REPRESENTATIO OF THE LAX SYSTEM FOR LEVI HIERARCHY\textsuperscript{*}

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In this letter, we shall study Levi eigenvalue problem:

\[ L_{\varphi} = \lambda/2 \cdot \varphi; \quad L(u) = \begin{pmatrix} \partial + \frac{q-r}{2} & q \\ -r & \partial + \frac{q-r}{2} \end{pmatrix}, \]  
\[(1)\]

where \( u = (q, r)^T, \varphi = (\varphi_1, \varphi_2)^T, \partial = \partial/\partial x. \)

\( L: u \mapsto L(u) \) is the mapping from potential function into differential operator.

\textbf{Definition.} The differential of the mapping \( L \) is defined as

\[ L_{\ast u}[\xi] = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} L(u + \varepsilon \xi). \]  
\[(2)\]

\textbf{Lemma.} The differential of \( L \) for Levi hierarchy is

\[ L_{\ast}[\xi] = \begin{pmatrix} (\xi^1 - \xi^2)/2 & \xi^1 \\ -\xi^2 & (\xi^1 - \xi^2)/2 \end{pmatrix}. \]  
\[(3)\]

and \( L_{\ast} \) is injective homomorphism.

\textbf{Proposition 1.} The functional gradient \( \nabla \lambda \) of the eigenvalue \( \lambda \) for Eq. \((1)\) is

\[ \nabla \lambda = \text{grad} \lambda \triangleq \left( \frac{\delta \lambda}{\delta q}, \frac{\delta \lambda}{\delta r} \right) = \left( \begin{pmatrix} \varphi_1 + \varphi_2 \end{pmatrix} \varphi_1, -\varphi_1 \right) \cdot \left( \int_{\Omega} \varphi_1 \cdot \varphi_2 dx \right)^{-1}, \]  
\[(4)\]

where \((\varphi_1, \varphi_2)\) is the eigenvalue function corresponding to \( \lambda \) of Eq. \((1)\). \( \Omega \) is the interval discussed in this letter.

\textbf{Proposition 2.} Let \( \lambda \) be an eigenvalue of Eq. \((1)\). Then \( \nabla \lambda \) is satisfied with

\[ K \nabla \lambda = \lambda \cdot J \nabla \lambda \]  
\[(5)\]

where

\[ K = \begin{pmatrix} -\varphi_1 \cdot \varphi_2 & \varphi_1 \cdot \varphi_2 \\ \varphi_1 \cdot \varphi_2 & \varphi_1 \cdot \varphi_2 \end{pmatrix}; \quad J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \]

\( k \) and \( J \) are skew-symmetric; \( J \) is a symplectic operator, \( K \) and \( J \) are called the pair of Lenard’s operator.

\textbf{Theorem 1.} Let \( G^{(1)}(x) \) and \( G^{(2)}(x) \) be arbitrary smooth functions, \( G = (G^{(1)}, G^{(2)})^T \). Let

\[ V = \begin{pmatrix} -\frac{1}{2} \partial (G^{(2)} + G^{(1)}) + (G^{(2)} \cdot G^{(1)}) \partial \\ \partial G^{(1)} \\ \frac{1}{2} \partial (G^{(1)} + G^{(2)}) + (G^{(2)} - G^{(1)}) \partial \end{pmatrix}, \]

then

\[ [V, L] \triangleq \nabla L - L V = L^*(KG) - L^*(JG) \cdot (2L). \]  
\[(6)\]

Define the Lenard sequence recursively: \( G_0 = \begin{pmatrix} 0 \end{pmatrix}, \quad KG_j = JG_{j+1}, \quad (j = -1, 0, 1, \ldots) \). \( G_j(x) \) is polynomial of \( q(x), r(x) \) and their derivatives\textsuperscript{2} and is unique if its constant term is required to be zero. \( X_m = JG_m \) \((m = 0, 1, \ldots)\) is the Levi vector field.
Theorem 2. Let \( G_j = (G_j^{(1)}, G_j^{(2)})^T \) be the Lenard sequence, and

\[
V_j = \begin{pmatrix}
-\frac{1}{2} \partial (G_j^{(1)} + G_j^{(2)}) + (G_j^{(2)} - G_j^{(1)}) \partial & -\partial G_j^{(2)} \\
\partial G_j^{(1)} & \frac{1}{2} \partial (G_j^{(1)} + G_j^{(2)}) + (G_j^{(2)} - G_j^{(1)}) \partial
\end{pmatrix}.
\]

Then

\[
[W_m, L] = L_\cdot (X_m),
\]

where \( W_m = \sum_{j=0}^m V_{j-1} (2L)^{m-j} \).

Corollary 1. The Levi equation \( u_r = \begin{pmatrix} q \\ r \end{pmatrix} = X_m \) if and only if \( L = [W_m, L] \) (\( m = 0, 1, \cdots \)).

Corollary 2. Potential function \( u(x) = \begin{pmatrix} q(x) \\ r(x) \end{pmatrix} \) is satisfied with stationary Levi system:

\[
X_N + a_1 X_{N-1} + \cdots + a_N X = 0 \text{ if and only if } \quad [w_N + a_1 W_{N-1} + \cdots + a_N W_0, L] = 0,
\]

where \( a_1, \cdots, a_N \) are constants.

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REFERENCES


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THE PSEUDO-EINSTEIN STRUCTURES ON THE CR MANIFOLD
WITH A POSITIVE WEBSTER-RICCI TENSOR

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Let \((M, \theta)\) be a strictly pseudo-convex CR manifold. Choose an admissible coframe \(\{\theta^a\}\), such that

\[
d\theta = i_{ah_{\bar{a}}} \theta^a \wedge \theta^{\bar{a}},
\]

where \((h_{a\bar{b}})\) is a positive Hermitian matrix. \(\theta\) is a pseudo-Einstein structure on \(M\) if the Webster-Ricci tensor \[^1\] is a scalar multiple of the Levi form, namely, \(R_{a\bar{b}} = f h_{a\bar{b}}\) for a function on \(M\).

Lee \[^2\] showed that if \(M\) is compact and strictly pseudo-convex, then in a neighborhood of every point of \(M\) there exists an associated pseudo-Einstein structure. For the global existence, he also showed that if the first Chern class of the holomorphic tangent bundle on \(M\) vanishes and the Webster Ricci tensor is nonnegative, then \(M\) admits a global pseudo-Einstein structure.