Theoretical and Mathematical Physics, Vol. 127, No. 3, pp. 827–834, 2001

THE r-MATRIX AND AN ALGEBRAIC-GEOMETRIC SOLUTION OF THE AKNS SYSTEM

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We construct an approach to finite-dimensional integrable systems with nonlinear evolution equations from the standpoint of the r-matrix and an algebraic-geometric solution, illustrating the method with the well-known AKNS equation. We present the r-matrix of the constrained AKNS flow and obtain the algebraic-geometric solution of the AKNS equation.

1. Introduction

The ideal aim for soliton equations or nonlinear evolution equations (NLEEs) is to obtain their explicit solutions. The Ablowitz–Kaup–Newell–Segur (AKNS) equations are a very important hierarchy of NLEEs in soliton theory [1]. It can yield the KdV, MKdV, NLS, sine-Gordon, sinh-Gordon equations, etc. All these equations are solvable by the inverse scattering transform (IST) [2] and usually have N-soliton solutions [3]. One of the research branches in this field is the periodic boundary value problem associated with those special NLEEs. Some of the early studies on this problem were done by Lax [4] and Dubrovin, Krichever, and Novikov [5]. They used the Bloch eigenfuctions and some analysis tools on a Riemann surface and successfully obtained the algebraic-geometric solutions (or finite-gap solutions) of some well-known nonlinear equations such as the KdV and Toda equations. But algebraic-geometric solutions of the AKNS equations were not given. In the present paper, we resolve this problem from the standpoint of the r-matrix and a constraint connecting finite-dimensional integrable systems with NLEEs.

We present notation used here: $dp \wedge dq$ denotes the standard symplectic structure in the Euclidean space $R^{2N} = \{(p,q):\ p = (p_1, \dots, p_N),\ q = (q_1, \dots, q_N)\}$, $N > 1$; $\langle \cdot, \cdot \rangle$ is the standard inner product in R^N; $[\cdot, \cdot]$ is the usual commutator; $\otimes$ is the tensor between two matrices; I is the 2×2 unit matrix; and $C^\infty(R)$ is the set of all C^∞-functions on the real field R. In $(R^{2N}, dp \wedge dq)$, the Poisson bracket of two Hamilton functions F and G is defined by

$$\{F, G\} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) = \left\langle \frac{\partial F}{\partial q}, \frac{\partial G}{\partial p} \right\rangle - \left\langle \frac{\partial F}{\partial p}, \frac{\partial G}{\partial q} \right\rangle;$$

$\lambda_1, \dots, \lambda_N$ are N arbitrarily given distinct constants; λ and μ are two different spectral parameters; $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$; and

$$\Gamma_j = \sum_{k=1, k\neq j}^{N} \frac{(p_j q_k - p_k q_j)^2}{\lambda_j - \lambda_k}, \quad j = 1, 2, \dots, N.$$

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Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 127, No. 3, pp. 488–496, June, 2001.

2. The constrained AKNS flow

We consider the traceless 2×2 matrix

$$L = L(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} p_j q_j & -q_j^2 \\ p_j^2 & -p_j q_j \end{pmatrix}, \tag{1}$$

which is called the Lax matrix. We then have

$$\frac{1}{2}\lambda^2 \operatorname{Tr} L^2(\lambda) = \lambda^2 + 2\lambda\langle p, q\rangle + \langle p, q\rangle^2 + 2H + \sum_{j=1}^{N} \frac{\lambda_j^2 E_j}{\lambda - \lambda_j}, \tag{2}$$

where

$$\begin{aligned} H &= \langle \Lambda p, q\rangle - \frac{1}{2}\langle q, q\rangle\langle p, p\rangle, \\ E_j &= 2p_j q_j - \Gamma_j, \quad j = 1, \ldots, N. \end{aligned}$$

The finite-dimensional Hamiltonian system generated by the above Hamilton function H is

$$(H): \quad \begin{cases} q_x = \dfrac{\partial H}{\partial p} = -\langle q, q\rangle p + \Lambda q, \\ p_x = -\dfrac{\partial H}{\partial q} = \langle p, p\rangle q - \Lambda p. \end{cases} \tag{3}$$

It can be easily seen that (H) is just the well-known Zakharov–Shabat–AKNS spectral problem [6]

$$y_x = \begin{pmatrix} \lambda & u \\ v & -\lambda \end{pmatrix} y \tag{4}$$

with the constraints

$$u = -\langle q, q\rangle, \qquad v = \langle p, p\rangle,$$

$\lambda = \lambda_j$, and $y = (q_j, p_j)^{\mathrm{T}}$. We therefore call the Hamiltonian system (H) the *constrained AKNS* (c-AKNS) *flow*, which coincides with the nonlinearized AKNS system via the Lax-pair nonlinearization method [7].

3. The r-matrix and integrability

Let $L_1(\lambda) = L(\lambda) \otimes I$ and $L_2(\mu) = I \otimes L(\mu)$. Then the following theorem holds.

Theorem 1. *The Lax matrix $L(\lambda)$ defined by Eq.* (1) *satisfies the fundamental Poisson bracket*

$$\{L(\lambda) \overset{\otimes}{,} L(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] - [r_{21}(\mu, \lambda), L_2(\mu)], \tag{5}$$

where $r_{12}(\lambda, \mu)$ and $r_{21}(\mu, \lambda)$ are the standard r-matrices

$$r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P, \qquad r_{21}(\mu, \lambda) = P r_{12}(\mu, \lambda) P \tag{6}$$

with

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. The fundamental Poisson bracket $\{L(\lambda) \overset{\otimes}{,} L(\mu)\}$ is a $4{\times}4$ matrix [8] whose entrices are given by

$$\{L(\lambda) \overset{\otimes}{,} L(\mu)\}_{kl,mn} = \{L(\lambda)_{km}, L(\mu)_{ln}\}.$$

A direct calculation therefore leads to (5). Equation (6) is called an r-matrix because it satisfies the Yang–Baxter equation

$$[r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{kj}, r_{ik}] = 0, \quad i, j, k = 1, 2, 3,$$

which completes the proof.

Remark 1. In fact, because r-matrix relation (5) is concerned only with the commutator, the r-matrix $r_{12}(\lambda, \mu)$ can be also chosen as

$$r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P + I \otimes \widetilde{S}, \qquad \widetilde{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the elements a, b, c, and d can be arbitrary functions $a(\lambda, \mu, p, q)$, $b(\lambda, \mu, p, q)$, $c(\lambda, \mu, p, q)$, and $d(\lambda, \mu, p, q)$ in $C^\infty(R)$ with respect to the spectral parameters λ and μ and the dynamic variables p and q. This shows that for a given Lax matrix, the associated r-matrix is not uniquely defined. Here, we give the simplest case: $a = b = c = d = 0$, i.e., the c-AKNS flow has standard r-matrix (6), which is obviously nondynamic.

An immediate consequence of Eq. (5) is

$$\{L^2(\lambda) \overset{\otimes}{,} L^2(\mu)\} = [\bar{r}_{12}(\lambda, \mu), L_1(\lambda)] - [\bar{r}_{21}(\mu, \lambda), L_2(\mu)], \tag{7}$$

where

$$\bar{r}_{ij}(\lambda, \mu) = \sum_{k=0}^{1} \sum_{l=0}^{1} L_1^{1-k}(\lambda) L_2^{1-l}(\mu) \cdot r_{ij}(\lambda, \mu) \cdot L_1^k(\lambda) L_2^l(\mu), \quad ij = 12, 21.$$

Thus, Eq. (7) leads to

$$4\{\operatorname{Tr} L^2(\lambda), \operatorname{Tr} L^2(\mu)\} = \operatorname{Tr}\{L^2(\lambda) \overset{\otimes}{,} L^2(\mu)\} = \operatorname{Tr}\{L_1^2(\lambda) \overset{\otimes}{,} L_2^2(\mu)\} = 0,$$

which guarantees the involutivity of those integrals of motion obtained in Eq. (2). Therefore, we have

$$\{E_i, E_j\} = \{H, E_j\} = \{F_s, E_j\} = 0, \quad i, j = 1, 2, \dots, N, \quad s = 0, 1, 2, \dots,$$

where

$$F_s = \sum_{j=1}^{N} \lambda_j^s E_j = 2\langle \Lambda^s p, q\rangle - \sum_{j+k=s-1} (\langle \Lambda^j p, p\rangle \langle \Lambda^k q, q\rangle - \langle \Lambda^j p, q\rangle \langle \Lambda^k q, p\rangle). \tag{8}$$

In addition, $E_1, E_2, \dots, E_N$ are functionally independent on a certain region of R^{2N}; hence, we obtain the following theorem.

Theorem 2. *The c-AKNS flow given by* (3) *is completely integrable in the Liouville sense.*

Remark 2. Here, the c-AKNS flow is proved to be integrable from the standpoint of the r-matrix and Lax matrix and not the Lax pair.

4. The AKNS hierarchy and its involutive solution

This section deals with the AKNS hierarchy of NLEEs. We connect the ZS spectral problem [6] and the AKNS hierarchy [2] with the finite-dimensional Hamiltonian systems (H) and (F_s), $s = 0, 1, \ldots$. We start from ZS spectral problem (4), where λ is an eigenvalue, $y = (y_1, y_2)^{\mathrm{T}}$ is the corresponding vector eigenfunction, and u and v are two potentials that either decay at infinity or have periodic boundary conditions. Then the AKNS hierarchy of NLEEs associated with (4) is derived as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_s} = JG_s, \quad s = 0, 1, 2, \ldots, \tag{9}$$

where $\{G_s = J^{-1}KG_{s-1}\}_{s=0}^{\infty}$ is the Lenard sequence with $G_{-1} = (0,0)^{\mathrm{T}}$ and $G_0 = (v, u)^{\mathrm{T}}$, the two symmetric operators K and J are

$$K = \begin{pmatrix} 2u\partial^{-1}u & \partial - 2u\partial^{-1}v \\ \partial - 2v\partial^{-1}u & -2v\partial^{-1}v \end{pmatrix}, \qquad J = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{10}$$

($\partial = \partial/\partial x$ and $\partial\partial^{-1} = \partial^{-1}\partial = 1$). A representative equation ($s = 2$) of (9) is

$$u_t = -\frac{1}{2}u_{xx} + u^2 v, \qquad v_t = \frac{1}{2}v_{xx} - v^2 u, \qquad t = t_2. \tag{11}$$

We consider the Hamilton functions F_s defined by (8). The Poisson bracket $\{F_s, H\} = 0$, $s = 0, 1, \ldots$, implies that all the canonical Hamiltonian systems (F_s) and (H) are completely integrable in the Liouville sense. Therefore, their Hamilton flows commute with each other.

Let $(p(x,t_s), q(x,t_s))^{\mathrm{T}}$ be a solution of the consistent canonical Hamilton equations (H) and (F_s), called the *involutive solution* [9]. We then have the following theorem.

Theorem 3. *Higher-order AKNS equations* (9) *are satisfied by*

$$u = -\langle q(x,t_s), q(x,t_s)\rangle, \qquad v = \langle p(x,t_s), p(x,t_s)\rangle, \quad s = 0, 1, \ldots.$$

In particular, Eq. (11) *is satisfied by the solution*

$$u = -\langle q(x,t_2), q(x,t_2)\rangle, \qquad v = \langle p(x,t_2), p(x,t_2)\rangle, \tag{12}$$

where $(p(x,t_2), q(x,t_2))^{\mathrm{T}}$ *is the involutive solution of the consistent Hamiltonian systems* (H) *and* (F_2).

Proof. Taking Eq. (3), $q_{t_s} = \partial F_s/\partial p$, $p_{t_s} = -\partial F_s/\partial q$, and the key equality

$$K\begin{pmatrix} \langle p(x,t_s), p(x,t_s)\rangle \\ -\langle q(x,t_s), q(x,t_s)\rangle \end{pmatrix} = J\begin{pmatrix} \langle \Lambda p(x,t_s), p(x,t_s)\rangle \\ -\langle \Lambda q(x,t_s), q(x,t_s)\rangle \end{pmatrix},$$

where the operators K and J are defined by (10), into account, we can verify that higher-order AKNS equations (9) are satisfied by

$$u = -\langle q(x,t_s), q(x,t_s)\rangle, \qquad v = \langle p(x,t_s), p(x,t_s)\rangle, \quad s = 0, 1, \ldots.$$

5. Algebraic-geometric solution

In the following procedure, we obtain an explicit expression for Eq. (12), i.e., we derive an algebraic-geometric solution of AKNS equation (11). For this, we rewrite Lax matrix (1) as

$$L = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix},$$

where

$$A(\lambda) = 1 + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} p_j q_j, \qquad B(\lambda) = -\sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} q_j^2, \qquad C(\lambda) = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} p_j^2.$$

We can substitute the following fractional forms for $B(\lambda)$ and $C(\lambda)$:

$$B(\lambda) \equiv -\frac{\langle q, q\rangle Q_B(\lambda)}{K(\lambda)}, \qquad C(\lambda) \equiv \frac{\langle p, p\rangle Q_C(\lambda)}{K(\lambda)},$$

where

$$\langle q, q\rangle Q_B(\lambda) = \sum_{j=1}^{N} q_j^2 \prod_{k=1, k\neq j}^{N} (\lambda - \lambda_k),$$

$$\langle p, p\rangle Q_C(\lambda) = \sum_{j=1}^{N} p_j^2 \prod_{k=1, k\neq j}^{N} (\lambda - \lambda_k), \qquad K(\lambda) = \prod_{j=1}^{N} (\lambda - \lambda_j).$$

Respectively choosing $N-1$ distinct real zero points $\mu_1^B, \ldots, \mu_{N-1}^B$ and $\mu_1^C, \ldots, \mu_{N-1}^C$ of $Q_B(\lambda)$ and $Q_C(\lambda)$ leads to

$$Q_B(\lambda) = \prod_{j=1}^{N-1} (\lambda - \mu_j^B), \qquad Q_C(\lambda) = \prod_{j=1}^{N-1} (\lambda - \mu_j^C),$$

$$\frac{\langle \Lambda q, q\rangle}{\langle q, q\rangle} = A_1 - \sum_{k=1}^{N-1} \mu_k^B, \tag{13}$$

$$\frac{\langle \Lambda p, p\rangle}{\langle p, p\rangle} = A_1 - \sum_{k=1}^{N-1} \mu_k^C, \tag{14}$$

$$\frac{\langle \Lambda^2 q, q\rangle}{\langle q, q\rangle} = A_1 \frac{\langle \Lambda q, q\rangle}{\langle q, q\rangle} - A_2 + \sum_{k,j=1, j<k}^{N-1} \mu_j^B \mu_k^B, \tag{15}$$

$$\frac{\langle \Lambda^2 p, p\rangle}{\langle p, p\rangle} = A_1 \frac{\langle \Lambda p, p\rangle}{\langle p, p\rangle} - A_2 + \sum_{k,j=1, j<k}^{N-1} \mu_j^C \mu_k^C, \tag{16}$$

where the two constants are

$$A_1 = \sum_{j=1}^{N} \lambda_j, \qquad A_2 = \sum_{k,j=1, j<k}^{N} \lambda_j \lambda_k.$$

Evidently, (13), (14), (15), (16) are equivalent to

$$
\begin{aligned}
&\sum_{k=1}^{N-1} \mu_k^B = A_1 - \frac{\langle \Lambda q, q\rangle}{\langle q, q\rangle}, \\
&\sum_{k=1}^{N-1} \mu_k^C = A_1 - \frac{\langle \Lambda p, p\rangle}{\langle p, p\rangle}, \\
&\left(A_1 - \sum_{k=1}^{N-1} \mu_k^B\right)^2 - \sum_{k=1}^{N-1} \left(\mu_k^B\right)^2 = 2A_2 - A_1^2 + 2\frac{\langle \Lambda^2 q, q\rangle}{\langle q, q\rangle}, \\
&\left(A_1 - \sum_{k=1}^{N-1} \mu_k^C\right)^2 - \sum_{k=1}^{N-1} \left(\mu_k^C\right)^2 = 2A_2 - A_1^2 + 2\frac{\langle \Lambda^2 p, p\rangle}{\langle p, p\rangle}.
\end{aligned}
\tag{17}
$$

On one hand, $u_x = -2\langle q, q_x\rangle = -2\langle q, \partial H/\partial p\rangle = -2\langle \Lambda q, q\rangle - 2uc_0(t)$, where the function $c_0(t)$ is only dependent on t. From Eq. (13), we thus obtain

$$\frac{\partial}{\partial x}\log u = 2A_1 - 2\sum_{k=1}^{N-1} \mu_k^B - 2c_0(t).$$

On the other hand, $u_{t_2} = -2\langle q, q_{t_2}\rangle = -2\langle q, \partial F_2/\partial p\rangle = -2\langle \Lambda^2 q, q\rangle$. Combined with Eq. (17), this gives the equality

$$\frac{\partial}{\partial t_2}\log u = \left(A_1 - \sum_{k=1}^{N-1} \mu_k^B\right)^2 - \sum_{k=1}^{N-1} (\mu_k^B)^2 - 2A_2 + A_1^2.$$

We thus obtain

$$
\begin{aligned}
u(x,t) = u(x_0,t_0)\exp\Bigg(&\int_{t_0}^{t}\left[\left(A_1 - \sum_{k=1}^{N-1} \mu_k^B\right)^2 - \sum_{k=1}^{N-1} \left(\mu_k^B\right)^2 - 2A_2 + A_1^2\right]dt + \\
&+ \int_{x_0}^{x}\left[2A_1 - 2\sum_{k=1}^{N-1} \mu_k^B - 2c_0(t)\right]dx\Bigg), \quad t = t_2,
\end{aligned}
\tag{18}
$$

where x_0 and t_0 are two fixed initial values. Similarly, $v(x,t)$ has the representation

$$
\begin{aligned}
v(x,t) = v(x_0,t_0)\exp\Bigg(&-\int_{t_0}^{t}\left[\left(A_1 - \sum_{k=1}^{N-1} \mu_k^C\right)^2 - \sum_{k=1}^{N-1} \left(\mu_k^C\right)^2 - 2A_2 + A_1^2\right]dt - \\
&- \int_{x_0}^{x}\left[2A_1 - 2\sum_{k=1}^{N-1} \mu_k^C - 2c_0(t)\right]dx\Bigg), \quad t = t_2.
\end{aligned}
\tag{19}
$$

Because Eqs. (18) and (19) solve nonlinear soliton equation (11), we only need to calculate the four key expressions $\sum_{j=1}^{N-1}(\mu_j^J)^k$, $J = B, C$ and $k = 1, 2$, to obtain their explicit form. For this, we follow the approach in the case of the Toda lattice equation [10, 11]. For the two sets of Darboux coordinates μ_j^J, $J = B, C$ and $j = 1, \ldots, N-1$, we then have the key equalities

$$\sum_{j=1}^{N-1}(\mu_j^J)^k = C_k(\Gamma) - \sum_{s=1}^{2} \operatorname*{Res}_{\lambda=\infty_s} \lambda^k d\log\Theta(A(P) - \phi - K_J), \quad J = B, C, \quad k = 1, \ldots, N-1,$$

where $C_k(\Gamma)$ is a constant [10, 12] only determined by the compact Riemann surface Γ of genus $N-1$, $\mu^2 = P(\lambda)K(\lambda)$,

$$P(\lambda) = K(\lambda) + \sum_{j=1}^{N} E_j \prod_{k\neq j,k=1}^{N} (\lambda - \lambda_k)$$

$$\infty_1 = (0, \sqrt{P(z^{-1})K(z^{-1})}|_{z=0}),$$

$$\infty_2 = (0, -\sqrt{P(z^{-1})K(z^{-1})}|_{z=0}),$$

$A(P) = \int_{P_0}^{P} \omega$ is the Abel map in which P_0 is an arbitrarily chosen point on Γ, $\omega = (\omega_1, \ldots, \omega_{N-1})^{\mathrm{T}}$,

$$\omega_j = \sum_{l=1}^{N-1} r_{j,l}\widetilde{\omega}_l = \sum_{l=1}^{N-1} r_{j,l} \frac{\prod_{k\neq l,k=1}^{N} (\lambda - \lambda_k)}{2\sqrt{K(\lambda)P(\lambda)}} d\lambda$$

is a normalized holomorphic differential form, and $r_{j,l}$ is the normalized factor. The jth component $\phi_j(x,t)$ of the $(N-1)$-dimensional vector ϕ is equal to

$$\sum_{l=1}^{N-1} r_{j,l} \left(Q_l^0 + \frac{\lambda_l x}{2} + \frac{\lambda_l^2 t}{2} + C_l(t) + \widetilde{C}_l(x) \right)$$

with the arbitrary constant Q_l^0 and functions $C_l(t), \widetilde{C}_l(x) \in C^\infty(R)$. The vectors K_B and K_C in $\mathbb{C}^{N-1}$ are the two Riemann constant vectors respectively associated with the Darboux coordinates μ_j^B and μ_j^C. The Riemann theta function [13] $\Theta(\xi)$ is defined on the Riemann surface Γ.

Calculating the residue at ∞_s, $s = 1, 2$, for $k = 1, 2$ yields

$$\sum_{j=1}^{N-1} \mu_j^J = C_1(\Gamma) - \frac{\partial}{\partial x} \log \frac{\Theta_1^J}{\Theta_2^J},$$

$$\sum_{j=1}^{N-1} (\mu_j^J)^2 = C_2(\Gamma) + \frac{\partial}{\partial t} \log \frac{\Theta_1^J}{\Theta_2^J} - \frac{\partial^2}{\partial x^2} \log \Theta_1^J \Theta_2^J,$$

where $\Theta_s^J = \Theta(\phi + K_J + \eta_s)$, $J = B, C$, and

$$\eta_{s,j} = \int_{\infty_s}^{P_0} \omega_j, \quad s = 1, 2,$$

is the jth component of the $(N-1)$-dimensional vector η_s.

Substituting the above equalities in (18) and (19) and sorting them, we obtain the explicit solution of soliton equation (11):

$$u(x,t) = u(x_0,t_0) e^{a(t-t_0)+2(b-c_0)(x-x_0)} \left. \frac{\Theta_1^B}{\Theta_2^B} \right|_{t=t_0} \left. \left(\frac{\Theta_2^B}{\Theta_1^B} \right)^2 \right|_{x=x_0} \times$$
$$\times \frac{\Theta_1^B}{\Theta_2^B} \exp\left(\int_{t_0}^{t} \left[\frac{\partial^2}{\partial x^2} \log \Theta_1^B \Theta_2^B + \left(b + \frac{\partial}{\partial x} \log \frac{\Theta_1^B}{\Theta_2^B} \right)^2 \right] dt \right), \tag{20}$$

$$v(x,t) = v(x_0,t_0) e^{-a(t-t_0)-2(b-c_0)(x-x_0)} \left. \frac{\Theta_2^C}{\Theta_1^C} \right|_{t=t_0} \left. \left(\frac{\Theta_1^C}{\Theta_2^C} \right)^2 \right|_{x=x_0} \times$$
$$\times \frac{\Theta_2^C}{\Theta_1^C} \exp\left(\int_{t_0}^{t} \left[\frac{\partial^2}{\partial x^2} \log \Theta_2^C \Theta_1^C + \left(b + \frac{\partial}{\partial x} \log \frac{\Theta_2^C}{\Theta_1^C} \right)^2 \right] dt \right), \tag{21}$$

where $a = A_1^2 - C_2(\Gamma) - 2A_2$ and $b = A_1 - C_1(\Gamma)$ are two constants, $c_0 = c_0(t) \in C^\infty(R)$ is a given function of t, and x_0 and t_0 are the initial values. Therefore, we obtain the following theorem.

Theorem 4. *AKNS equations* (11) *have a pair of explicit solutions* (20) *and* (21) *given by the form of the Riemann theta function, which are called the* algebraic-geometric solution.

An analogous calculation process can also yield the algebraic-geometric solution of higher-order AKNS equations (9). But that is a more complicated case, which we omit here.

6. Conclusion

A motivation for writing this paper originates from our work on finite-dimensional integrable systems described in [10]. Indeed, with the example of the AKNS equations, we show that a procedure for algebraic-geometric solutions is successfully extended from finite-dimensional integrable systems to integrable NLEEs or soliton equations. This procedure can be applied into other NLEEs. Zhou [12] gave the algebraic-geometric solution of the Jaulent–Miodek equation. Afterwards, Zhang [14] and Du [15] also obtained the algebraic-geometric solutions of some soliton equations using this method. Of course, there are other methods for solving soliton equations or NLEEs. Recently, Deift, Its, and Zhou [16] obtained the Θ-function solutions of some NLEEs such as the KdV, MKdV, and nonlinear Schrödinger equation using the Riemann–Hilbert asymptotic method. All these methods are still under development.

Acknowledgments. The author sincerely thanks the Fachbereich 17 of the University-GH Kassel and Professor Strampp and Professor Varnhorn in particular for their warm invitation and hospitality.

This work was supported by the Research Program of the Los Alamos National Laboratory (USA), the Alexander von Humboldt Foundation (Germany), the Chinese National Basic Reseach Project "Nonlinear Science," the Special Grant of Chinese Excellent Doctoral Dissertation and Doctoral Program Foundation of the Education Ministry of China.

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