{\partial t_m}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \dfrac{\partial F_m}{\partial p} \\ -\dfrac{\partial F_m}{\partial q} \end{pmatrix} = I\nabla F_m, \qquad I = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \tag{4.1}$$

where I_N is an $N \times N$ unit matrix. Denote the solution operator of its initial value problem by $g_m^{t_m}$; then its solution can be expressed as

$$\begin{pmatrix} q(t_m) \\ p(t_m) \end{pmatrix} = g_m^{t_m}\begin{pmatrix} q(0) \\ p(0) \end{pmatrix}.$$

Since any two F_x, F_l are in involution, $(F_k, F_l) = 0$, we have [1].

Proposition 4.1. Any two canonical systems (F_k), (F_l) are compatible; the Hamiltonian phase-flows $g_k^{t_l}$, $g_l^{t_l}$ commute.

Denote the flow variables of (F_0) and (F_m) by $x = t_0$, and t_m respectively. Define

$$\begin{pmatrix} q(x,t_m) \\ p(x,t_m) \end{pmatrix} = g_0^x g_m^{t_m}\begin{pmatrix} q(0,0) \\ p(0,0) \end{pmatrix}. \tag{4.2}$$

The commutativity of g_0^x, $g_m^{t_m}$ implies that it is a smooth function of (x, t_m), which is called the involutive solution of the consistent system of equations (F_0), (F_m).

Theorem 4.2. Let $(q(x,t_m), p(x,t_m))$ be an involutive solution of the consistent system (F_0), (F_m) defined by (4.2). Let $u(x,t_m) = \langle q,q\rangle - \langle p,p\rangle$, $v(x,t_m) = \langle q+p, q+p\rangle$. Then

1) The flow equations (F_0), (F_m) are reduced to the spatial part and the time part respectively of the Lax pair for the higher order nonlinear evolution equation in relation to (1.1) with u and v as their potentials (c_j are independent of x):

$$L\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} v-\partial & u+v \\ v-u & v+\partial \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} = A\begin{pmatrix} q \\ p \end{pmatrix}, \tag{4.3}$$

$$\frac{\partial}{\partial t_m}\begin{pmatrix} q \\ p \end{pmatrix} = (W_m + c_1 W_{m-1} + \cdots + c_m W_0)\begin{pmatrix} q \\ p \end{pmatrix}. \tag{4.4}$$

2) $u(x,t_m) = \langle q,q\rangle - \langle p,p\rangle$, $v(x,t_m) = \langle q+p, q+p\rangle$ satisfy the higher order nonlinear evolution equation:

$$\frac{\partial}{\partial t_m}\begin{pmatrix} u \\ v \end{pmatrix} = X_m + c_1 X_{m-1} + \cdots + c_m X_0. \tag{4.5}$$

Proof. From expression (3.11) for F_0, we have

$$(F_0) \qquad q_x = \frac{\partial F_0}{\partial p} = -Aq + vq + (u+v)p,$$

$$p_x = -\frac{\partial F_0}{\partial q} = Ap - vp + (u-v)q.$$

Obviously (F_0) implies (4.3). From expression (3.12) for F_m, we have

$$(F_m) \qquad q_{t_m} = \frac{\partial F_m}{\partial p} = -A^{m+1}q + 2\langle q,q\rangle A^m p + 2\langle q,p\rangle A^m p + \langle A^m p,p\rangle q$$
$$+ \langle A^m q,q\rangle q + 2\langle A^m q,q\rangle p + 2\sum_{j=1}^{m-1}\left[\langle A^j q,q\rangle\ A^{m-j}p - \langle A^i q,p\rangle\ A^{m-j}q\right],$$
$$p_{t_m} = -\frac{\partial F_m}{\partial q} = A^{m+1}p - 2\langle p,p\rangle\ A^m q - 2\langle q,p\rangle\ A^m q - \langle A^m q,q\rangle\ p$$
$$-\langle A^m p,p\rangle p - 2\langle A^m p,p\rangle q - 2\sum_{j=1}^{m-1}\left[\langle A^j p,p\rangle\ A^{m-j}q - \langle A^j q,p\rangle\ A^{m-j}p\right].$$

Through direct calculation (F_m) can be written in the following form

$$q_{t_m} = \sum_{j=1}^{m}\Bigg[\frac{1}{2}(\langle A^{j-1}p,p\rangle - \langle A^{j-1}q,q\rangle)_x\ A^{m-j}q + \frac{1}{2}(\langle A^{j-1}p,p\rangle - \langle A^{j-1}q,q\rangle)_x A^{m-j}p$$
$$-\frac{1}{2}\langle A^{j-1}(q+p);q+p\rangle_x\ A^{m-j}p + \langle A^{j-1}(q+p),q+p\rangle\ A^{m-j}q_x\Bigg] + A^m q_x, \tag{4.6}$$

$$p_{t_m} = \sum_{j=1}^{m}\Bigg[-\frac{1}{2}(\langle A^{j-1}p,p\rangle - \langle A^{j-1}q,q\rangle)_x\ A^{m-j}q - \frac{1}{2}\langle A^{j-1}(q+p),q+p\rangle_x A^{m-j}q$$
$$-\frac{1}{2}(\langle A^{j-1}p,p\rangle - \langle A^{j-1}q,q\rangle)_x A^{m-j}p + \langle A^{j-1}(q+p),q+p\rangle A^{m-j}p_x\Bigg] + A^m p_x. \tag{4.7}$$

By using Lemma 3.1, (4.6) and (4.7) are reduced to ($c_0 = 1$):

$$q_{t_m} = \sum_{l=0}^{m} C_l \sum_{j=l}^{m}\left[\frac{1}{2}G^{(1)}_{j-l-1,x}A^{m-j}q + \frac{1}{2}\left(G^{(1)}_{j-l-1} - G^{(2)}_{j-l-1}\right)_x A^{m-j}p + G^{(2)}_{j-l-1}\partial A^{m-j}q\right],$$

$$P_{t_m} = \sum_{l=0}^{m} c_l \sum_{j=l}^{m}\left[-\frac{1}{2}(G^{(1)}_{j-l-1} + G^{(2)}_{j-l-1})_x A^{m-j}q - \frac{1}{2}G^{(1)}_{j-l-1,x}A^{m-j}p + G^{(2)}_{j-l-1}\partial A^{m-j}p\right].$$

So we have

$$\begin{pmatrix} q \\ p \end{pmatrix}_{t_m} = \sum_{l=0}^{m} C_l W_{m-l}\begin{pmatrix} q \\ p \end{pmatrix}.$$

(4.5) is obtained through direct calculation from (F_m):

$$\frac{\partial u}{\partial t_m} = 2\langle q, q_{t_m}\rangle - 2\langle p, p_{t_m}\rangle$$
$$= 2\langle A^m(q+p),q+p\rangle\ \langle q+p,q+p\rangle\ - 2(\langle A^{m+1}q,q\rangle\ +\ \langle A^{m+1}p,p\rangle)$$
$$= -\partial(\langle A^m p,p\rangle - \langle A^m q,q\rangle),$$
$$\frac{\partial v}{\partial t_m} = 2\langle q+p,(q+p)_{t_m}\rangle$$
$$= 2(\langle q,q\rangle - \langle p,p\rangle)\langle A^m(q+p),q+p\rangle + 2(\langle A^{m+1}p.p\rangle - \langle A^{m+1}q,q\rangle)$$
$$= \partial\langle A^m(q+p),q+p\rangle.$$

Hence

$$\frac{\partial}{\partial t_m}\begin{pmatrix} u \\ v \end{pmatrix} = J\begin{pmatrix} \langle A^m p, p\rangle - \langle A^m q, q\rangle \\ \langle A^m (q+p), q+p\rangle \end{pmatrix} = J\sum_{s=0}^{m+1} c_s G_{m-s} = \sum_{s=0}^{m} c_s X_{m-s}.$$

Remark 4.3. In the above sense, under the constraints $u = \langle q,q\rangle - \langle p,p\rangle$, $v = \langle q+p, q+p\rangle$, the spatial and the time parts of the Lax pair for the higher order evolution equation are nonlinearized to the canonical equations (F_0), (F_m) respectively. Both of them are completely integrable in Liouville sense.

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