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TRAVELING WAVE SOLUTIONS OF THE GENERALIZED BBM EQUATION

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Abstract

This paper suggests an approach to solve the generalized Benjamin-Bona-Mahony (BBM) equation. The approach is given through improving the tanh function method based on an auxiliary ordinary differential equation, and is used to construct explicit traveling wave solutions of nonlinear evolution equations. We will show the efficiency of our method through the generalized BBM equation. The results we have got in the present paper indicate that our approach not only offers the existing solutions in literature, but also some new solitary wave solutions and triangular periodic wave solutions.

Key words: Tanh function method; Generalized BBM equation; exact solution

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1. Introduction

Recently, the following two auxiliary equations

$$(\frac{d\varphi}{d\xi})^2 = c_2\varphi^2(\xi) + c_3\varphi^3(\xi) + c_4\varphi^4(\xi), \tag{1}$$

$$(\frac{d\varphi}{d\xi})^2 = c_2\varphi^2(\xi) + c_3\varphi^4(\xi) + c_4\varphi^6(\xi), \tag{2}$$

were used for solving nonlinear evolution equations [1–3]. Some nonlinear physical models are investigated and new traveling wave solutions are explicitly obtained [1–3]. Then, a natural question arises here: can we combine the exact solution of Eq. (1) with Eq. (2)'s to form a uniform solution for both equations. Furthermore, which equation is more powerful and useful in solving traveling wave solutions for nonlinear evolution equations? The present paper is going to address the questions.

This paper is organized as follows. In section 2, a new auxiliary ordinary differential equation is proposed, and uniform of exact solutions of Eqs.(1) and (2) is provided. In section 3, the procedure of the improved tanh function method is described in details. In section 4, by means of the improved tanh function method and the uniform solutions of the auxiliary equations, the traveling wave solutions for the generalized BBM equation with any power order are presented. Section 5 gives some remarks and conclusions.

2. Exact Solutions of the Uniform Auxiliary Ordinary Differential Equation

Combining Eq. (1) with Eq. (2), we construct the following uniform of auxiliary ordinary differential equation:

$$\left(\frac{d\varphi}{d\xi}\right)^2 = c_2\varphi^2(\xi) + c_3\varphi^{p+2}(\xi) + c_4\varphi^{2p+2}(\xi), \tag{3}$$

where c_2, c_3, c_4 are constant coefficients and $p = 1,2,3,\cdots$. Apparently, Eqs.(1) and (2) are two special cases of $p = 1$ and $p = 2$ in Eq. (3). In general, it is very difficult to find exact solutions of Eq. (3) [4]. In this paper, based on our previous experience [2], we obtain new exact solutions of Eq. (3). For brevity, we omit the procedure of solving Eq. (3) and list exact solutions of Eq. (3) below:

Case1. When $c_2 > 0$, and $(D)^{1/P}$ makes sense for arbitrary negative number D, Eq.(3) has the following solution:

$$\varphi_1(\xi) = \left(\frac{-c_2c_3\,\mathrm{sec}h^2(\pm\frac{p\sqrt{c_2}}{2}\xi)}{c_3^2 - c_2c_4(1-\tanh(\pm\frac{p\sqrt{c_2}}{2}\xi))^2}\right)^{\frac{1}{p}}, \tag{4}$$

Case2. When $c_2 > 0$, $c_4 > 0$ and $(D)^{1/P}$ makes sense for arbitrary negative number D, Eq.(3) obtains the following solutions:

$$\varphi_2(\xi) = \left(\frac{c_2\,\mathrm{csc}h^2\frac{p\sqrt{c_2}}{2}\xi}{c_3 + 2\sqrt{c_2c_4}\coth\frac{p\sqrt{c_2}}{2}\xi}\right)^{\frac{1}{p}}, \tag{5}$$

$$\varphi_3(\xi)=\left(\frac{4c_2(\cosh p\sqrt{c_2}\xi+\sinh p\sqrt{c_2}\xi)}{4c_2c_4-(c_3+\cosh p\sqrt{c_2}\xi+\sinh p\sqrt{c_2}\xi)^2}\right)^{\frac{1}{p}}, \tag{6}$$

$$\varphi_4(\xi)=\left(\frac{8c_2^2\operatorname{sec}hp\sqrt{c_2}\xi}{c_3^2+4c_2(c_2-c_4)-4c_2c_3\operatorname{sec}hp\sqrt{c_2}\xi+(c_3^2-4c_2(c_2+c_4))\tanh p\sqrt{c_2}\xi}\right)^{\frac{1}{p}} \tag{7}$$

$$\varphi_5(\xi)=\left(\frac{c_2(-1+(\tanh p\sqrt{c_2}\xi\pm i\operatorname{sec}hp\sqrt{c_2}\xi)^2)}{c_3+2\sqrt{c_2c_4}(\tanh p\sqrt{c_2}\xi\pm i\operatorname{sec}hp\sqrt{c_2}\xi)}\right)^{\frac{1}{p}}, \tag{8}$$

$$\varphi_6(\xi)=\left(\frac{c_2\operatorname{csc}h\frac{p\sqrt{c_2}}{2}\xi}{c_3\sinh\frac{p\sqrt{c_2}}{2}\xi+2\sqrt{c_2c_4}\cosh\frac{p\sqrt{c_2}}{2}\xi}\right)^{\frac{1}{p}}, \tag{9}$$

$$\varphi_7(\xi)=\left(\frac{c_2\operatorname{sec}h\frac{p\sqrt{c_2}}{2}\xi}{2\sqrt{c_2c_4}\sinh\frac{p\sqrt{c_2}}{2}\xi-c_3\cosh\frac{p\sqrt{c_2}}{2}\xi}\right)^{\frac{1}{p}}, \tag{10}$$

Case3. When $c_2>0$, $c_3^2-4c_2c_4>0$ and $(D)^{1/P}$ makes sense for arbitrary negative number D, Eq.(3) admits the following solution:

$$\varphi_8(\xi)=\left(\frac{2c_2}{-c_3\pm\sqrt{c_3^2-4c_2c_4}\cosh p\sqrt{c_2}\xi}\right)^{\frac{1}{p}}, \tag{11}$$

Case4. When $c_2>0$, $c_3^2-4c_2c_4<0$ and $(D)^{1/P}$ makes sense for arbitrary negative number D, Eq.(3) has the following solution:

$$\varphi_9(\xi)=\left(\frac{2c_2\operatorname{csc}hp\sqrt{c_2}\xi}{\pm\sqrt{4c_2c_4-c_3^2}-c_3\operatorname{csc}hp\sqrt{c_2}\xi}\right)^{\frac{1}{p}}, \tag{12}$$

Case5. When $c_2>0$, $c_3^2-4c_2c_4=0$ and $(D)^{1/P}$ makes sense for arbitrary negative number D, Eq.(3) obtains the following solutions:

$$\varphi_{10}(\xi)=\left(-\frac{c_2}{c_3}(1\pm\tanh\frac{p\sqrt{c_2}}{2}\xi)\right)^{\frac{1}{p}}, \tag{13}$$

$$\varphi_{11}(\xi)=(-\frac{c_2}{c_3}(1\pm\coth\frac{p\sqrt{c_2}}{2}\xi))^{\frac{1}{p}}, \tag{14}$$

Case6. When $c_2<0, c_4>0$ and $(D)^{1/P}$ makes sense for arbitrary negative number D, Eq.(3) admits the following solutions:

$$\varphi_{12}(\xi)=(\frac{2c_2}{-c_3\pm\sqrt{c_3^2-4c_2c_4}\sin p\sqrt{-c_2}\xi})^{\frac{1}{p}}, \tag{15}$$

$$\varphi_{13}(\xi)=(\frac{2c_2}{-c_3\pm\sqrt{c_3^2-4c_2c_4}\cos p\sqrt{-c_2}\xi})^{\frac{1}{p}}, \tag{16}$$

$$\varphi_{14}(\xi)=(\frac{c_2\sec^2\frac{p\sqrt{-c_2}}{2}\xi}{-c_3+2\sqrt{-c_2c_4}\tan\frac{p\sqrt{-c_2}}{2}\xi})^{\frac{1}{p}}, \tag{17}$$

$$\varphi_{15}(\xi)=(\frac{c_2\csc^2\frac{p\sqrt{-c_2}}{2}\xi}{-c_3+2\sqrt{-c_2c_4}\cot\frac{p\sqrt{-c_2}}{2}\xi})^{\frac{1}{p}}, \tag{18}$$

$$\varphi_{16}(\xi)=(\frac{-c_2(1+(\tan p\sqrt{-c_2}\xi\pm\sec p\sqrt{-c_2}\xi)^2)}{c_3-2\sqrt{-c_2c_4}(\tan p\sqrt{-c_2}\xi\pm\sec p\sqrt{-c_2}\xi)})^{\frac{1}{p}}, \tag{19}$$

$$\varphi_{17}(\xi)=(\frac{-c_2\csc\frac{p\sqrt{-c_2}}{2}\xi}{c_3\sin\frac{p\sqrt{-c_2}}{2}\xi+2\sqrt{-c_2c_4}\cos\frac{p\sqrt{-c_2}}{2}\xi})^{\frac{1}{p}}, \tag{20}$$

$$\varphi_{18}(\xi)=(\frac{c_2\sec\frac{p\sqrt{-c_2}}{2}\xi}{2\sqrt{-c_2c_4}\sin\frac{p\sqrt{-c_2}}{2}\xi-c_3\cos\frac{p\sqrt{-c_2}}{2}\xi})^{\frac{1}{p}}, \tag{21}$$

Case7. When $c_3 = 0$ and $(D)^{1/P}$ makes sense for arbitrary negative number D, Eq.(3) has the following solutions:

$$\varphi_{19}(\xi) = (\pm\sqrt{\frac{c_2}{c_4}}\operatorname{csch} p\sqrt{c_2}\xi)^{\frac{1}{p}}, \qquad (c_2 > 0, c_4 > 0) \tag{22}$$

$$\varphi_{20}(\xi) = (\pm\sqrt{-\frac{c_2}{c_4}}\operatorname{sech} p\sqrt{c_2}\xi)^{\frac{1}{p}}, \qquad (c_2 > 0, c_4 < 0) \tag{23}$$

$$\varphi_{21}(\xi) = (\pm\sqrt{-\frac{c_2}{c_4}}\csc p\sqrt{-c_2}\xi)^{\frac{1}{p}}, \qquad (c_2 < 0, c_4 > 0) \tag{24}$$

Case8. When $c_4 = 0$ and $(D)^{1/P}$ makes sense for arbitrary negative number D, Eq.(3) obtains the following solutions:

$$\varphi_{22}(\xi) = (-\frac{c_2}{c_3}\operatorname{sech}^2 \frac{p\sqrt{c_2}}{2}\xi)^{\frac{1}{p}}, \qquad (c_2 > 0) \tag{25}$$

$$\varphi_{23}(\xi) = (\frac{c_2}{c_3}\operatorname{csch}^2 \frac{p\sqrt{c_2}}{2}\xi)^{\frac{1}{p}}, \qquad (c_2 > 0) \tag{26}$$

$$\varphi_{24}(\xi) = (-\frac{c_2}{c_3}\sec^2 \frac{p\sqrt{-c_2}}{2}\xi)^{\frac{1}{p}}. \qquad (c_2 < 0) \tag{27}$$

Remark: The solutions (4)-(14), (22),(23),(25) and (26) are solitary wave solutions that include bell-profile and kink-profile solutions, and the solutions (15)-(21), (24) and (27) are triangular periodic wave solutions. The solutions (11), (13) and (15) coincide with the results in Refs. [5-8] and others are new, which do not yet appear in literature within our knowledge.

3. The Improved tanh Function Method

The procedure of the improved tanh function method is described as follows

Step1.For a given nonlinear evolution equation with physical fields $u(x,t)$

$$H(u, u_t, u_x, u_{xx} \cdots) = 0, \tag{28}$$

By using the wave transformation

$$u(x,t) = u(\xi), \xi = x - \lambda t\ , \tag{29}$$

where λ are constant to be determined later. Then Eq.(28) is reduced to nonlinear ordinary differential equations (NODE)

$$H(u, u', u'', \cdots) = 0\ . \tag{30}$$

Step2. We take advantage of Eq.(3) and use its solutions to replace tanh function in tanh function method [9]. Namely, expand the solution of Eq.(30) in the form

$$u(x,t) = u(\xi) = \sum_{i=0}^{n} a_i \varphi^i\ , \qquad (n \geq 1) \tag{31}$$

where a_i $(i = 0,1,\ldots,n)$ is constant and the variable $\varphi = \varphi(\xi)$ satisfies Eq.(3).The derivatives with respect to the variable ξ become the derivatives with respect to the variable φ as :

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \to \pm\sqrt{c_2\varphi^2(\xi) + c_3\varphi^{p+2}(\xi) + c_4\varphi^{2p+2}(\xi)}\,\frac{\mathrm{d}}{\mathrm{d}\varphi}, \tag{32}$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \to & \frac{1}{2}(2c_2\varphi(\xi) + (p+2)c_3\varphi^{p+1}(\xi) + (2p+2)c_4\varphi^{2p+1}(\xi))\frac{\mathrm{d}}{\mathrm{d}\varphi} \\ & + (c_2\varphi^2(\xi) + c_3\varphi^{p+2}(\xi) + c_4\varphi^{2p+2}(\xi))\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}, \quad \cdots\end{aligned} \tag{33}$$

Step3. Supposing the order of the highest derivative term and the degree of the nonlinear terms in Eq.(30) respectively are m and r , and substituting the expression (31) into Eq.(30) and balancing the highest derivative term with the nonlinear terms in Eq.(30) by making use of the expressions (32) and (33), we can obtain the expression

$$n = \frac{mp}{r-1}\ . \tag{34}$$

For simpler and more powerful solving nonlinear evolution equations, we take p as minimum.

Again, for $n \geq 1$, we obtain

$$r \leq mp + 1\ . \tag{35}$$

If we take Eq.(1) or Eq.(2) as an auxiliary equation, we can not solve the nonlinear evolution equations with the order of nonlinear terms more than $m+1$ or $2m+1$.Therefore, reducing the order of nonlinear terms in nonlinear evolution equations by proper function transformation, we can make use of Eq.(1) or Eq.(2) or other auxiliary equation ($p = 3,4,\cdots$) to solve the nonlinear evolution equations with nonlinear terms of any order.

Step4. Substituting the expansion (31) into Eq.(30) and setting the coefficients of all powers of $\varphi^k(\pm\sqrt{c_2\varphi^2(\xi)+c_3\varphi^{p+2}(\xi)+c_4\varphi^{2p+2}(\xi)})^\nu$ $(\nu = 0,1; k = 0,1,2,\ldots)$ to zero, we will get a system of nonlinear algebraic equations with respect to a_i and c_j $(i = 0,1,\ldots,n,\ j = 2,3,4)$,then solve the system of algebraic equations with the aid of the Mathematica to obtain a_i and c_j .

Step5.Substituting the constants c_j into Eq.(3), its all the possible solutions are obtained. By means of a_i obtained in Step4 and solutions of Eq.(3), we can obtain the solution of Eq.(28).

4. The Traveling Wave Solutions of the Generalized Benjamin-Bona-Mahony (BBM) Equation

The well-known generalized BBM equation with nonlinear terms of any order power [5]

$$u_t + au^\gamma u_x + bu^{2\gamma}u_x - \delta u_{xxt} = 0 \quad (\delta \neq 0, \gamma > 0) \tag{36}$$

with constants a, b, δ, γ , has been used in the surface waves of long wavelength in liquids, hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, and acoustic gravity waves in compressible fluids [10]. When $a = \gamma = 1, b = 0,$ Eq. (3) becomes the BBM equation. We perform the following transformation:

$$u(x,t) = u(\xi), \xi = x - \lambda t\,, \tag{37}$$

where λ is a constant. Substituting the expression (37) into Eq.(36) and integrating it with respect to ξ and then taking the integration constant as zero yield an NODE

$$u''(\xi) - \frac{1}{\delta}u(\xi) + \frac{a}{\delta\lambda(\gamma+1)}u^{\gamma+1}(\xi) + \frac{b}{\delta\lambda(2\gamma+1)}u^{2\gamma+1}(\xi) = 0\,. \tag{38}$$

By means of the expression (35), we obtained $\gamma \le p$ ($p = 1,2,3,\cdots$).Evidently, we can not directly solve Eq.(38) with auxiliary equation (3).Thus by using the function transformation:

$$u(\xi) = [v(\xi)]^{1/\gamma}, \tag{39}$$

Eq.(38) becomes:

$$\begin{aligned} &-\gamma^2(1+\gamma)(1+2\gamma)\lambda v^2(\xi) + \gamma^2 a(1+2\gamma)v^3(\xi) + \gamma^2 b(1+\gamma)v^4(\xi) \\ &-(\gamma^2-1)(1+2\gamma)\delta\lambda v'^2(\xi) + \gamma(1+\gamma)(1+2\gamma)\delta\lambda v(\xi)v''(\xi) = 0. \end{aligned} \tag{40}$$

In Eq.(40), by means of the expression (34), we obtain $n = p$. For simpler and more powerful solving Eq.(40), we take $n = p = 1$. Thus the solution of Eq.(40) can be taken as

$$v(\xi) = a_0 + a_1\varphi(\xi),\ (a_1 \ne 0) \tag{41}$$

where variable $\varphi(\xi)$ satisfies Eq.(1)(the case of $p = 1$ in Eq.(3)). Substituting the expression (41) into Eq.(40) along with Eq.(1) leads to the following system of algebraic equations:

$$\gamma^2 a_0^2(-(1+3\gamma+2\gamma^2)\lambda + (1+2\gamma)aa_0 + b(1+\gamma)a_0^2) = 0,$$

$$\gamma a_0 a_1(3a\gamma(1+2\gamma)a_0 + 4b\gamma(1+\gamma)a_0^2 - (1+3\gamma+2\gamma^2)\lambda(2\gamma - \delta c_2)) = 0,$$

$$\begin{aligned} &\frac{1}{2}a_1(2a_1(3a\gamma^2(1+2\gamma)a_0 + 6b\gamma^2(1+\gamma)a_0^2 - (1+3\gamma+2\gamma^2)\lambda(\gamma^2 - \delta c_2)) \\ &+ 3\gamma(1+3\gamma+2\gamma^2)\delta\lambda a_0 c_3) = 0, \end{aligned}$$

$$\begin{aligned} &\frac{1}{2}a_1(2\gamma^2(a + 2a\gamma + 4b(1+\gamma)a_0)a_1^2 + (2+7\gamma+7\gamma^2+2\gamma^3)\delta\lambda a_1 c_3 \\ &+ 4\gamma(1+3\gamma+2\gamma^2)\delta\lambda a_0 c_4) = 0, \end{aligned}$$

$$(1+\gamma)a_1^2(b\gamma^2 a_1^2 + (1+3\gamma+2\gamma^2)\delta\lambda c_4) = 0.$$

By use of the Mathematica, solving the over-determined algebraic equations, we have the following results:

Case1: $a_0 = 0$, $a_1 = -\dfrac{(\gamma^2+3\gamma+2)\delta\lambda c_3}{2a\gamma^2}$, $c_2 = \dfrac{\gamma^2}{\delta}$, $c_4 = -\dfrac{(\gamma+1)(\gamma+2)^2 b\delta\lambda c_3^2}{4a^2\gamma^2(2\gamma+1)}$. (42)

Case2: $$\lambda = -\frac{3a^2}{16b}, a_0 = -\frac{3a}{4b}, a_1 = -\frac{9a\delta c_3}{8b}, c_2 = 0, c_4 = \frac{9\delta c_3^2}{8};\ (\gamma = 1) \tag{43}$$

$$\lambda = -\frac{a^2}{6b}, a_0 = -\frac{a}{2b},\ a_1 = \pm\sqrt{\frac{a^2\delta c_4}{b^2}}, c_2 = -\frac{1}{2\delta}, c_3 = 0;\ (\gamma = 1) \tag{44}$$

$$\lambda = -\frac{5a^2}{36b}, a_0 = -\frac{5a}{6b},\ a_1 = -\frac{5a\delta c_3}{8b}, c_2 = 0, c_4 = \frac{3\delta c_3^2}{4};\ (\gamma = 2) \tag{45}$$

$$\lambda = -\frac{25a^2}{192b}, a_0 = -\frac{5a}{8b},\ a_1 = \pm\frac{5}{16}\sqrt{\frac{5a^2\delta c_4}{b^2}}, c_2 = -\frac{4}{5\delta}, c_3 = 0;\ (\gamma = 2) \tag{46}$$

$$\lambda = -\frac{(2\gamma+1)a^2}{(\gamma+1)(\gamma+2)^2 b}, a_0 = -\frac{(2\gamma+1)a}{(\gamma+2)b}, a_1 = -\frac{(2\gamma+1)a\delta c_3}{2(\gamma+2)b\gamma^2}, c_2 = \frac{\gamma^2}{\delta},$$
$$c_4 = \frac{\delta c_3^2}{4\gamma^2} \qquad (\gamma \neq 1, \gamma \neq 2,) \tag{47}$$

Substituting the expression (42) with the expressions (4)-(27), respectively, into the expressions (41) and (39), we obtain the following solitary wave solutions and triangular periodic wave solutions to Eq.(36):

$$u_1(x,t) = \left[\frac{(\gamma^2 + 3\gamma + 2)\lambda \sec h^2(\pm\frac{1}{2\sqrt{\delta}}\gamma\xi)}{2a + \frac{(\gamma+1)(\gamma+2)^2 b\lambda}{2(2\gamma+1)a}(1 - \tanh(\pm\frac{1}{2\sqrt{\delta}}\gamma\xi))^2}\right]^{\frac{1}{\gamma}}, \tag{48}$$

where $\delta > 0, \xi = x - \lambda t$;

$$u_2(x,t) = \left[\frac{-(\gamma+1)\lambda \operatorname{csc} h^2 \frac{\gamma}{2\sqrt{\delta}}\xi}{\frac{2a}{\gamma+2} + 2\frac{a|c_3|}{c_3|a|}\sqrt{-\frac{(\gamma+1)b\lambda}{2\gamma+1}}\coth\frac{\gamma}{2\sqrt{\delta}}\xi}\right]^{\frac{1}{\gamma}}, \tag{49}$$

where $\delta > 0, b\lambda < 0,\ \xi = x - \lambda t$;

$$u_3(x,t)=\left[\frac{-2(\gamma^2+3\gamma+2)\lambda c_3(\cosh\frac{\gamma}{\sqrt{\delta}}\xi+\sinh\frac{\gamma}{\sqrt{\delta}}\xi)}{-\frac{(\gamma+1)(\gamma+2)^2 b\lambda c_3^2}{a(2\gamma+1)}-a(c_3+\cosh\frac{\gamma}{\sqrt{\delta}}\xi+\sinh\frac{\gamma}{\sqrt{\delta}}\xi)^2}\right]^{\frac{1}{\gamma}}, \quad (50)$$

where $\delta>0, b\lambda<0,\ \xi=x-\lambda t$;

$$u_4(x,t)=[(-4(\gamma^2+3\gamma+2)\lambda c_3\frac{\gamma^2}{a\delta}\sec h\frac{\gamma}{\sqrt{\delta}}\xi)/(c_3^2-4\frac{\gamma^2}{\delta}c_3\sec h\frac{\gamma}{\sqrt{\delta}}\xi+4\frac{\gamma^4}{\delta^2}$$
$$+\frac{(\gamma+1)(\gamma+2)^2 b\lambda c_3^2}{a^2(2\gamma+1)}+(c_3^2-4\frac{\gamma^4}{\delta^2}+\frac{(\gamma+1)(\gamma+2)^2 b\lambda c_3^2}{a^2(2\gamma+1)})\tanh\frac{\gamma}{\sqrt{\delta}}\xi)]^{\frac{1}{\gamma}}, \quad (51)$$

where $\delta>0, b\lambda<0,\ \xi=x-\lambda t$;

$$u_5(x,t)=\left[\frac{-(\gamma+1)\lambda(-1+(\tanh\frac{\gamma}{\sqrt{\delta}}\xi\pm i\sec h\frac{\gamma}{\sqrt{\delta}}\xi)^2)}{\frac{2a}{\gamma+2}+2\frac{a|c_3|}{|a|c_3}\sqrt{-\frac{(\gamma+1)b\lambda}{(2\gamma+1)}}(\tanh\frac{\gamma}{\sqrt{\delta}}\xi\pm i\sec h\frac{\gamma}{\sqrt{\delta}}\xi)}\right]^{\frac{1}{\gamma}}, \quad (52)$$

where $\delta>0, b\lambda<0,\ \xi=x-\lambda t$;

$$u_6(x,t)=\left[\frac{-(\gamma+1)\lambda\csc h\frac{\gamma}{2\sqrt{\delta}}\xi}{\frac{2a}{\gamma+2}\sinh\frac{\gamma}{2\sqrt{\delta}}\xi+2\frac{a|c_3|}{|a|c_3}\sqrt{-\frac{(\gamma+1)b\lambda}{2\gamma+1}}\cosh\frac{\gamma}{2\sqrt{\delta}}\xi}\right]^{\frac{1}{\gamma}}, \quad (53)$$

where $\delta>0, b\lambda<0,\ \xi=x-\lambda t$;

$$u_7(x,t)=\left[\frac{-(\gamma+1)\lambda\sec h\frac{\gamma}{2\sqrt{\delta}}\xi}{2\frac{a|c_3|}{|a|c_3}\sqrt{-\frac{(\gamma+1)b\lambda}{(2\gamma+1)}}\sinh\frac{\gamma}{2\sqrt{\delta}}\xi-\frac{2a}{\gamma+2}\cosh\frac{\gamma}{2\sqrt{\delta}}\xi}\right]^{\frac{1}{\gamma}}, \quad (54)$$

where $\delta>0, b\lambda<0,\ \xi=x-\lambda t$;

$$u_8(x,t)=\left[\frac{-(\gamma^2+3\gamma+2)\lambda}{-a\pm a\frac{|c_3|}{c_3}\sqrt{1+\frac{(\gamma+1)(\gamma+2)^2 b\lambda}{a^2(2\gamma+1)}}\cosh\frac{\gamma}{\sqrt{\delta}}\xi}\right]^{\frac{1}{\gamma}}, \tag{55}$$

where $\delta>0$, $\lambda>\frac{-(2\gamma+1)a^2}{(\gamma+1)(\gamma+2)^2 b}$, $\xi=x-\lambda t$;

$$u_9(x,t)=\left[\frac{-(\gamma^2+3\gamma+2)\lambda\operatorname{csc}h\frac{\gamma}{\sqrt{\delta}}\xi}{\pm a\frac{|c_3|}{c_3}\sqrt{-\frac{(\gamma+1)(\gamma+2)^2 b\lambda}{a^2(2\gamma+1)}-1}-a\operatorname{csc}h\frac{\gamma}{\sqrt{\delta}}\xi}\right]^{\frac{1}{\gamma}}, \tag{56}$$

where $\delta>0$, $\lambda<\frac{-(2\gamma+1)a^2}{(\gamma+1)(\gamma+2)^2 b}$, $\xi=x-\lambda t$;

$$u_{10}(x,t)=\left[\frac{-(2\gamma+1)a}{2(\gamma+2)b}\left(1\pm\tanh\frac{\gamma}{2\sqrt{\delta}}\xi\right)\right]^{\frac{1}{\gamma}}, \tag{57}$$

where $\delta>0$, $\lambda=\frac{-(2\gamma+1)a^2}{(\gamma+1)(\gamma+2)^2 b}$, $\xi=x-\lambda t$;

$$u_{11}(x,t)=\left[\frac{-(2\gamma+1)a}{2(\gamma+2)b}\left(1\pm\coth\frac{\gamma}{2\sqrt{\delta}}\xi\right)\right]^{\frac{1}{\gamma}}, \tag{58}$$

where $\delta>0$, $\lambda=\frac{-(2\gamma+1)a^2}{(\gamma+1)(\gamma+2)^2 b}$, $\xi=x-\lambda t$;

$$u_{12}(x,t)=\left[\frac{-(\gamma^2+3\gamma+2)\lambda}{-a\pm\frac{a|c_3|}{c_3}\sqrt{1+\frac{(\gamma+1)(\gamma+2)^2 b\lambda}{a^2(2\gamma+1)}}\sin\frac{\gamma}{\sqrt{-\delta}}\xi}\right]^{\frac{1}{\gamma}}, \tag{59}$$

where $\delta<0$, $b\lambda>0$, $\xi=x-\lambda t$;

$$u_{13}(x,t)=\left[\frac{-(\gamma^2+3\gamma+2)\lambda}{-a\pm\frac{a|c_3|}{c_3}\sqrt{1+\frac{(\gamma+1)(\gamma+2)^2 b\lambda}{a^2(2\gamma+1)}}cos\frac{\gamma}{\sqrt{-\delta}}\xi}\right]^{\frac{1}{\gamma}}, \tag{60}$$

where $\delta < 0$, $b\lambda > 0$, $\xi = x - \lambda t$;

$$u_{14}(x,t) = \left[\frac{-(\gamma+1)\lambda \sec^2 \frac{\gamma}{2\sqrt{-\delta}}\xi}{\frac{-2a}{\gamma+2} + 2\frac{a|c_3|}{|a|c_3}\sqrt{\frac{(\gamma+1)b\lambda}{(2\gamma+1)}}\tan\frac{\gamma}{2\sqrt{-\delta}}\xi}\right]^{\frac{1}{\gamma}}, \tag{61}$$

where $\delta < 0$, $b\lambda > 0$, $\xi = x - \lambda t$;

$$u_{15}(x,t) = \left[\frac{-(\gamma+1)\lambda \csc^2 \frac{\gamma}{2\sqrt{-\delta}}\xi}{\frac{-2a}{\gamma+2} + 2\frac{a|c_3|}{|a|c_3}\sqrt{\frac{(\gamma+1)b\lambda}{(2\gamma+1)}}\cot\frac{\gamma}{2\sqrt{-\delta}}\xi}\right]^{\frac{1}{\gamma}}, \tag{62}$$

where $\delta < 0$, $b\lambda > 0$, $\xi = x - \lambda t$;

$$u_{16}(x,t) = \left[\frac{(\gamma+1)\lambda(1+(\tan\frac{\gamma}{\sqrt{-\delta}}\xi \pm \sec\frac{\gamma}{\sqrt{-\delta}}\xi)^2)}{\frac{2a}{\gamma+2} - 2\frac{a|c_3|}{|a|c_3}\sqrt{\frac{(\gamma+1)b\lambda c_3^2}{(2\gamma+1)}}(\tan\frac{\gamma}{\sqrt{-\delta}}\xi \pm \sec\frac{\gamma}{\sqrt{-\delta}}\xi)}\right]^{\frac{1}{\gamma}}, \tag{63}$$

where $\delta < 0$, $b\lambda > 0$, $\xi = x - \lambda t$;

$$u_{17}(x,t) = \left[\frac{(\gamma+1)\lambda \csc\frac{\gamma}{2\sqrt{-\delta}}\xi}{\frac{2a}{\gamma+2}\sin\frac{\gamma}{2\sqrt{-\delta}}\xi + 2\frac{a|c_3|}{|a|c_3}\sqrt{\frac{(\gamma+1)b\lambda}{(2\gamma+1)}}\cos\frac{\gamma}{2\sqrt{-\delta}}\xi}\right]^{\frac{1}{\gamma}}, \tag{64}$$

where $\delta < 0$, $b\lambda > 0$, $\xi = x - \lambda t$;

$$u_{18}(x,t) = \left[\frac{-(\gamma+1)\lambda \sec\frac{\gamma}{2\sqrt{-\delta}}\xi}{2\frac{a|c_3|}{|a|c_3}\sqrt{\frac{(\gamma+1)b\lambda}{2\gamma+1}}\sin\frac{\gamma}{2\sqrt{-\delta}}\xi - \frac{2a}{\gamma+2}\cos\frac{\gamma}{2\sqrt{-\delta}}\xi}\right]^{\frac{1}{\gamma}}, \tag{65}$$

where $\delta < 0$, $b\lambda > 0$, $\xi = x - \lambda t$.

Remark: In above expressions, $a \neq 0, b \neq 0$ and $(D)^{1/\gamma}$ makes sense for arbitrary negative number D.The solutions (55) and (57) coincide with the results in Ref. [5], and others are new.

We can also obtain the solitary wave solutions and the triangular periodic wave solutions to Eq. (36) in the case of the expressions (43) - (47).

5. Conclusions

In this paper, we improve the tanh function method and apply it to the uniform auxiliary ordinary differential equation. The proposed method is adopted to solve the nonlinear evolution equations with nonlinear terms of any order power under some suitable function transformation. As an example, the generalized BBM equation is investigated and exact traveling wave solutions are obtained, including new solitary wave solutions and triangular periodic wave solutions. In addition to the generalized BBM equation considered in this paper, the proposed method is also available to other nonlinear evolution equations, including the compound KdV-type equation, the generalized modified Boussinesq equation without dissipative term, the generalized one–dimensional Klein-Gordon equation, the generalized Zakharov equations, the generalized (2+1) dimensional Klein-Gordon equation, the Rangwala-Rao (RR) equation, the Ablowitz (A) equation and the Gerdjikov-Ivanov (GI)equation [5]. Tracking our procedure, the exact solutions of a given nonlinear evolution equation depend on the explicit solvability of Eq.(3) and the system of nonlinear algebraic equations with respect to a_i and c_j.Thus in order to solve nonlinear evolution equations in a simpler procedure, we always take p in Eq. (3) and n in the expression (31) as minimum. Recently, we also studied the peaked soliton equations, including the Camassa-Holm hierarchy [13,18], the Degasperis-Procesi hierarchy [12,16,17], and new cusp and M/W-shape peak soliton equations [11,14,15,]. How do we apply our improved tanh function method to those cusp and peaked soliton equations? We will think about them in near future.

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