A NOTE ON $r$-MATRIX OF THE PEAKON DYNAMICS

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Abstract
This paper deals with the $r$-matrix of the peakon dynamical systems. Our result shows that there does not exist constant $r$-matrix for the peakon dynamical system.

Keywords: $r$-matrix Structure, Peakon Systems.

AMS Subject: 35Q53; 58F07; 35Q35

PACS: 03.40.Gc; 03.40Kf; 47.10.+g

In 1993, Camassa and Holm proposed a shallow water equation and discussed the peaked-soliton (peakon) solution of the equation [1]. Later in 1996, Ragnisco and Bruschi [2] showed the integrability of the finite-dimensional peakon system through constructing a constant $r$-matrix. Their starting point is the following Lax matrix (1). The $r$-matrix is usually dynamical in the framework of the $r$-matrix approach located in the fundamental Poisson bracket [3]. Ragnisco and Bruschi claimed that for a particular choice of the relevant parameters in the Hamiltonian (the one corresponding to the pure peakons case) the $r$-matrix becomes essentially constant [2]. In ref. [4], Qiao extended the Camassa-Holm (CH) equation to the whole integrable CH hierarchy, including positive and negative members in the hierarchy, and studied $r$-matrix structures of the constrained CH systems and algebraic-geometric solutions on a symplectic submanifold through using the constraint approach [5]. In this note, what we want to show is no constant $r$-matrix for the CH peakon system. Let us discuss below.

For the peakon system, let us consider the Lax matrix which is given in ref. [2]:

$$L = \sum_{i,j=1}^{N} L_{ij} E_{ij}$$  \hspace{1cm} (1)

where

$$L_{ij} = \sqrt{p_i p_j} A_{ij},$$  \hspace{1cm} (2)

$$A_{ij} = A(q_i - q_j) = e^{-\frac{1}{2}|q_i - q_j|}.$$  \hspace{1cm} (3)

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In Eq. (3),
\[ A(x) = e^{-\frac{1}{2}|x|}, \tag{4} \]
and \( A(x) \) has the following properties:
\[
\begin{align*}
A'(x) &= -\frac{1}{2} \text{sgn}(x) A(x), \\
A_{ij} &= A_{ji}, \quad A_{ii} = 1, \\
A'_{ij} &= A'(q_i - q_j) = -A'(q_j - q_i) = -A_{ji}', \quad A_{ii}' = 0,
\end{align*}
\]
\[
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) A(x) A(y) = A'(x) A(y) + A(x) A'(y) \\
&= -\frac{1}{2} A(x) A(y) [\text{sgn}(x) + \text{sgn}(y)], \\
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) A(x) A(y)|_{y=-x} &= 0.
\]

We work with the matrix basis \( E_{ij} \):
\[
(E_{ij})_{kl} = \delta_{ik} \delta_{jl}, \quad i, j, k, l = 1, \ldots, N.
\]

To have the r-matrix structure, we consider the so-called fundamental Poisson bracket [3]:
\[
\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2], \tag{5}
\]
where
\[
\begin{align*}
L_1 &= L \otimes 1 = \sum_{i,j=1}^{N} L_{ij} E_{ij} \otimes 1, \\
L_2 &= 1 \otimes L = \sum_{k,l=1}^{N} L_{kl} 1 \otimes E_{kl}, \\
r_{12} &= \sum_{l,k=1}^{N} r_{lk} E_{lk} \otimes (E_{lk} + E_{kl}), \\
r_{21} &= \sum_{l,k=1}^{N} r_{lk} (E_{lk} + E_{kl}) \otimes E_{lk}, \\
\{L_1, L_2\} &= \sum_{i,j,k,l=1}^{N} \{L_{ij}, L_{kl}\} E_{ij} \otimes E_{kl}.
\end{align*}
\]
Here \( \{L_{ij}, L_{kl}\} \) is of sense under the standard Poisson bracket of two functions, \( 1 \) is the \( N \times N \) unit matrix, and \( r_{lk} \) are to be determined. In Eq. (5), \([\cdot, \cdot]\) means the usual commutator of matrix.
Now, let us calculate the left hand side of Eq. (5).

\[
\frac{\partial L_{ij}}{\partial q_m} = \sqrt{p_i p_j} A'_{ij} (\delta_{im} - \delta_{jm})
\]
\[
\frac{\partial L_{kl}}{\partial p_m} = \frac{A_{kl}}{2 \sqrt{p_k p_l}} (p_l \delta_{km} + p_k \delta_{lm})
\]

\[
\{L_{ij}, L_{kl}\} = \sum_{m=1}^{N} \left( \frac{\partial L_{ij}}{\partial q_m} \frac{\partial L_{kl}}{\partial q_m} - \frac{\partial L_{ij}}{\partial q_m} \frac{\partial L_{kl}}{\partial p_m} \right)
\]
\[
= \frac{1}{2} \sum_{j,k,l=1}^{N} \left[ \sqrt{p_j p_l} A'_{ij} A_{kl} (\delta_{im} - \delta_{jm}) (p_l \delta_{km} + p_k \delta_{lm})
\right.
\]
\[
- \sqrt{p_k p_l} A'_{ij} A_{kl} (\delta_{im} - \delta_{jm}) (p_l \delta_{km} + p_k \delta_{lm})
\left. \right] + \frac{1}{2} \sqrt{p_j p_l} \delta_{ik} \left( \sqrt{p_i p_k} A'_{ij} A_{kl} + \sqrt{p_i p_l} A'_{ij} A_{kl} \right)
\]
\[
- \frac{1}{2} \sqrt{p_k p_l} \delta_{ij} \left( \sqrt{p_i p_k} A'_{ij} A_{kl} + \sqrt{p_i p_l} A'_{ij} A_{kl} \right) + \frac{1}{2} \sqrt{p_j p_l} \delta_{ij} \left( \sqrt{p_i p_k} A'_{ij} A_{kl} - \sqrt{p_i p_l} A'_{ij} A_{kl} \right),
\]

where the subscript \( \cdot \) means \( A'(x) \) with the argument.

Thus, we obtain

\[
\{L_1, L_2\} = \sum_{i,j,k,l=1}^{N} \{L_{ij}, L_{kl}\} E_{ij} \otimes E_{kl}
\]
\[
= \frac{1}{2} \sum_{j,k,l=1}^{N} \left[ \sqrt{p_j p_l} A'_{ij} A_{kl} (E_{ij} \otimes E_{kl} - E_{kl} \otimes E_{ij})
\right.
\]
\[
+ \sqrt{p_k p_l} A'_{ij} A_{kl} (E_{ik} \otimes E_{lj} - E_{lj} \otimes E_{ik})
\]
\[
+ \sqrt{p_k p_l} A'_{ij} A_{kl} (E_{ik} \otimes E_{lj} - E_{lj} \otimes E_{ik})
\left. \right].
\]

Next, we compute the right hand side of Eq. (5). Before doing that, let us give some simple tensor product of the matrix basis \( E_{ij} \):

\[
(E_{ij} \otimes E_{st}) (E_{kl} \otimes 1) = \delta_{jk} E_{il} \otimes E_{st},
\]
\[
(E_{kl} \otimes 1) (E_{ij} \otimes E_{st}) = \delta_{il} E_{kj} \otimes E_{st},
\]
\[
(E_{ij} \otimes E_{st}) (1 \otimes E_{kl}) = \delta_{ks} E_{ij} \otimes E_{lt},
\]
\[
(1 \otimes E_{kl}) (E_{ij} \otimes E_{st}) = \delta_{ls} E_{ij} \otimes E_{kt},
\]
\[
E_{kl} E_{st} = \delta_{ls} E_{kt}.
\]

So, we have

\[
[r_{12}, L_1] - [r_{21}, L_2]
\]
\[
= \sum_{i,j,k,l=1}^{N} r_{jk} L_{ij} \left( [E_{ik} \otimes (E_{ik} + E_{kl}), E_{ij} \otimes 1] - [(E_{ik} + E_{kl}) \otimes E_{ik}, 1 \otimes E_{ij}] \right)
\]
\[
= \sum_{i,j,k,l=1}^{N} r_{ik} L_{ij} \left( \delta_{ik} E_{lj} \otimes (E_{lk} + E_{kl}) - \delta_{ik} (E_{lk} + E_{kl}) \otimes E_{lj} \\
+ \delta_{jl} (E_{lk} + E_{kl}) \otimes E_{ik} - \delta_{jl} E_{ik} \otimes (E_{lk} + E_{kl}) \right)
\]
\[
= \sum_{j,k,l=1}^{N} r_{lk} L_{kj} \left( E_{lj} \otimes (E_{lk} + E_{kl}) - (E_{lk} + E_{kl}) \otimes E_{lj} \right) \\
+ \sum_{j,k,l=1}^{N} r_{kl} L_{jk} \left( - E_{jl} \otimes (E_{lk} + E_{kl}) + (E_{lk} + E_{kl}) \otimes E_{jl} \right)
\]
\[
= \sum_{j,k,l=1}^{N} r_{lk} L_{jk} \left( E_{lj} \otimes (E_{lk} + E_{kl}) - (E_{lk} + E_{kl}) \otimes E_{lj} \right),
\]
where we set \( r_{ik} = -r_{kl} \) and used \( L_{jk} = L_{kj} \).

After comparing both sides of the fundamental Poisson bracket (5), we should have the following 2 equalities:

\[
\begin{align*}
r_{lk} &= \frac{1}{2} \frac{A'_{kl} A_{jl}}{A_{jk}}, \\
r_{lj} - r_{lk} &= \frac{1}{2} \frac{(A_{lk} A_{jl})'}{A_{jk}}.
\end{align*}
\]

In fact, the 2nd one is a natural result derived from the 1st one. Thus, for the CH peakons case we have

\[
\begin{align*}
r_{lk} &= \frac{1}{2} \frac{A'_{kl} A_{jl}}{A_{jk}} \\
&= \frac{1}{4} \operatorname{sgn}(q_k - q_l) \frac{A_{kl} A_{jl}}{A_{jk}} \\
&= \frac{1}{4} \operatorname{sgn}(q_l - q_k) e^{-\frac{1}{2}(|q_l - q_k| + |q_j - q_l| - |q_j - q_k|)}, \forall j \in \mathbb{Z}^+.
\end{align*}
\]

This equality holds for arbitrary \( j \in \mathbb{Z}^+ \). Obviously, only in the cases of \( j > l > k \) or \( j < l < k \) Eq. (8) becomes constant, namely, \( \pm \frac{1}{4} \). But for other \( j \), apparently Eq. (8) is NOT constant.

So, we think that the constant matrix given in ref. [2]

\[
r_{12} = a \sum_{l,k=1}^{N} \operatorname{sgn}(q_l - q_k) E_{lk} \otimes (E_{lk} + E_{kl}), \quad a = \text{constant}
\]
is not an \( r \)-matrix for the CH peakon dynamical system.

Discussions for more general case of Lax matrix are seen in ref. [6].

Acknowledgments

This work has been supported by the UTPA-FRC.
References


