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On peaked and smooth solitons for the Camassa-Holm equation

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Abstract. – This letter presents all possible explicit single soliton solutions for the Camassa-Holm (CH) equation $m_t + m_x u + 2mu_x = 0$, $m = u - u_{xx}$. This equation is studied under the boundary condition $u \to A$ (A is a constant) as $x \to \pm\infty$. Regular peakon solutions correspond to the case of $A = 0$. For the case of $A \neq 0$, both new peaked solitons and new type of smooth solitons, which are expressed in terms of trigonometric and hyperbolic functions, are tremendously given through investigating a Newton equation with a new potential. Mathematical analysis and numeric graphs are provided for those smooth soliton and new peaked soliton solutions.

Both the Camassa-Holm (CH) equation [1]

$$m_t + m_x u + 2mu_x = 0, \qquad m = u - \alpha^2 u_{xx}, \qquad x \in \mathbb{R} \tag{1}$$

and the unidirectional shallow-water wave equation [2]

$$\tilde{m}_t + \tilde{m}_x \tilde{u} + 2\tilde{m}\tilde{u}_x = -c_0 \tilde{u}_x + \gamma \tilde{u}_{xxx}, \qquad \tilde{m} = \tilde{u} - \alpha^2 \tilde{u}_{xx}, \qquad x \in \mathbb{R} \tag{2}$$

have excited much attraction in recent years. The CH equation was implied by the work of Fokas and Fuchssteiner (1981) on hereditary symmetries [3]. It came to be remarkable in the work of Camassa and Holm (1993) where the peakon was described [1]. A peakon is a weak solution with non-smooth property at some points. A discussion of mathematical details is given in several literatures: by Beals *et al.* (1998) [4]; Contantin, Escher, and McKean (1998, 1999) [5–7], Alber *et al.* (2001) [8], Johnson (2002) [9], Qiao (2003) [10], and Gesztesy and Holden (2003) [11].

Both equations are integrable and have peaked solitons and infinite number of conservation laws [1,2]. Moreover, the two wave equations are transformable one into the other by a simple transformation, namely

$$\tilde{m}(x,t) \;=\; m\left(x - \frac{t}{2}(3\gamma - c_0), t\right) + \frac{1}{2}(\gamma - c_0), \quad \tilde{u}(x,t) = u\left(x - \frac{t}{2}(3\gamma - c_0), t\right) + \frac{1}{2}(\gamma - c_0).$$

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Because the parameter α can be rescaled to unity, without any loss of generality, we set $\alpha = 1$ and consider a cleaned CH equation

$$m_t + m_x u + 2mu_x = 0, \qquad m = u - u_{xx}, \qquad x \in \mathbb{R}. \tag{3}$$

Parker [12] studied the bilinear form for the CH equation and gave solitary-wave solutions, regular peaked solitons and some solution given in an implicit form. Dullin, Gottwald and Holm [13] dealt with the traveling-wave solutions of a generalized version of the CH equation and gave solutions also in an implicit form. The present letter provides an approach to obtain *all possible explicit single soliton solutions* for the CH equation (3). Our strategy is to use the discontinuity of the first-order derivative and the Dirac distribution skills for the CH equation (see eq. (11) and Theorem 1). In the letter we find new explicit solutions of the CH equation through setting the traveling-wave solution under the boundary condition $u \to A$ (A is a constant) as $x \to \pm\infty$. In particular, regular peakon solutions of the CH equation correspond to the case of $A = 0$. The most interesting case is $A \neq 0$, where smooth soliton solutions and new peaked soliton solutions are tremendously obtained in this letter.

Let us consider the traveling-wave solution of the CH equation (3) through the setting $u(x,t) = U(x-ct)$, where c is the wave speed. Let $\xi = x-ct$, then $u(x,t) = U(\xi)$. Substituting it into the CH equation (3) yields

$$(U-c)(U-U'')' + 2U'(U-U'') = 0, \tag{4}$$

where $U' = U_\xi$, $U'' = U_{\xi\xi}$, $U''' = U_{\xi\xi\xi}$.

The CH equation has the peakon solution [1] $u(x,t) = U(\xi) = ce^{-|x-ct-\xi_0|}$ ($\xi_0 = x_0 - ct_0$) with the following properties:

$$U(\xi_0) = c, \qquad U(\pm\infty) = 0, \qquad U'(\xi_0-) = c, \qquad U'(\xi_0+) = -c, \tag{5}$$

where $U'(\xi_0-)$ and $U'(\xi_0+)$ represent the left-derivative and the right-derivative at ξ_0, respectively.

Proposition 1. *The CH equation (3) has the following weak traveling-wave solution:*

$$u(x,t) = -a\sinh(|x-ct-\xi_0|) + ce^{-|x-ct-\xi_0|}, \tag{6}$$

where $a \in \mathbb{R}$ is an arbitrary constant, c is the wave speed, and $\xi_0 = x_0 - ct_0$ is an arbitrarily real constant.

In particular, if we take $a = 0$ in this theorem, then (6) exactly gives the regular peakon solution $u(x,t) = ce^{-|x-ct-\xi_0|}$ which was described by Camassa and Holm [1].

Let us assume that U is *neither* a constant function *nor* satisfies $U - U'' = 0$. Then eq. (4) can be changed to

$$\frac{(U-U'')'}{U-U''} = \frac{2U'}{c-U}. \tag{7}$$

Let us find all possible soliton solutions of eq. (3) with the boundary condition

$$\lim_{\xi\to\pm\infty} U = A. \tag{8}$$

where A is a non-zero constant. After taking double integrations on both sides of eq. (7), we obtain

$$U'^2 = U^2 + \frac{2A(A-c)^2}{U-c} + A(2c-3A) = (U-A)^2\frac{U-c+2A}{U-c}. \tag{9}$$

This equation has a physical meaning and coincides with the Newton equation of a particle in the potential $V(U) = U^2 + 2A\frac{(A-c)^2}{U-c}$. Actually, it is $U'^2 = V(U) - V(A)$. The fact that both sides of eq. (9) are non-negative implies $\frac{U-c+2A}{U-c} \geq 0$.

We consider the single soliton solutions, namely, the solution satisfies the boundary condition (8) and has only one peak point where the discontinuity of the first derivative probably appears. Assume that ξ_0 is the unique peak point, then eq. (9) can be selected as

$$U' = -\mathrm{sign}(\xi - \xi_0)(U - A)\sqrt{\frac{U - c + 2A}{U - c}}. \tag{10}$$

Because of the translation invariance of the differential equation (4), without any loss of generality, we choose the peak point ξ_0 as vanishing, $\xi_0 = 0$. Thus, eq. (10) reads:

$$U' = -\mathrm{sign}(\xi)(U - A)\sqrt{\frac{U - c + 2A}{U - c}}. \tag{11}$$

Let us assume that $A \neq 0$ (if $A = 0$, the traveling-wave solutions are regular peakons [1]). Let $X = \sqrt{\frac{U-c}{U-c+2A}}$, then $U = (c-2A) + \frac{2A}{1-X^2}$, $\mathrm{d}U = \frac{4AX}{(X^2-1)^2}\mathrm{d}X$, and eq. (11) is converted to

$$f(X)\mathrm{d}X \equiv \frac{4AX^2}{(X^2 - 1)[(3A - c)X^2 - (A - c)]}\mathrm{d}X = \mathrm{sign}(\xi)\mathrm{d}\xi. \tag{12}$$

Since $\lim_{\xi\to\pm\infty} U = A$, we know that there is no solution for eq. (9) if A falls in the gap between $c - 2A$ and c, that is,

Proposition 2. *If* $0 < A \leq c < 3A$ *or* $3A < c \leq A < 0$, *then there is no solution for eq.* (9) satisfying the boundary condition (8).

Case I: $A > 0, c < A$ *or* $A < 0, c > A$.
1) If $A > 0$, $c < A$, then $U > c$ and $0 < X < 1$. Let $a = \sqrt{\frac{A-c}{3A-c}}$, then $0 < a < 1$. Integrating eq. (12) on the interval $[0, \xi]$ (or $[\xi, 0]$) leads to the following implicit solutions:

$$F(X) \equiv a \ln\left|\frac{X + a}{X - a}\right| - \ln\frac{1 + X}{1 - X} = |\xi|. \tag{13}$$

Since $F'(X) = f(X)$, we know that $F(X)$ strictly increases on the interval $0 < X < a$ with $F(0) = 0, F(a-) = \infty$. Denote

$$F_1(X) = F|_{(0,a)}(X) = a \ln\frac{a + X}{a - X} - \ln\frac{1 + X}{1 - X} = a\ \mathrm{arctanh}\ \frac{X}{a} - \mathrm{arctanh}\ X. \tag{14}$$

Then $F_1(X)$ has the inverse on the interval $(0, a)$. So, eq. (13) can be solved uniquely for X on the interval $(0, a)$. Therefore, we define $F_1(X)$'s inverse as $X_1(\xi) = F_1^{-1}(|\xi|)$.
2) If $A < 0$, $c > A$, then we have $U < c$, $0 < X < 1$ and $a < 1$. This case is completely similar to the case of $A > 0$, $c < A$.

Case II: $c > 3A > 0$ *or* $c < 3A < 0$.
1) If $c > 3A > 0$, then $U < c - 2A$, $X > 1$, $a > 1$. On the interval $a < X < \infty$, $U > A$ and eq. (9) is equivalent to

$$\frac{1}{U - A}\sqrt{\frac{c - U}{c - 2A - U}}\mathrm{d}U = -\mathrm{sign}(\xi)\mathrm{d}\xi. \tag{15}$$

Solving this equation yields the following implicit solution of eq. (9):

$$F_2(X) \equiv a \ln\frac{X + a}{X - a} - \ln\frac{X + 1}{X - 1} = |\xi|. \tag{16}$$

By $F_2'(X) = f(X)$, we know that $F_2(X)$ strictly decreases on the interval $a < X < \infty$ with $F_2(a+) = \infty$, $F_2(\infty) = 0$, which implies that $F_2(X)$ is invertible on the interval (a, ∞). Denote the inverse by $X_2(\xi) = F_2^{-1}(|\xi|)$.
2) If $c < 3A < 0$, then $U > c - 2A$, $X > 1$ and $a > 1$. This case is completely analogous to the case of $A > 0$, $c > 3A$. Let

$$U_j = (c - 2A) + \frac{2A}{1 - (F_j^{-1}(|\xi|))^2}, \qquad j = 1, 2; \tag{17}$$

and we denote the left-hand side of eq. (4) by

$$L(U) \equiv (U - c)(U - U'')' + 2U'(U - U''). \tag{18}$$

A direct calculation leads to the following Dirac distribution theorem.

Theorem 1. *Let U be either U_1 or U_2, then*

$$L(U) = 2(U(0) - A)(U(0) - c)\sqrt{\frac{U(0) - c + 2A}{U(0) - c}}\delta'(\xi), \tag{19}$$

where $\delta(\xi)$ is the delta function, and U is a weak solution of eq. (4) *iff*

$$U(0) = A \quad or \quad U(0) = c \quad or \quad U(0) = c - 2A. \tag{20}$$

Notice that $U(0) = A$ corresponds to the trivial solution $U \equiv A$. What we are interested in is the other two cases: $U(0) = c$ or $U(0) = c - 2A$. Here is our main result.

Theorem 2. *Assume that the single soliton solution $u(x,t) = U(x - ct)$ (0 is the unique peak point of U) of the CH equation* (3) *satisfies the boundary condition* (8). *Then we have*

1) *if $A = 0$, the only single soliton solution $u(x,t)$ is the following peakon:*

$$u(x,t) = U(x - ct) = ce^{-|x-ct|},$$

with the properties

$$U(0) = c, \qquad U(\pm\infty) = 0, \qquad U'(0+) = -c, \qquad U'(0-) = c;$$

2) *if $A > 0$, $A \le c \le 3A$ or $A < 0$, $3A \le c \le A$, there is no solitary-wave solution $U(x-ct)$ for the CH equation* (3);

3) *if $A > 0, c < A$ or $A < 0, c > A$, the solitary-wave solutions of the CH equation* (3) *have peak points (see figs.* 1*) and can be expressed as*

$$u(x,t) = U(x - ct) = (c - 2A) + \frac{2A}{1 - (F_1^{-1}(|x - ct|))^2},$$

with the properties

$$U(0) = c, \qquad U(\pm\infty) = A, \qquad U'(0+) = \mathrm{sign}(A)\infty, \qquad U'(0-) = -\mathrm{sign}(A)\infty,$$

where F_1 is defined by eq. (14);

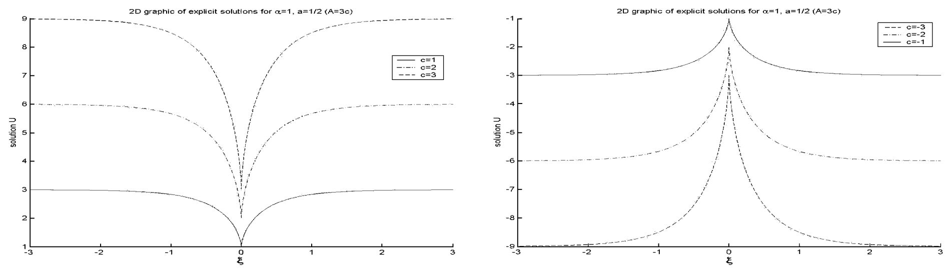

Fig. 1 – 2D graphics for peaked soliton solutions with $a = 1/2$.

4) *if* $0 < 3A < c$ *or* $c < 3A < 0$, *the solitary-wave solution of the CH equation* (3) *is smooth with the following form (see figs.* 2*):*

$$u(x,t) = U(x-ct) = (c-2A) + \frac{2A}{1-(F_2^{-1}(|x-ct|))^2}$$

with the properties

$$U(0) = c - 2A, \qquad U(\pm\infty) = A, \qquad U'(0) = 0,$$

where F_2 *is defined by eq.* (16).

As shown above, we provide the exact traveling-wave solutions of the CH equation (3). However, because of the implicit form of F_1^{-1} and F_2^{-1} (see Theorem 2), in general, we cannot get an explicit form from F_1^{-1} and F_2^{-1}. But, for some special a's, we do have explicit solutions. Let us work on the inverse functions of F_1 and F_2.

$F_1(X) = |\xi|$ implies that

$$\ln \frac{1+X}{1-X} + a \ln \frac{a-X}{a+X} = -|\xi|, \qquad 0 < X < a,$$

which is equivalent to

$$\frac{1+X}{1-X}\left(\frac{a-X}{a+X}\right)^a = e^{-|\xi|}.$$

Let us discuss some special cases below.

Case $A = 3c$.

In this case, we have $a = \sqrt{\frac{A-c}{3A-c}} = \frac{1}{2}$ and

$$\left(\frac{1+X}{1-X}\right)^2 \frac{1-2X}{1+2X} = e^{-2|\xi|},$$

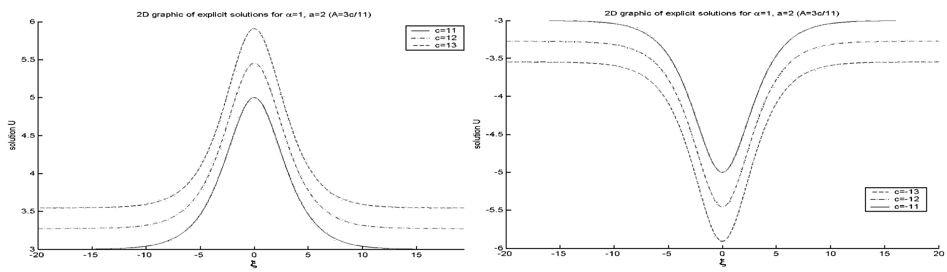

Fig. 2 – 2D graphics for smooth soliton solutions with $a = 2$.

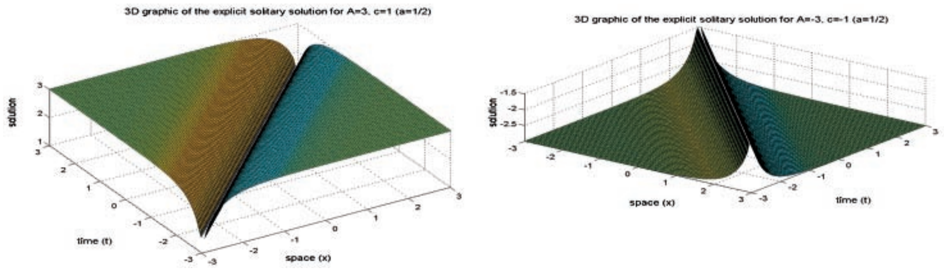

Fig. 3 – 3D pictures for the explicit peaked soliton solutions with $a = 1/2$.

which can be reduced to $X^3 + 3bX^2 - b = 0$ where $b = \frac{1}{2}\tanh|\xi|$. Solving this equation gives us $X(\xi) = \frac{1}{2}\tanh|\xi|\left[\tanh^{2/3}\frac{|\xi|}{2} + \coth^{2/3}\frac{|\xi|}{2} - 1\right]$. So, we obtain a family of explicit peaked soliton solutions of the CH equation (3),

$$U(\xi) = c + 6c\frac{\left[\tanh^{2/3}\frac{|\xi|}{2} + \coth^{2/3}\frac{|\xi|}{2} - 1\right]^2}{4\coth^2|\xi| - \left[\tanh^{2/3}\frac{|\xi|}{2} + \coth^{2/3}\frac{|\xi|}{2} - 1\right]^2} \equiv cV_1(\xi), \tag{21}$$

where the wave speed c is an arbitrary nonzero constant, and

$$V_1(\xi) = 1 + 6\frac{\left[\tanh^{2/3}\frac{|\xi|}{2} + \coth^{2/3}\frac{|\xi|}{2} - 1\right]^2}{4\coth^2\frac{|\xi|}{\alpha} - \left[\tanh^{2/3}\frac{|\xi|}{2} + \coth^{2/3}\frac{|\xi|}{2} - 1\right]^2}$$

is the bounded peaked soliton solution corresponding to $c = 1$ (see fig. 3).

Case $A = 3c/11$.

In this case $a = 2$, and from $F_2(X) = |\xi|$, we obtain $\frac{X+1}{X-1}\left(\frac{X-2}{X+2}\right)^2 = e^{-|\xi|}$ which gives the following explicit form of X in terms of ξ: $X(\xi) = \coth\frac{|\xi|}{2}\left[1 + 2\cos\left(\frac{2}{3}\operatorname{arccos}\operatorname{sech}\frac{|\xi|}{2}\right)\right]$. Apparently, $X(\xi) = \frac{1}{\sin\frac{\theta(\xi)}{3}} > 2$. Thus we obtain a family of explicit smooth soliton solutions of the CH equation (3),

$$U(\xi) = \frac{c}{11}\,\frac{11\cos\left(\frac{2}{3}\operatorname{arccos}\operatorname{sech}\frac{|\xi|}{2}\right) - 1}{\cos\left(\frac{2}{3}\operatorname{arccos}\operatorname{sech}\frac{|\xi|}{2}\right) + 1} \equiv \frac{c}{11}V_2(\xi), \tag{22}$$

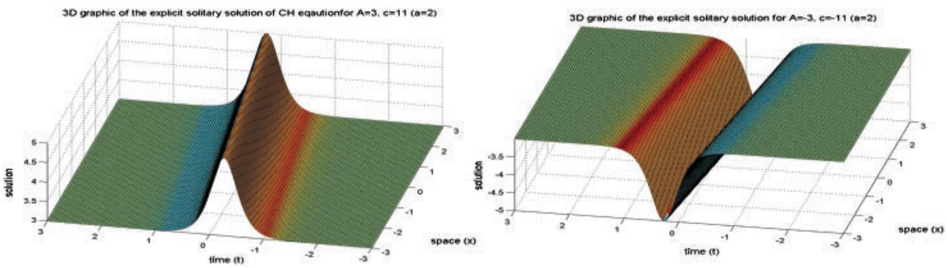

Fig. 4 – 3D pictures for the explicit smooth soliton solutions with $a = 2$.

where the wave speed c is an arbitrary nonzero constant, and

$$V_2(\xi) = \frac{11\cos\left(\frac{2}{3}\arccos\operatorname{sech}\frac{|\xi|}{2}\right) - 1}{\cos\left(\frac{2}{3}\arccos\operatorname{sech}\frac{|\xi|}{2}\right) + 1}$$

is the smooth soliton solution corresponding to $c = 11$ (see fig. 4).

In this letter we provide an approach to obtain all possible explicit solutions of the Camassa-Holm (CH) equation $m_t + m_x u + 2mu_x = 0$, $m = u - u_{xx}$ under the boundary condition $u \to A$ (A is a constant) as $x \to \pm\infty$. In particular, for the first time we have got new peaked solitons and new type of smooth solitons, which are expressed in terms of trigonometric and hyperbolic functions (see eqs. (21) and (22)), for the CH equation. Consequently, we solve the Newton equation $U'^2 = V(U) - V(A)$ with a new potential $V(U)$ (see eq. (9)) for all possible single soliton solutions on the base of the mathematical skills and proof since this equation is generated from the CH equation. This is very helpful for us to deal with physical equations. New peaked solitons and new type of smooth soliton solutions are expected to apply in nonlinear shallow-water wave theory and Newton motion theory because they have a very close relation to the Newton equation (9).

Actually, this approach can be also applied to other types of nonlinear PDEs. What we are interested in is to find new solutions of nonlinear equations regardless of integrability. We are applying this method to all b-equations: $m_t + m_x u + bmu_x = 0$, $m = u - u_{xx}$ and already get some new solutions, which we will report in another paper.

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