Front. Math. China 2013, 8(5): 1185–1196
DOI 10.1007/s11464-013-0314-x

RESEARCH ARTICLE

Integrable peakon systems with weak kink and kink-peakon interactional solutions

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Abstract We report two integrable peakon systems that have weak kink and kink-peakon interactional solutions. Both peakon systems are guaranteed integrable through providing their Lax pairs. The peakon and multi-peakon solutions of both equations are studied. In particular, the two-peakon dynamic systems are explicitly presented and their collisions are investigated. The weak kink solution is studied, and more interesting, the kink-peakon interactional solutions are proposed for the first time.

Keywords Integrable system, Lax pair, peakon, weak kink, kink-peakon
MSC 37K10, 35Q51, 35Q58

1 Introduction

In recent years, the Camassa-Holm (CH) equation [2]

$$m_t - bu_x + 2mu_x + m_x u = 0, \quad m = u - u_{xx}, \tag{1}$$

where b is an arbitrary constant, has attracted much attention in the theory of soliton and integrable system [1,3–6,9,11,13,14,17,21,22]. The most interesting feature of the CH equation (1) is to admit peaked soliton (peakon) solutions in the case of $b = 0$. In addition to the CH equation, other integrable models with peakon solutions have been found, such as the Degasperis-Procesi equation [7,8,18,19] and the cubic nonlinear peakon equations [10,12,15,16,20,23–25].

In this paper, we study the following equation with both quadratic and cubic nonlinearity:

$$m_t = bu_x + \frac{1}{2}k_1[m(u^2 - u_x^2)]_x + \frac{1}{2}k_2(2mu_x + m_x u), \quad m = u - u_{xx}, \tag{2}$$

Received April 18, 2013; accepted June 3, 2013
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(with k_1 and k_2 being two arbitrary constants) and its two-component extension:

$$\begin{cases} m_t = bu_x + \frac{1}{2}[m(uv - u_x v_x)]_x - \frac{1}{2} m(uv_x - u_x v), \\ n_t = bv_x + \frac{1}{2}[n(uv - u_x v_x)]_x + \frac{1}{2} n(uv_x - u_x v), \\ m = u - u_{xx}, \\ n = v - v_{xx}. \end{cases} \tag{3}$$

Equation (2) is actually a linear combination of CH equation (1) and cubic nonlinear equation

$$m_t = bu_x + [m(u^2 - u_x^2)]_x, \quad m = u - u_{xx}, \tag{4}$$

which was derived independently by Fokas [10], Fuchssteiner [12], Olver and Rosenau [21], and Qiao [23], where the equation was derived from the two-dimensional Euler system, and Lax pair, the M/W-shape solitons and peakon/cuspon solutions were presented. Apparently, the two-component system (3) we propose is reduced to the CH equation (1), the cubic CH equation (4), and the generalized CH equation (2) as $v = 2$, $v = 2u$, and $v = k_1 u + k_2$, respectively.

Both (2) and (3) are proven integrable through their Lax pairs, bi-Hamiltonian structures, and infinitely many conservation laws. In the case of $b = 0$, we show that systems (2)–(4) admit the single-peakon as well as multi-peakon solutions. In particular, we explicitly solve the two-peakon dynamic systems and study their collisions in details. In the case of $b \neq 0$, we find that (4) and (3) possess the weak kink solutions. More interesting, the kink-peakon interactional solutions are for the first time proposed for equation (4) in the case of $b \neq 0$.

2 Lax pair, bi-Hamiltonian structure, and conservation laws

Equation (3) arises as a compatibility condition

$$U_t - V_x + [U, V] = 0$$

of a pair of linear spectral problems

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad U = \frac{1}{2}\begin{pmatrix} -\alpha & \lambda m \\ -\lambda n & \alpha \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = V \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad V = -\frac{1}{2}\begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{6}$$

where

$$m = u - u_{xx}, \quad n = v - v_{xx},$$

b is an arbitrary constant, λ is a spectral parameter,

$$\alpha = \sqrt{1-\lambda^2 b},$$

and

$$\begin{aligned} A &= \lambda^{-2}\alpha + \frac{\alpha}{2}(uv - u_x v_x) + \frac{1}{2}(uv_x - u_x v), \\ B &= -\lambda^{-1}(u - \alpha u_x) - \frac{1}{2}\lambda m(uv - u_x v_x), \\ C &= \lambda^{-1}(v + \alpha v_x) + \frac{1}{2}\lambda n(uv - u_x v_x). \end{aligned} \tag{7}$$

Since equation (3) is reduced to the generalized CH equation (2) as $v = k_1 u + k_2$, we obtain the Lax pair of (2) by substituting $v = k_1 u + k_2$ into (5) and (6). Thus, both (2) and (3) are integrable in the sense of Lax pair.

Proposition 1 *Equation* (2) *has the following bi-Hamiltonian structure*:

$$m_t = J\frac{\delta H_1}{\delta m} = K\frac{\delta H_2}{\delta m}, \tag{8}$$

where

$$J = k_1 \partial m \partial^{-1} m \partial + \frac{1}{2}k_2(\partial m + m\partial) + b\partial, \quad H_1 = \frac{1}{2}\int_{-\infty}^{+\infty}(u^2 + u_x^2)\mathrm{d}x, \tag{9}$$

$$\begin{aligned} &K = \partial - \partial^3, \\ &H_2 = \frac{1}{8}\int_{-\infty}^{+\infty}\Big(k_1 u^4 + 2k_1 u^2 u_x^2 - \frac{1}{3}k_1 u_x^4 + 2k_2 u^3 + 2k_2 u u_x^2 + 4bu^2\Big)\mathrm{d}x. \end{aligned} \tag{10}$$

Proposition 2 *Equation* (3) *can be rewritten as the following bi-Hamiltonian form*:

$$(m_t, n_t)^{\mathrm{T}} = J\Big(\frac{\delta H_1}{\delta m}, \frac{\delta H_1}{\delta n}\Big)^{\mathrm{T}} = K\Big(\frac{\delta H_2}{\delta m}, \frac{\delta H_2}{\delta n}\Big)^{\mathrm{T}}, \tag{11}$$

where

$$\begin{aligned} &J = \begin{pmatrix} \partial m \partial^{-1} m \partial - m\partial^{-1} m & \partial m \partial^{-1} n \partial + m\partial^{-1} n + 2b\partial \\ \partial n \partial^{-1} m \partial + n\partial^{-1} m + 2b\partial & \partial n \partial^{-1} n \partial - n\partial^{-1} n \end{pmatrix}, \\ &H_1 = \frac{1}{2}\int_{-\infty}^{+\infty}(uv + u_x v_x)\mathrm{d}x, \quad K = \begin{pmatrix} 0 & \partial^2 - 1 \\ 1 - \partial^2 & 0 \end{pmatrix}, \\ &H_2 = \frac{1}{4}\int_{-\infty}^{+\infty}[(u^2 v_x + u_x^2 v_x - 2uu_x v)n + 2b(uv_x - u_x v)]\mathrm{d}x. \end{aligned} \tag{12}$$

Based on a standard treatment, from the Lax pairs (5) and (6), we may construct the following infinitely many conserved densities and the associated fluxes of equation (3):

$$\rho_0 = \sqrt{-mn}, \quad F_0 = \frac{1}{2}\sqrt{-mn}\,(uv - u_x v_x), \quad \rho_1 = \frac{mn_x - m_x n - 2mn}{2mn},$$

$$F_1 = -\frac{1}{2}\,(uv - u_x v_x + uv_x - u_x v) + \frac{1}{2}\,\rho_1(uv - u_x v_x), \tag{13}$$

$$\rho_j = m\omega_j, \quad F_j = (u - u_x)\omega_{j-2} + \frac{1}{2}\,\rho_j(uv - u_x v_x), \quad j \geqslant 2,$$

where ω_j is given by

$$\omega_0 = \sqrt{-\frac{n}{m}}, \quad \omega_1 = \frac{mn_x - m_x n - 2mn}{2m^2 n}, \tag{14}$$

and the recursion relation

$$\omega_{j+1} = \frac{1}{m\omega_0}\left[\omega_j - \omega_{j,x} - \frac{1}{2}\,m \sum_{i+k=j+1,\, i,k\geqslant 1} \omega_i\omega_k\right], \quad j \geqslant 1. \tag{15}$$

The infinitely many conservation laws of equation (2) may be obtained by substituting $v = k_1 u + k_2$ into (13).

3 Peakon solutions in case of $b = 0$

3.1 Peakon solutions of cubic CH equation (4)

One can directly check that the single-peakon solution of equation (4) with $b = 0$ is given by

$$u = \pm\sqrt{\frac{3c}{2}}\,\mathrm{e}^{-|x+ct|}.$$

In general, we make the ansatz for N-peakons

$$u(x,t) = \sum_{j=1}^{N} p_j(t)\mathrm{e}^{-|x-q_j(t)|}, \tag{16}$$

which implies

$$m = 2\sum_{j=1}^{N} p_j\delta(x - q_j).$$

Substituting them into equation (4) yields the following evolution equations for the peak positions and amplitudes:

$$\begin{cases} p_{j,t} = 0, \\ q_{j,t} = \dfrac{1}{3}\,p_j^2 - \displaystyle\sum_{i,k=1}^{N} p_i p_k (1 - \mathrm{sgn}(q_j - q_i)\mathrm{sgn}(q_j - q_k))\mathrm{e}^{-|q_j-q_i|-|q_j-q_k|}. \end{cases} \tag{17}$$

For $N = 2$, (17) can be solved with the explicit solutions:

$$\begin{cases} p_1(t) = c_1, \quad p_2(t) = c_2, \\ q_1(t) = \text{sgn}(t)\dfrac{3c_1c_2}{|\,(c_1^2 - c_2^2)\,|}\,(\mathrm{e}^{-|2(c_1^2-c_2^2)t/3|} - 1) - \dfrac{2}{3}\,c_1^2 t, \\ q_2(t) = \text{sgn}(t)\dfrac{3c_1c_2}{|\,(c_1^2 - c_2^2)\,|}\,(\mathrm{e}^{-|2(c_1^2-c_2^2)t/3|} - 1) - \dfrac{2}{3}\,c_2^2 t, \end{cases} \tag{18}$$

where c_1 and c_2 are arbitrary constants. The two-peakon collision occurs at the moment $t = 0$, since $q_1(0) = q_2(0) = 0$. Without loss of generality, let us suppose $0 < c_1 < c_2$. From formula (18), we know that for $t < 0$, the tall and fast peakon (with the amplitude c_2 and peak position q_2) chases after the short and slow peakon (with the amplitude c_1 and peak position q_1). At the moment of $t = 0$, the two-peakon collides and overlaps. After the collision ($t > 0$), the two-peakon departs, and the tall and fast peakon surpasses the short and slow one. See Fig. 1 (a) for the developments of this kind of two-peakon.

Remark 1 Our results show that the collision of two-peakon of equation (4) is very different from the case of CH equation (1). For the CH equation (1), the collision happens between peakon and anti-peakon [2]. For the cubic CH equation (4), the collision of two-peakon occurs in the case that the tall peakon 'chase' the short one as described above.

3.2 Peakon solutions of generalized CH system (2)

It is easy to verify that the single-peakon solution of equation (2) with $b = 0$ take the form of

$$u = C\mathrm{e}^{-|x-ct|}, \tag{19}$$

where C is determined by

$$\frac{1}{3}\,k_1C^2 + \frac{1}{2}\,k_2C + c = 0. \tag{20}$$

If $k_1 = 0$, $k_2 = -2$, then $C = c$. Thus, we recover the single-peakon solution $u = c\mathrm{e}^{-|x-ct|}$ of the CH equation (1) with $b = 0$. For $k_1 = 2$ and $k_2 = 0$, we reduce to the single-peakon solution of the cubic nonlinear CH equation (4) with $b = 0$. In general, for $k_1 \neq 0$, we may obtain

$$C = \frac{-3\big(\sqrt{3}\,k_2 \pm \sqrt{3k_2^2 - 16k_1c}\,\big)}{4\sqrt{3}\,k_1}. \tag{21}$$

If $3k_2^2 - 16k_1c \geqslant 0$, then C is a real number. If $3k_2^2 - 16k_1c < 0$, then C is a complex number. This means that we may have a peakon solution with complex coefficient.

Let us assume that the N-peakons are the same form as (16). Then we obtain the following N-peakon dynamic system:

$$p_{j,t} = -\frac{1}{2}\,k_2p_j\sum_{k=1}^{N} p_k\text{sgn}(q_j - q_k)\mathrm{e}^{-|q_j-q_k|},$$

$$q_{j,t} = -\frac{1}{2}k_2\sum_{k=1}^{N} p_k \mathrm{e}^{-|q_j-q_k|} + \frac{1}{2}k_1\Big(\frac{1}{3}p_j^2 - \sum_{i,k=1}^{N} p_i p_k (1-\mathrm{sgn}(q_j-q_i)\mathrm{sgn}(q_j-q_k))\mathrm{e}^{-|q_j-q_i|-|q_j-q_k|}\Big). \tag{22}$$

For $N = 2$, selecting $k_1 = k_2 = -2$ may yield the following special solution:

$$\begin{aligned} p_1(t) &= \coth t, \quad q_1(t) = \frac{8}{3(\mathrm{e}^{2t}-1)} + \log(\mathrm{e}^{2t}+1) - \frac{1}{3}t - \log 2, \\ p_2(t) &= -\coth t, \quad q_2(t) = \frac{8}{3(\mathrm{e}^{2t}-1)} - \log(\mathrm{e}^{2t}+1) + \frac{5}{3}t + \log 2. \end{aligned} \tag{23}$$

Thus, we arrive at the following peakon-antipeakon solution:

$$u(x,t) = \coth t(\mathrm{e}^{-|x-q_1(t)|} - \mathrm{e}^{-|x-q_2(t)|}), \tag{24}$$

where $q_1(t)$ and $q_2(t)$ are shown in (23). In spite of

$$\lim_{t\to 0} p_1(t) = -\lim_{t\to 0} p_2(t) = \infty, \quad \lim_{t\to 0} q_1(t) = \lim_{t\to 0} q_2(t) = \infty, \tag{25}$$

from (24), we still have

$$\lim_{t\to 0} u(x,t) = 0, \quad \forall\, x\in\mathbb{R}, \tag{26}$$

which indicates that the peakon and the antipeakon vanish when they overlap. Guided by the above results, we may describe the dynamics of peakon-antipeakon solution (24) as follows. For $t < 0$, the peak is at $q_2(t)$ and the trough is at $q_1(t)$. The peak and the trough approach each other as t goes to 0. At the moment of $t = 0$, the peakon and the antipeakon collide and vanish. After their collision ($t > 0$), they separate and reemerge with the trough at $q_2(t)$ and the peak at $q_1(t)$. Fig. 1 (b) shows the peakon-antipeakon interactional dynamics.

Remark 2 The amplitudes $p_1(t)$ and $p_2(t)$ in formula (23) are the same as those of the CH equation [2], but the peak positions $q_1(t)$ and $q_2(t)$ are different. In the CH equation, only $p_1(t)$ and $p_2(t)$ become infinite at the instant of collision [2,3]. In the new equation (2), both ($p_1(t)$,$p_2(t)$) and ($q_1(t)$,$q_2(t)$) become infinite at the instant of collision. However, in both cases, the peakon-antipeakon vanishes when the overlap occurs.

3.3 Peakon solutions of two-component system (3)

By a direct calculation, we find the single peakon solutions of (3) with $b = 0$ take the form of

$$u = c_1\mathrm{e}^{-|x+\frac{1}{3}c_1c_2t|}, \quad v = c_2\mathrm{e}^{-|x+\frac{1}{3}c_1c_2t|}, \tag{27}$$

where c_1 and c_2 are two arbitrary constants. In general, N-peakon solution is cast in the following form:

$$u(x,t)=\sum_{j=1}^{N} p_j(t)\mathrm{e}^{-|x-q_j(t)|}, \quad v(x,t)=\sum_{j=1}^{N} r_j(t)\mathrm{e}^{-|x-q_j(t)|}. \tag{28}$$

Substituting (28) into (3) with $b=0$, we are able to obtain the following N-peakon dynamic system:

$$\begin{cases} p_{j,t}=\dfrac{1}{2}\,p_j\displaystyle\sum_{i,k=1}^{N} p_i r_k\left(\mathrm{sgn}(q_j-q_k)-\mathrm{sgn}(q_j-q_i)\right)\mathrm{e}^{-|q_j-q_k|-|q_j-q_i|}, \\ q_{j,t}=\dfrac{1}{6}\,p_j r_j-\dfrac{1}{2}\displaystyle\sum_{i,k=1}^{N} p_i r_k\left(1-\mathrm{sgn}(q_j-q_i)\mathrm{sgn}(q_j-q_k)\right)\mathrm{e}^{-|q_j-q_i|-|q_j-q_k|}, \\ r_{j,t}=-\dfrac{1}{2}\,r_j\displaystyle\sum_{i,k=1}^{N} p_i r_k\left(\mathrm{sgn}(q_j-q_k)-\mathrm{sgn}(q_j-q_i)\right)\mathrm{e}^{-|q_j-q_k|-|q_j-q_i|}. \end{cases} \tag{29}$$

For $N=2$, we have the following explicit solution of (29):

$$\begin{gathered} p_1(t)=B\mathrm{e}^{\frac{3(A_2D^2-A_1)}{2D(A_1-A_2)}\,\mathrm{e}^{-|(A_1-A_2)t|/3}}, \quad p_2(t)=\frac{p_1}{D}, \\ r_1(t)=\frac{A_1}{p_1}, \quad r_2(t)=\frac{A_2}{p_2}, \\ q_1(t)=-\frac{1}{3}\,A_1t+\frac{3(A_2D^2+A_1)}{2D(A_1-A_2)}\mathrm{sgn}[(A_1-A_2)t](\mathrm{e}^{-|(A_1-A_2)t|/3}-1), \\ q_2(t)=-\frac{1}{3}\,A_2t+\frac{3(A_2D^2+A_1)}{2D(A_1-A_2)}\mathrm{sgn}[(A_1-A_2)t](\mathrm{e}^{-|(A_1-A_2)t|/3}-1), \end{gathered} \tag{30}$$

where A_1, A_2, B, and D are integration constants. Choosing special

$$A_1=1, \quad A_2=4, \quad B=1, \quad D=1$$

leads to

$$\begin{cases} p_1(t)=p_2(t)=\mathrm{e}^{-3\mathrm{e}^{-|t|}/2}, \\ r_1(t)=\mathrm{e}^{3\mathrm{e}^{-|t|}/2}, \quad r_2(t)=4\mathrm{e}^{3\mathrm{e}^{-|t|}/2}, \\ q_1(t)=-\dfrac{1}{3}\,t+\dfrac{5}{2}\,\mathrm{sgn}(t)(\mathrm{e}^{-|t|}-1), \\ q_2(t)=-\dfrac{4}{3}\,t+\dfrac{5}{2}\,\mathrm{sgn}(t)(\mathrm{e}^{-|t|}-1), \end{cases} \tag{31}$$

which generate the following two-peakon solution of (3):

$$\begin{cases} u(x,t)=\mathrm{e}^{-3\mathrm{e}^{-|t|}/2}(\mathrm{e}^{-|x+\frac{1}{3}t-\frac{5}{2}\,\mathrm{sgn}(t)(\mathrm{e}^{-|t|}-1)|}+\mathrm{e}^{-|x+\frac{4}{3}t-\frac{5}{2}\,\mathrm{sgn}(t)(\mathrm{e}^{-|t|}-1)|}), \\ v(x,t)=\mathrm{e}^{3\mathrm{e}^{-|t|}/2}(\mathrm{e}^{-|x+\frac{1}{3}t-\frac{5}{2}\,\mathrm{sgn}(t)(\mathrm{e}^{-|t|}-1)|}+4\mathrm{e}^{-|x+\frac{4}{3}t-\frac{5}{2}\,\mathrm{sgn}(t)(\mathrm{e}^{-|t|}-1)|}). \end{cases} \tag{32}$$

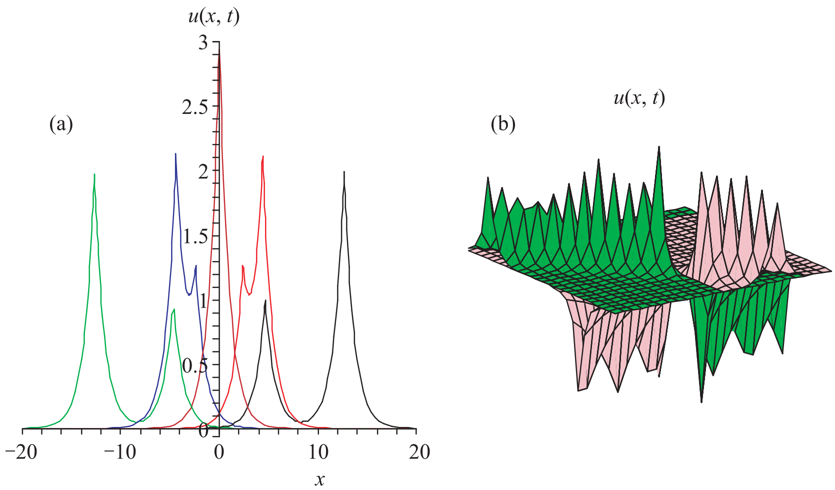

Fig. 1 (a) Two-peakon solution determined by (18) with $c_1 = 1$, $c_2 = 2$. Black line: $t = -4$; red line: $t = -1$; brown line: $t = 0$ (collision); blue line: $t = 1$; green line: $t = 4$.

(b) Peakon-antipeakon solution (24). Pink: peakon (and antipeakon) with peak (and trough) position q_2; green: antipeakon (and peakon) with trough (and peak) position q_1.

Apparently, the two-peakon solution of $u(x,t)$ possesses the same amplitude $\mathrm{e}^{-3\mathrm{e}^{-|t|}/2}$, which reaches the minimum value at the moment of collision ($t = 0$). Fig. 2 (a) shows the profile of the two-peakon dynamics for $u(x,t)$. The two-peakon solution of $v(x,t)$ with the amplitudes $\mathrm{e}^{3\mathrm{e}^{-|t|}/2}$ and $4\mathrm{e}^{3\mathrm{e}^{-|t|}/2}$ also collides at $t = 0$. At this moment, the amplitudes attain the maximum value and the two-peakon overlaps into one peakon $5\mathrm{e}^{3/2}\mathrm{e}^{-|x|}$, which is much higher than other moments. See Fig. 2 (b) for a 3-dimensional graph of the two-peakon dynamics for $v(x,t)$.

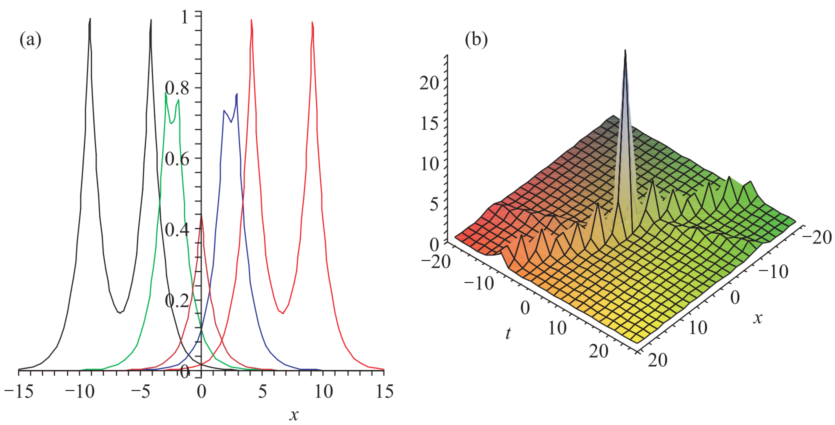

Fig. 2 (a) Two-peakon solution $u(x,t)$ in (32). Red line: $t = -5$; blue line: $t = -1$; brown line: $t = 0$ (collision); green line: $t = 1$; black line: $t = 5$.

(b) 3-dimensional graph for two-peakon solution $v(x,t)$ in (32).

4 Weak kink solutions of systems (4) and (3) in case of $b \neq 0$

Let us seek the kink solution of equation (4) in the form of

$$u = C\mathrm{sgn}(x - ct)(\mathrm{e}^{-|x-ct|} - 1), \tag{33}$$

where the wave speed c and the constant C are to be determined. The first order partial derivatives of (33) read

$$u_x = -C\mathrm{e}^{-|x-ct|}, \quad u_t = cC\mathrm{e}^{-|x-ct|}. \tag{34}$$

The second order partial derivatives of (33) do not exist at $x = ct$. Therefore, like the case of peakon solutions, the kink solution in the form of (33) should also be understood in the distribution sense. (33) is called a weak kink solution of equation (4). Substituting (33) and (34) into (4) yields

$$c = -\frac{1}{2}\,b, \quad C = \pm\sqrt{\frac{-b}{2}}. \tag{35}$$

See Fig. 3 (a) for the profile of this weak kink wave solution with $b = -2$.

Similarly, the two-component system (3) with $b \neq 0$ admits the following weak kink solution:

$$u = C_1\mathrm{sgn}\Big(x + \frac{1}{2}\,bt\Big)(\mathrm{e}^{-|x+\frac{1}{2}bt|} - 1), \quad v = C_2\mathrm{sgn}\Big(x + \frac{1}{2}\,bt\Big)(\mathrm{e}^{-|x+\frac{1}{2}bt|} - 1), \tag{36}$$

where $C_1C_2 = -b$.

Remark 3 In formula (35), $c = -b/2$ means that the kink wave speed is exactly $-b/2$. This is very different from the single-peakon solution whose wave speed is usually taken as an arbitrary constant c. The multi-peakon solutions take the form of superpositions of single-peakon solutions. However, by direct calculations, we find that the two systems (4) and (3) with $b \neq 0$ do not allow the multi-kink solution in the form of the superpositions of single-kink solutions.

5 Weak kink-peakon interactional solutions of equation (4)

Let us make the following ansatz of solution to equation (4):

$$u = p_1(t)\mathrm{sgn}(x - q_1(t))(\mathrm{e}^{-|x-q_1(t)|} - 1) + p_2(t)\mathrm{e}^{-|x-q_2(t)|}, \tag{37}$$

which actually describes a new phenomena of weak kink-peakon interactional dynamics in soliton theory. Substituting (37) into (4) and integrating in the distribution sense, we obtain

$$\begin{cases} p_1 = \pm\sqrt{\dfrac{-b}{2}}, \\ p_{2,t} = 2p_1^2 p_2 \mathrm{sgn}(q_2 - q_1)\mathrm{e}^{-|q_1-q_2|}, \\ q_{1,t} = -\dfrac{1}{2}\, b - 2p_1 p_2 \mathrm{sgn}(q_2 - q_1)\mathrm{e}^{-|q_1-q_2|}, \\ q_{2,t} = -\dfrac{2}{3}\, p_2^2 - p_1^2 + 2(p_1^2 - p_1 p_2 \mathrm{sgn}(q_2 - q_1))\mathrm{e}^{-|q_1-q_2|} + 2\mathrm{sgn}(q_2 - q_1)p_1 p_2. \end{cases} \tag{38}$$

Let us choose $b = -2$ and $p_1 = 1$. To solve the above system, let us make an assumption $q_1 < q_2$. After integrating equation (38), we obtain

$$\begin{cases} q_1 = t - p_2 + A_1, \\ q_2 = t - p_2 - \log\left|\dfrac{1}{9}\, p_2^2 - \dfrac{1}{2}\, p_2 + 1 + \dfrac{A_2}{2p_2}\right| + A_1, \\ p_{2,t} = \dfrac{2}{9}\, p_2^3 - p_2^2 + 2p_2 + A_2, \end{cases} \tag{39}$$

where A_1 and A_2 are integration constants. Letting $A_2 = 0$, we may solve the third equation of (39) for p_2 with the following implicit form:

$$\log|p_2| - \frac{1}{2}\log\left(p_2^2 - \frac{9}{2}\, p_2 + 9\right) + \frac{3\sqrt{7}}{7}\arctan\frac{4p_2 - 9}{3\sqrt{7}} = 2t + A_3. \tag{40}$$

See Fig. 3 (b) for the profile of the weak kink-peakon interactional solution with $A_1 = A_2 = A_3 = 0$.

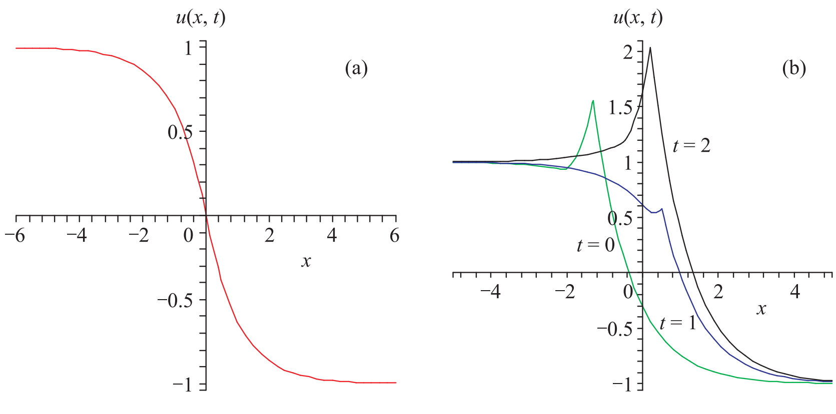

Fig. 3 (a) Weak kink solution given by (33) and (35) at $t = 0$.
(b) Weak kink-peakon interactional solution.

In general, we may assume the following ansatz of the solution to equation (4):

$$u = p_0(t)\mathrm{sgn}(x - q_0(t))(\mathrm{e}^{-|x-q_0(t)|} - 1) + \sum_{j=1}^{N} p_j(t)\mathrm{e}^{-|x-q_j(t)|}, \tag{41}$$

which can be viewed as the interaction of single weak kink and N-peakon solutions. Through a very lengthy calculation, we are able to arrive at the following interactional dynamical system of single weak kink and N-peakon:

$$\begin{cases} p_0 = \pm\sqrt{-\dfrac{b}{2}}, \\ q_{0,t} = p_0^2 + 2p_0 \displaystyle\sum_{i=1}^{N} p_i \mathrm{sgn}(q_0 - q_i)\mathrm{e}^{-|q_0-q_i|} \\ \qquad + \displaystyle\sum_{i,k=1}^{N} p_i p_k \mathrm{sgn}(q_i - q_k)(\mathrm{sgn}(q_k - q_0) - \mathrm{sgn}(q_i - q_0))\mathrm{e}^{-|q_i-q_k|}, \\ p_{j,t} = 2p_0^2 p_j \mathrm{sgn}(q_j - q_0)\mathrm{e}^{-|q_0-q_j|} \\ \qquad + 2p_0 p_j \displaystyle\sum_{i=1}^{N} p_i \mathrm{sgn}(q_j - q_i)\mathrm{sgn}(q_j - q_0)\mathrm{e}^{-|q_j-q_i|}, \\ q_{j,t} = \dfrac{1}{3} p_j^2 - p_0^2(1 - 2\mathrm{e}^{-|q_0-q_j|}) \\ \qquad - \displaystyle\sum_{i,k=1}^{N} p_i p_k (1 - \mathrm{sgn}(q_j - q_i)\mathrm{sgn}(q_j - q_k))\mathrm{e}^{-|q_j-q_i|-|q_j-q_k|} \\ \qquad - 2p_0 \displaystyle\sum_{i=1}^{N} p_i (\mathrm{sgn}(q_j - q_0)(\mathrm{e}^{-|q_0-q_j|} - 1)\mathrm{e}^{-|q_i-q_j|} \\ \qquad - \mathrm{sgn}(q_j - q_i)\mathrm{e}^{-|q_0-q_j|-|q_i-q_j|}). \end{cases} \tag{42}$$

The above system is not presented in the canonical Hamiltonian system. We still do not know whether this system is integrable for $N \geqslant 2$ under a Poisson structure.

Acknowledgements This work was partially supported by the U. S. Army Research Office (Contract/Grant No. W911NF-08-1-0511) and the Texas Norman Hackerman Advanced Research Program (Grant No. 003599-0001-2009).

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