REPRESENTATIO OF THE LAX SYSTEM FOR LEVI HIERARCHY*

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In this letter, We shall study Levi eigenvalue problem:

$$L_{\varphi}=\lambda/2 \cdot \varphi;\ L(u)=\begin{pmatrix} \partial+\frac{q-r}{2} & q \\ -r & \partial+\frac{q-r}{2} \end{pmatrix}, \tag{1}$$

where $u=(q, r)^T$, $\varphi=(\varphi_1, \varphi_2)^T$, $\partial=\partial/\partial x$.

L: $u \mapsto L(u)$ is the mapping from potential function into differential operator.

Definition[1]. The differential of the mapping L is defined as

$$L_{*u}[\xi]=\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L(u+\varepsilon\xi). \tag{2}$$

Lemma. *The differential of L for Levi hierarchy is*

$$L_*[\xi]=\begin{pmatrix} (\xi^1-\xi^2)/2 & \xi^1 \\ -\xi^2 & (\xi^1-\xi^2)/2 \end{pmatrix}, \tag{3}$$

and L_* *is injective homomorphism.*

Proposition 1. *The functional gradient* $\nabla\lambda$ *of the eigenvalue* λ *for Eq.* (1) *is*

$$\nabla\lambda=\mathrm{grad}\,\lambda \triangleq \begin{pmatrix} \delta\lambda/\delta q \\ \delta\lambda/\delta r \end{pmatrix}=\begin{pmatrix} (\varphi_1+\varphi_2)\varphi_2 \\ -(\varphi_1+\varphi_2)\varphi_1 \end{pmatrix} \cdot \left(\int_\Omega \varphi_1 \cdot \varphi_2 dx\right)^{-1}, \tag{4}$$

where (φ_1, φ_2) *is the eigenvalue function corresponding to* λ *of Eq.* (1), Ω *is the interval discussed in this letter.*

Proposition 2. *Let* λ *be an eigenvalue of Eq.* (1). *Then* $\nabla\lambda$ *is satisfied with*

$$K\nabla\lambda=\lambda \cdot J\nabla\lambda \tag{5}$$

where

$$K=\begin{pmatrix} -q\partial-\partial q & -\partial^2+\partial q-r\partial \\ \partial^2-\partial r+q\partial & r\partial+\partial r \end{pmatrix};\ J=\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix},$$

k and J are skew-symmetric; J is a symplectic operator, K and J are called the pair of Lenard's operater.

Theorem 1. *Let* $G^{(1)}(x)$ *and* $G^{(2)}(x)$ *be arbitrary smooth functions,* $G=(G^{(1)},G^{(2)})^T$. *Let*

$$V=\begin{pmatrix} -\frac{1}{2}\partial(G^{(2)}+G^{(1)})+(G^{(2)}-G^{(1)})\partial & -\partial G^{(2)} \\ \partial G^{(1)} & \frac{1}{2}\partial(G^{(1)}+G^{(2)})+(G^{(2)}-G^{(1)})\partial \end{pmatrix},$$

then

$$[V, L] \triangleq VL-LV=L^*(KG)-L^*(JG)\cdot(2L). \tag{6}$$

Define the Lenard sequence recursively: $G_{-1}=\binom{0}{1}$, $KG_j=JG_{j+1}$, $(j=-1, 0, 1,\cdots)$, $G_j(x)$ *is polynomial of* $q(x)$, $r(x)$ *and their derivatives*[2] *and is unique if its constant term is required to be zero.* $X_m=JG_m(m=0, 1,\cdots)$ *is the Levi vector field.*

Theorem 2. *Let $G_j=(G_j^{(1)},G_j^{(2)})^T$ be the Lenard sequence, and*

$$V_j=\begin{pmatrix} -\frac{1}{2}\partial(G_j^{(1)}+G_j^{(2)})+(G_j^{(2)}-G_j^{(1)})\partial & -\partial G_j^{(2)} \\ \partial G_j^{(1)} & \frac{1}{2}\partial(G_j^{(1)}+G_j^{(2)})+(G_j^{(2)}-G_j^{(1)})\partial \end{pmatrix}.$$

Then

$$[W_m, L]=L_*(X_m), \tag{7}$$

where $W_m=\sum_{j=0}^{m}V_{j-1}(2L)^{m-j}$.

Corollary 1. *The Levi equation* $u_t=\begin{pmatrix} q \\ r \end{pmatrix}_t=X_m$ *if and only if* $L_t=[W_m, L]$. $(m=0, 1,\cdots)$.

Corollary 2. *Potential function* $u(x)=\begin{pmatrix} q(x) \\ r(x) \end{pmatrix}$ *is satisfied with stationary Levi system:* $X_N+a_1X_{N-1}+\cdots+a_NX=0$ *if and only if*

$$[w_N+a_1W_{N-1}+\cdots+a_NW_0, L]=0, \tag{8}$$

where $a_1, \cdots, a_N$ *are constants.*

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THE PSEUDO-EINSTEIN STRUCTURES ON THE CR MANIFOLD WITH A POSITIVE WEBSTER-RICCI TENSOR

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Let (M, θ) be a strictly pseudo-convex CR manifold. Choose an admissible coframe $\{\theta^\alpha\}$, such that

$$d\theta=ih_{\alpha\bar{\beta}}\theta^\alpha\wedge\theta^{\bar{\beta}},$$

where $(h_{\alpha\bar{\beta}})$ is a positive Hermitian matrix. θ is a pseudo-Einstein structure on M if the Webster-Ricci tensor[1] is a scalar multiple of the Levi form, namely, $R_{\alpha\bar{\beta}}=f\,h_{\alpha\bar{\beta}}$ for a function on M.

Lee[2] showed that if M is compact and strictly paeudo-convex, then in a neighborhood of every point of M there exists an associated pseudo-Einstein structure. For the global existence, he also showed that if the first Chern class of the holomorphic tangent bundle on M vanishes and the Webster Ricci tensor is nonnegative, then m admits a global pseudo-Einstein structure.