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C. Neumann System Associated with the Levi Hierarchy*)

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Abstract A C. Neumann system is presented in this paper. This system is generated through the nonlinearization of the Levi spectral problem under the so-called C. Neumann constraint. We prove that the C. Neumann system is completely integrable in the Liouville sense. The involutive solutions of the Levi hierarchy are given. Particularly, on the tangent bundle $TS^{N-1}=\{(p,q)\in R^{2N}\mid F\equiv\frac{1}{2}(\langle q,q\rangle-1)=0,\ G\equiv\langle p,q\rangle=0\}$ of the sphere S^{N-1}, the involutive solutions of the well-known Burgers equation $u_t=u_{xx}-2uu_x$ and MKdV equation $u_t=-u_{xxx}+6u^2u_x$ are obtained.

Key Words and Phrases C. Neumann System, Involutive Solution, Levi Hierarchy

§1. Introduction

Usually a soliton equation can be expressed as the compatible condition of an over-determined system, the so-called Lax pair. We have already known that the two linear equations in the Lax pair can be nonlinearized to be two compatible, completely integrable Hamiltonian systems in the Liouville sense under some certain constraint conditions between the eigenfunctions and the potentials (see [1]). Among these constraint conditions are there the Bargmann constraint (see [1, 2, 4]), the C. Neumann constraint (see [1, 4]), and the symmetry constraint (see [5, 6]). The method of nonlinearization (see [2, 3]) is quite an effective approach for obtaining finite-dimensional completely integrable systems. According to this approach, quite a few new finite-dimensional completely integrable Liouville's systems in explicit form are successfully found. Another important application of the method of nonlinearization is

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that the solution of soliton equation can be reduced to solving two compatible ordinary differential equations.

This paper is devoted to the application of nonlinearization to investigate the completely integrable Hamiltonian system on some symplectic submanifold. In ref. [7], we gave a completely integrable Bargmann system, produced by the Bargmann constraint of the Levi spectral problem. In the present paper, we shall study the C. Neumann system associated with the Levi hierarchy, and discuss the integrability of C. Neumann system and the involutive solution of Levi hierarchy.

§ 2. Finite-dimensional Involutive Systems

The Poisson bracket of two Hamiltonian functions E,F in the symplectic space $(R^{2N}, dp\wedge dq=\sum_{i=1}^{N}dp_i\wedge dq_i)$ is referred as

$$(E,F)=\sum_{i=1}^{N}\left(\frac{\partial E}{\partial q_i}\frac{\partial F}{\partial p_i}-\frac{\partial E}{\partial p_i}\frac{\partial F}{\partial q_i}\right) \tag{1}$$

which is skew-symmetric, bilinear, and satisfies the Jacobi identity and the Leibnitz rule: $(EF,H)=E(F,H)+F(E,H)$. A hierarchy of smooth functions $\{f_i\}$ are called in involution, if $(f_i, f_j)=0$, for any i,j.

Some direct calculations lead to the following propositions.

Proposition 1

$$\begin{aligned}E_k=&-\frac{1}{2}p_kq_k+\frac{1}{2}\langle q,q\rangle p_k^2+\frac{1}{2}\langle p,q\rangle(q_k^2-2p_kq_k)\\&+\frac{1}{2}\sum_{j=1,j\neq k}^{N}\frac{\lambda_j(p_kq_j-p_jq_k)^2}{\lambda_k-\lambda_j},\quad k=1,\cdots,N\end{aligned} \tag{2}$$

make up an N-involutive system, where $\lambda_1,\cdots,\lambda_N$ are N distinct constants, and $\langle\cdot,\cdot\rangle$ stands for the standard inner-product in R^N.

Proposition 2 Let $F_m=\sum_{k=1}^{N}\lambda_k^{m+1}E_k$, $m=0,1,2,\cdots$, $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_N)$. Then

$$\begin{aligned}H\equiv F_0=&-\frac{1}{2}\langle\Lambda p,q\rangle+\frac{1}{2}\langle q,q\rangle\langle\Lambda p,p\rangle\\&+\frac{1}{2}\langle p,q\rangle(\langle\Lambda q,q\rangle-2\langle\Lambda p,q\rangle)\end{aligned} \tag{3}$$

$$\begin{aligned}F_m=&\frac{1}{2}\langle\Lambda^{m+1}p,q\rangle+\frac{1}{2}\langle q,q\rangle\langle\Lambda^{m+1}p,p\rangle\\&+\frac{1}{2}\langle p,q\rangle(\langle\Lambda^{m+1}q,q\rangle-2\langle\Lambda^{m+1}p,q\rangle)\\&+\frac{1}{2}\sum_{j=0}^{m-1}(\langle\Lambda^{j+1}q,q\rangle\langle\Lambda^{m-j}p,p\rangle)-\langle\Lambda^{j+1}p,q\rangle\langle\Lambda^{m-j}p,q\rangle)\end{aligned} \tag{4}$$

and $(F_m,F_n)=(H,F_m)=0$, $\forall\, m,n\in Z^+$.

Here F_m are either different from or inequivalent to the ones in ref. [7].

Proposition 3 The Hamiltonian systems

$$q_t = \frac{\partial F_m}{\partial p},\ p_t = \frac{\partial F_m}{\partial q},\quad m = 0,1,2,\cdots, \tag{5}$$

are completely integrable. Particularly, the Hamiltonian system $(R^{2N}, dp\wedge dq, H=F_0)$ is i le.

§ 3. C. Neumann System Associated with the Levi Hierarchy

Consider the eigenvalue problem

$$\psi_x = M\psi,\quad \psi = (\psi_1,\psi_2)^T,$$
$$M = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}(u-v) & u \\ v & \frac{1}{2}\lambda - \frac{1}{2}(u-v)\end{pmatrix}. \tag{6}$$

(6) produces the Levi hierarchy of NLEEs:

$$(u,v)_t^T = J\mathscr{L}^m G_0 = J\mathscr{L}^{m+1}G_{-1},\quad m = 0,1,2,\cdots, \tag{7}$$

where

$$J = \begin{pmatrix} 0 & \partial \\ \partial & 0\end{pmatrix},\ \mathscr{L} = \begin{pmatrix} \partial - v + \partial^{-1}u\partial & v + \partial^{-1}v\partial \\ -u-\partial^{-1}u\partial & -\partial + u - \partial^{-1}v\partial\end{pmatrix}, \tag{8}$$
$$G_0 = (-v, -u)^T,\ G_{-1} = (0, -1)^T.$$

The representative equations and Lax representations of (7) were given in ref. [8].

Let $\psi_j = (y_{1j}, y_{2j})^T$ be the eigenfuctions corresponding to the eigenvalues $\lambda_j (j=1,2,\cdots,N)$, and $p = (y_{21},\cdots,y_{2N})^T$, $q = (y_{11}+y_{21},\cdots,y_{1N}+y_{2N})^T$. Then under the C. Neumann constraint

$$\langle q,q\rangle = 1,\ \langle p,q\rangle = 0,\ u = \langle \Lambda(p-q),\ p-q\rangle,\ v = -\langle \Lambda p,p\rangle, \tag{9}$$

(6) is nonlinearized as a C. Neumann system

$$(\mathrm{C})\qquad \begin{cases} q_x = -\frac{1}{2}\Lambda q + \Lambda p + \frac{1}{2}(\langle \Lambda q,q\rangle - 2\langle \Lambda p,q\rangle)q, \\ p_x = \frac{1}{2}\Lambda p - \langle \Lambda p,p\rangle q - \frac{1}{2}(\langle \Lambda q,q\rangle - 2\langle \Lambda p,q\rangle)p \end{cases} \tag{10}$$

which is in fact restricted in the tangent bundle TS^{N-1}

$$TS^{N-1} = \{(p,q)\in R^{2N} \mid F \equiv \frac{1}{2}(\langle q,q\rangle - 1) = 0,\ G \equiv \langle p,q\rangle = 0\} \tag{11}$$

of the sphere S^{N-1}. Thus, by use of the method of Lagrange multiplier and the propositions described in last section, it is not difficult to show the following theorem.

Theorem 1 C. Neumann system (C) is a completely integrable Hamiltonian system $(TS^{N-1}, dp\wedge dq|_{TS^{N-1}}, H^* = H)$ in Liouville's sense.

Proposition 4 Let $(p,q)^T$ be a solution of the C. Neumann system (C). Then u and v defined by (9) satisfy a stationary nonlinear Levi equation

$$J\mathscr{L}^N G_0 + \sum_{j=1}^{N-1}\beta_{N-j}J\mathscr{L}^j G_0 = 0, \tag{12}$$

where $\beta_j(\beta_N = 1,\ j=1,2,\cdots,N)$ are determined by the constants $\lambda_1,\cdots,\lambda_N$.

§ 4. The Involutive Solutions under the C. Neumann System

Under the Bargmann system the involutive solution of Levi hierarchy (7) was obtained in ref. [7]. How about the case of C. Neumann system? Since the Poisson bracket $(H^*=H, F_m^*=F_m)|_{TS^{N-1}}=0$, let us on $TS^{N-1}=0$ define the involutive solution of consistent Hamiltonian equations (H^*) and (F_m^*)

$$\begin{pmatrix} q(x,t_m) \\ p(x,t_m) \end{pmatrix} \stackrel{\text{def}}{=} g_0^x g_m^{t_m} \begin{pmatrix} q(0,0) \\ p(0,0) \end{pmatrix} \tag{13}$$

which is smooth about (x,t_m) on $TS^{N-1}=0$, where g_0^x, $g_m^{t_m}$ are the solution operators of their initial-value problems. After doing some tedious calculations and using the features of operators J, $\mathscr{L}$, we can get Theorem 2.

Theorem 2 Let $q=q(x,t_m), p=p(x,t_m)$ be an involutive solution of consistent systems (H^*) and (F_m^*) on $TS^{N-1}=0$. Then

$$u(x,t_m) = \langle \Lambda(p-q), q-p\rangle, \quad v(x,t_m) = \langle \Lambda p, p\rangle \tag{14}$$

satisfy the nonlinear Levi equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = J\mathscr{L}^m G_0 = J\mathscr{L}^{m+1}G_{-1}, \quad m = 0,1,2,\cdots \tag{15}$$

As special applications of Theorem 2, we have the following two corollaries.

Corollary 1 Let $q=q(x,t_1)$, $p=p(x,t_1)$ be an involutive solution of consistent systems (H^*) and (F_1^*) on $TS^{N-1}=0$. Then the well-known Burgers equation

$$u_t = u_{xx} - 2uu_x \tag{16}$$

has the involutive solution

$$u(x,t_1) = \langle \Lambda(p-q), q-p\rangle = 2\langle \Lambda p, q\rangle - \langle \Lambda q, q\rangle \tag{17}$$

on $TS^{N-1}=0$.

Corollary 2 Let $q=q(x,t_2)$, $p=p(x,t_2)$ be an involutive solution of consistent systems (H^*) and (F_2^*) on $TS^{N-1}=0$. Then the well-known MKdV equation

$$u_t = -u_{xxx} + 6u^2u_x \tag{18}$$

possesses the involutive solution

$$u(x,t_2) = \langle \Lambda(p-q), q-p\rangle = \langle \Lambda p, p\rangle \tag{19}$$

on $TS^{N-1}=0$.

Here the involutive solutions of Burgers equation (16) and MKdV equation (19) are different from the ones under the Bargmann constraint in ref. [7]. Thus it is concluded that one nonlinear equation may have different involutive solution under the different constraint.

We know that some 2×2 spectral problems are gauge equivalent with each other. The gauge transformation between the equivalent spectral problems is usually nonlinear. A natural question is: whether or not their nonlinearized systems are equivalent yet? The answer is not sure, because from r-matrix viewpoint we have recently found that the two nonlinearied systems generated by a pair of gauge equivalent spectral problems have the different r-matrix (see [9]). Some further informations are being investigated.

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