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A NOTE ON r-MATRIX OF THE PEAKON DYNAMICS

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Abstract

This paper deals with the r-matrix of the peakon dynamical systems. Our result shows that there does not exist constant r-matrix for the peakon dynamical system.

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In 1993, Camassa and Holm proposed a shallow water equation and discussed the peaked-soliton (peakon) solution of the equation [1]. Later in 1996, Ragnisco and Bruschi [2] showed the integrability of the finite-dimensional peakon system through constructing a constant r-matrix. Their starting point is the following Lax matrix (1). The r-matrix is usually dynamical in the framework of the r-matrix approach located in the fundamental Poisson bracket [3]. Ragnisco and Bruschi claimed that for a particular choice of the relevant parameters in the Hamiltonian (the one corresponding to the pure peakons case) the r-matrix becomes essentially constant [2]. In ref. [4], Qiao extended the Camassa-Holm (CH) equation to the whole integrable CH hierarchy, including positive and negative members in the hierarchy, and studied r-matrix structures of the constrained CH systems and algebraic-geometric solutions on a symplectic submanifold through using the constraint approach [5]. In this note, what we want to show is no constant r-matrix for the CH peakon system. Let us discuss below.

For the peakon system, let us consider the Lax matrix which is given in ref. [2]:

$$L = \sum_{i,j=1}^{N} L_{ij}E_{ij} \tag{1}$$

where

$$L_{ij} = \sqrt{p_i p_j}A_{ij}, \tag{2}$$
$$A_{ij} = A(q_i - q_j) = e^{-\frac{1}{2}|q_i-q_j|}. \tag{3}$$

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In Eq. (3),

$$A(x) = e^{-\frac{1}{2}|x|}, \tag{4}$$

and $A(x)$ has the following properties:

$$\begin{aligned}
A'(x) &= -\frac{1}{2} sgn(x) A(x), \\
A_{ij} &= A_{ji}, \ A_{ii} = 1, \\
A'_{ij} &= A'(q_i - q_j) = -A'(q_j - q_i) = -A'_{ji}, \ A'_{ii} = 0,
\end{aligned}$$

$$\begin{aligned}
(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) A(x)A(y) &= A'(x)A(y) + A(x)A'(y) \\
&= -\frac{1}{2} A(x)A(y) \left[sgn(x) + sgn(y) \right], \\
(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) A(x)A(y)|_{y=-x} &= 0.
\end{aligned}$$

We work with the matrix basis E_{ij}:

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}, \ i, j, k, l = 1, ..., N.$$

To have the r-matrix structure, we consider the so-called fundamental Poisson bracket [3]:

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2], \tag{5}$$

where

$$\begin{aligned}
L_1 &= L \otimes \mathbf{1} = \sum_{i,j=1}^{N} L_{ij} E_{ij} \otimes \mathbf{1}, \\
L_2 &= \mathbf{1} \otimes L = \sum_{k,l=1}^{N} L_{kl} \mathbf{1} \otimes E_{kl}, \\
r_{12} &= \sum_{l,k=1}^{N} r_{lk} E_{lk} \otimes (E_{lk} + E_{kl}), \\
r_{21} &= \sum_{l,k=1}^{N} r_{lk} (E_{lk} + E_{kl}) \otimes E_{lk}, \\
\{L_1, L_2\} &= \sum_{i,j,k,l=1}^{N} \{L_{ij}, L_{kl}\} E_{ij} \otimes E_{kl}.
\end{aligned}$$

Here $\{L_{ij}, L_{kl}\}$ is of sense under the standard Poisson bracket of two functions, $\mathbf{1}$ is the $N \times N$ unit matrix, and r_{lk} are to be determined. In Eq. (5), $[\cdot, \cdot]$ means the usual commutator of matrix.

Now, let us calculate the left hand side of Eq. (5).

$$\begin{aligned}\frac{\partial L_{ij}}{\partial q_m} &= \sqrt{p_i p_j}A'_{ij}(\delta_{im}-\delta_{jm})\\ \frac{\partial L_{kl}}{\partial p_m} &= \frac{A_{kl}}{2\sqrt{p_k p_l}}(p_l\delta_{km}+p_k\delta_{lm})\end{aligned}$$

$$\begin{aligned}\{L_{ij},L_{kl}\} &= \sum_{m=1}^{N}\left(\frac{\partial L_{ij}}{\partial q_m}\frac{\partial L_{kl}}{\partial p_m}-\frac{\partial L_{kl}}{\partial q_m}\frac{\partial L_{ij}}{\partial p_m}\right)\\ &= \frac{1}{2}\sum_{m=1}^{N}\Big[\sqrt{p_i p_j}A'_{ij}\frac{A_{kl}}{\sqrt{p_k p_l}}(\delta_{im}-\delta_{jm})(p_l\delta_{km}+p_k\delta_{lm})\\ &\quad -\sqrt{p_k p_l}A'_{kl}\frac{A_{ij}}{\sqrt{p_i p_j}}(\delta_{km}-\delta_{lm})(p_j\delta_{im}+p_i\delta_{jm})\Big]\\ &= \frac{1}{2}\sqrt{p_j p_l}\delta_{ik}\left(\sqrt{\frac{p_i}{p_k}}A'_{ij}A_{kl}-\sqrt{\frac{p_k}{p_i}}A'_{kl}A_{ij}\right)+\frac{1}{2}\sqrt{p_j p_k}\delta_{il}\left(\sqrt{\frac{p_i}{p_l}}A'_{ij}A_{kl}+\sqrt{\frac{p_l}{p_i}}A'_{kl}A_{ij}\right)\\ &\quad -\frac{1}{2}\sqrt{p_i p_l}\delta_{jk}\left(\sqrt{\frac{p_j}{p_k}}A'_{ij}A_{kl}+\sqrt{\frac{p_k}{p_j}}A'_{kl}A_{ij}\right)-\frac{1}{2}\sqrt{p_i p_k}\delta_{jl}\left(\sqrt{\frac{p_j}{p_l}}A'_{ij}A_{kl}-\sqrt{\frac{p_l}{p_j}}A'_{kl}A_{ij}\right),\end{aligned}$$

where the supscript $'$ means $A'(x)$ with the argument.

Thus, we obtain

$$\begin{aligned}\{L_1,L_2\} &= \sum_{i,j,k,l=1}^{N}\{L_{ij},L_{kl}\}E_{ij}\otimes E_{kl}\\ &= \frac{1}{2}\sum_{j,k,l=1}^{N}\Big[\sqrt{p_k p_j}A'_{kl}A_{jl}(E_{jl}\otimes E_{kl}-E_{kl}\otimes E_{jl})\\ &\quad +\sqrt{p_k p_j}A'_{lk}A_{lj}(E_{lk}\otimes E_{lj}-E_{lj}\otimes E_{lk})\\ &\quad +\sqrt{p_k p_j}(A_{lk}A_{jl})'(E_{lk}\otimes E_{jl}-E_{jl}\otimes E_{lk})\Big].\end{aligned}$$

Next, we compute the right hand side of Eq. (5). Before doing that, let us give some simple tensor product of the matrix basis E_{ij}:

$$\begin{aligned}(E_{ij}\otimes E_{st})(E_{kl}\otimes \mathbf{1}) &= \delta_{jk}E_{il}\otimes E_{st},\\ (E_{kl}\otimes \mathbf{1})(E_{ij}\otimes E_{st}) &= \delta_{il}E_{kj}\otimes E_{st},\\ (E_{ij}\otimes E_{st})(\mathbf{1}\otimes E_{kl}) &= \delta_{tk}E_{ij}\otimes E_{sl},\\ (\mathbf{1}\otimes E_{kl})(E_{ij}\otimes E_{st}) &= \delta_{ls}E_{ij}\otimes E_{kt},\\ E_{kl}E_{st} &= \delta_{ls}E_{kt}.\end{aligned}$$

So, we have

$$\begin{aligned}&[r_{12},L_1]-[r_{21},L_2]\\ =&\sum_{i,j,k,l=1}^{N}r_{lk}L_{ij}\Big([E_{lk}\otimes(E_{lk}+E_{kl}),E_{ij}\otimes\mathbf{1}]-[(E_{lk}+E_{kl})\otimes E_{lk},\mathbf{1}\otimes E_{ij}]\Big)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j,k,l=1}^{N} r_{lk}L_{ij}\Big(\delta_{ik}E_{lj}\otimes(E_{lk}+E_{kl})-\delta_{ik}(E_{lk}+E_{kl})\otimes E_{lj} \\
&\quad +\delta_{jl}(E_{lk}+E_{kl})\otimes E_{ik}-\delta_{jl}E_{ik}\otimes(E_{lk}+E_{kl})\Big) \\
&= \sum_{j,k,l=1}^{N} r_{lk}L_{kj}\Big(E_{lj}\otimes(E_{lk}+E_{kl})-(E_{lk}+E_{kl})\otimes E_{lj}\Big) \\
&\quad + \sum_{j,k,l=1}^{N} r_{kl}L_{jk}\Big(-E_{jl}\otimes(E_{lk}+E_{kl})+(E_{lk}+E_{kl})\otimes E_{jl}\Big) \\
&= \sum_{j,k,l=1}^{N} r_{lk}L_{jk}\Big((E_{lj}+E_{jl})\otimes(E_{lk}+E_{kl})-(E_{lk}+E_{kl})\otimes(E_{lj}+E_{jl})\Big),
\end{aligned}
$$

where we set $r_{lk}=-r_{kl}$ and used $L_{jk}=L_{kj}$.

After comparing both sides of the fundamental Poisson bracket (5), we should have the following 2 equalities:

$$
r_{lk} = \frac{1}{2}\frac{A'_{kl}A_{jl}}{A_{jk}}, \tag{6}
$$

$$
r_{lj}-r_{lk} = \frac{1}{2}\frac{(A_{lk}A_{jl})'}{A_{jk}}. \tag{7}
$$

In fact, the 2nd one is a natural result derived from the 1st one. Thus, for the **CH peakons case** we have

$$
\begin{aligned}
r_{lk} &= \frac{1}{2}\frac{A'_{kl}A_{jl}}{A_{jk}} \\
&= -\frac{1}{4}sgn(q_k-q_l)\frac{A_{kl}A_{jl}}{A_{jk}} \\
&= \frac{1}{4}sgn(q_l-q_k)e^{-\frac{1}{2}(|q_l-q_k|+|q_j-q_l|-|q_j-q_k|)},\ \forall \mathbf{j}\in \mathbf{Z}^+.
\end{aligned} \tag{8}
$$

This equality holds for arbitrary $j\in Z^+$. Obviously, only in the cases of $j>l>k$ or $j<l<k$ Eq. (8) becomes constant, namely, $\pm\frac{1}{4}$. But for other j, apparently Eq. (8) is NOT constant.

So, we think that the constant matrix given in ref. [2]

$$
r_{12} = a\sum_{l,k=1}^{N} sgn(q_l-q_k)E_{lk}\otimes(E_{lk}+E_{kl}),\ a=constant
$$

is not an r-matrix for the CH peakon dynamical system.

Discussions for more general case of Lax matrix are seen in ref. [6].

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References

[1] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71**(1993), 1661-1664.

[2] O. Ragnisco and M. Bruschi, Peakons, r-matrix and Toda Lattice, *Physica A* **228**(1996), 126-149.

[3] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer-Verlag, Berlin, 1987).

[4] Zhijun Qiao, The Camassa-Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold, *Comm. Math. Phys.* **239**, 309-341.

[5] Zhijun Qiao, Generalized r-matrix structure and algebro-geometric solutions for integrable systems, *Rev. Math. Phys.* **13**(2001), 545-586.

[6] Zhijun Qiao, r-matrix structure for general even functions in finite-dimensional Hamiltonian systems, preprint, 2008.