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Commutator representations of nonlinear evolution equations: Harry-Dym and Kaup-Newell cases

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Abstract

A general structure of commutator representations for the hierarchy of nonlinear evolution equations (NLEEs) is proposed. As two concrete examples, the Harry-Dym and Kaup-Newell cases are discused.

Recently, the commutator representations of the hierarchy of nonlinear evolution integrable equations (NLEEs) and the related Lax operator algebra properties have been intensively discussed [1–6]. It is well-known for the spectral problem $L\psi \equiv L(u)\psi = \lambda\psi$ ($u = (u_1, ..., u_N)^T$ is a potential vector, λ is a constant parameter) that if its hierarchy of evolution equations possesses commutator representations, then its key lies in solving an operator equations of the differential operator $V = V(G)$ [1, 3, 5]

$$[V, L] = L_*(KG) - L_*(JG)L \tag{1}$$

where K, J are the pair of Lenard's operators corresponding to the spectral problem $L(u)\psi = \lambda\psi$, $G = (G^{(1)}, ..., G^{(N)})^T$ is an arbitrary given vector function,

$$L_*(\xi) \stackrel{\triangle}{=} \frac{d}{d\varepsilon}|_{\varepsilon=0} L(u + \varepsilon\xi).$$

Now, consider the spectral problem

$$\psi_x = U(u, \lambda)\psi \tag{2}$$

where $\psi = (\psi_1, ..., \psi_n)^T$, each element of the $n \times n$ matrix $U(u, \lambda)$ is the polynomial of λ, λ^{-1} and the coefficients of its every term depend on u.

According to the methods proposed in ref. [7, 8], we can always acquire the spectral gradient $\nabla_u \lambda = (\delta\lambda/\delta u_1, ..., \delta\lambda/\delta u_N)^T$ of the spectral parameter λ with respect to the potential vector u. Generally, $\nabla_u \lambda$ is related to λ, u, and the special function ψ. The integro-differential operators $K = K(u)$, $J = J(u)$ depending on the potential vector an satisfying the following linear relation

$$K\nabla_u \lambda = \lambda^\theta \cdot J\nabla_u \lambda \qquad (\theta \quad \text{is a fixed constant}) \tag{3}$$

are called the pair of Lenard's operators of (2). The operators K, J can be obtained with (2) and the concrete expression of $\nabla_u \lambda$ after some delicate calculations.

As $U(u, \lambda)$ is linear on λ, (2) can always lead to

$$L\,\psi \equiv L(u)\,\psi = \lambda\,\psi. \tag{4}$$

Otherwise, (2) can't read the form like (4). Nevertheless, because each element of the $n \times n$ matrix $U(u, \lambda)$ is the polynomial in λ, λ^{-1} and the coefficients of its every term depend on u, the spectral problem (2) can be usually rewritten as

$$L\,\psi \equiv L(u, \lambda)\,\psi = \lambda^\gamma\,\psi \tag{5}$$

where γ is the highest order of λ in $U(u, \lambda)$, $L = L(u, \lambda)$ is a differential operator related to u and λ. A basic problem is: what is conditions under which the isospectral hierarchy of evolution equations (5) posesses the commutator representations?

For the spectral problem of its form like (5), here we construct a wider operator equation with the differential operator $V = V(G)$ than (1)

$$[V, L] = L_*(KG)\,L^\beta - L_*(JG)\,L^\alpha \tag{6}$$

where $[\cdot, \cdot]$ stands for the commutator; $L = L(u, \lambda)$; K, J are the pair of Lenard's operators determined by (3); $G = (G^{(1)}, \cdots, G^{(N)})^T$ is an arbitrary given vector function,

$$L_*(\xi) \stackrel{\triangle}{=} \frac{d}{d\varepsilon}\,|_{\varepsilon=0}\, L(u + \varepsilon\xi, \lambda), \quad \xi = (\xi_1, ..., \xi_N)^T;$$

α, β are two fixed constants associated with (5) and $\beta < \alpha$.

Let $\eta = \alpha - \beta$, choose $G_{-\eta} \in \text{Ker} \quad J = \{G \,|\, JG = 0\}$, and define Lenard's recursive secuence $\{G_{j\eta}\}$:

$$KG_{(j-1)\eta} = JG_{j\eta}, \quad j = 0, 1, 2, ... \,. \tag{7}$$

The NLEEs $u_t = X_m(u) \quad (m = 0, 1, 2, ...)$ produced by the vector field $X_m \stackrel{\triangle}{=} JG_{m\eta}$ $(m = 0, 1, 2, ...)$ and called the hierarchy of evolution equations (5).

The following two theorems give a simple and clear approach that the hierarchy of isospectral evolution equations $u_t = X_m(u) \quad (m = 0, 1, 2, ...)$ of (5) owns the commutator representations.

Theorem 1 *Let* $\{G_{j\eta}\}_{j=-1}^{\infty}$ *be the Lenard's recursive sequence of (5). For any* $G_{j\eta}$*, the operator equation (6) has the commutator solution* $V_j = V(G_{j\eta})$*. Then the operator* $W_m = \sum_{j=0}^{m} V_{j-1} L^{(m-j)\eta-\beta}$ *is the Lax operator (4) of the vector field* $X_m(u)$*, that is,* W_m *satisfies*

$$[W_m, L] = L_*(X_m), \quad m = 0, 1, 2, ... \,. \tag{8}$$

Proof.

$$\begin{aligned}[W_m, L] &= \sum_{j=0}^{m} [V_{j-1}, L]\, L^{(m-j)\eta-\beta} \\ &= \sum_{j=0}^{m} (L_*(KG_{(j-1)\eta})\, L^{(m-j)\eta} - L_*(JG_{(j-1)\eta})\, L^{(m-j+1)\eta}) \\ &= L_*(JG_{m\eta}) \\ &= L_*(X_m).\end{aligned}$$

From this theorem, we can also further discuss the Lax operator algebra generated by the Lax operator W_m which is left to a later paper.

Theorem 2 *Let the conditions in Theorem 1 be satisfied, and the Gateaux derivative mapping L_*: $\xi \to L_*(\xi)$ of the spectral operator L in the direction ξ is an injective homomorphism. Then the isospectral hierarchy of evolution equations $u_t = X_m(u)$ of (5) possesses the commutator representations*

$$L_t = [W_m, L], \quad m = 0, 1, 2, \dots . \tag{9}$$

Proof. $L_t = L_*(u_t)$,

$$L_t - [W_m, L] = L_*(u_t) - L_*(X_m(u)) = L_*(u_t - X_m(u)).$$

The above equality implies Theorem 2 holds because L_* is injective.

By Theorem 1 and Theorem 2, we can evidently see that in order to secure the commutator representations (9) of NLEEs $u_t = X_m(u)$, its key lies in constructing the corresponding operator equation (6) according to the form of (5) and finding an operator solution of (6).

Corollary The potential $u = (u_1, ..., u_N)^T$ satisfies a stationary nonlinear equation $\sum_{k=0}^{1} \alpha_k X_{1-k} - 0$ if and only if $[\sum_{k=0}^{1} \alpha_k W_{1-k}, L] = 0$, where α_k $(k = 0, 1, 2, ..., l)$ are some constants, $l \in Z^+$.

In the following, as two concrete examples of the above approach, we shall discuss the Harry-Dym and Kaup-Newell hierarchies, present the corresponding operator equation (6), solve it, and finally give the commutator representations of these two hierarchies.

1. Consider the spectral problem

$$\psi_x = U(u,\lambda)\,\psi, \quad U(u,\lambda) = \begin{pmatrix} -i\lambda & (u-1)\lambda \\ -\lambda & i\lambda \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad i^2 = -1. \tag{10}$$

(10) is equivalent to the famous Sturm-Liouville equation

$$-\partial^2 y = \mu u y, \tag{11}$$

via the transformations $\psi = iy - \lambda^{-1} y_x$, $\psi_2 = y$, $\mu = \lambda^2$. The isospectral property of the Harry-Dym hierarchy was studied in [9], and the nonlinearization of the Lax pair for the Harry-Dym equation $u_t = (u^{-(1/2)})_{xxx}$ was discussed in [10]. In the present paper, using the above skeleton, we further give the commutator representations of each equation in the Harry-Dym hierarchy including the Harry-Dym equation $u_t = (u^{-(1/2)})_{xxx}$.

The spectral gradient $\nabla_u\lambda$ of (10) with regard to u is

$$\nabla_u\lambda = \lambda\psi_2^2 \Big(\int\limits_\Omega (2i\psi_1\psi_2 - u\psi_2^2 - \psi_1^2)\,dx\Big)^{-1}. \tag{12}$$

Noticing the relation $\partial^{-1}u\partial\psi_2^2 = 2i\psi_1\psi_2 + \psi_2^2 - \psi_1^2$ and (9), only choosing Lenard's operators

$$K = \partial^3, \qquad J = -2\,(\partial u + u\partial), \tag{13}$$

we have

$$K\nabla_u\lambda = \lambda^2 \cdot J\nabla_u\lambda. \tag{14}$$

Let $G_{-2} = u^{-(1/2)} \in \operatorname{Ker} J$, define the Lenard recursive sequence $\{G_{2j}\}$ of (10): $KG_{2(j-1)} = JG_{2j},\ \ j = 0, 1, 2, \ldots$. The Harry-Dym vector fields $X_j\,(u) \stackrel{\triangle}{=} JG_{2j}$ yield the isospectral hierarchy of NLEEs (10): $u_t = X_j(u)\ \ (j = 0, 1, 2, \ldots)$, in which the first system is the well-known Harry-Dym equation $u_t = KG_{-2} = (u^{-(1/2)})_{xxx}$.

(10) can be rewritten as

$$L\psi = \lambda\psi, \qquad L = L\,(u) = \frac{1}{u}\begin{pmatrix} i & 1-u \\ 1 & -i \end{pmatrix}\partial, \quad \partial = \partial/\partial x. \tag{15}$$

The Gateaux derivative mapping L_* of L in the direction ξ is

$$L_*(\xi) = \frac{\xi}{u^2}\begin{pmatrix} -i & -1 \\ -1 & i \end{pmatrix}\partial = \frac{\xi}{u}\begin{pmatrix} 0 & -i \\ 0 & -1 \end{pmatrix} L \tag{16}$$

and L_* is an injective homomorphism.

Let $G\,(x)$ be an arbitrary smooth function. For the spectral problem (15), we establish the corresponding operator equation of $V = V\,(G)$ as follows

$$[\,V,\, L\,] = L_*\,(KG)\,L^{-1} - L_*\,(JG)\,L \tag{17}$$

which is equivalent to (6) with $\alpha = 1,\ \ \beta = -1$.

Theorem 3 *The operator equation (17) has the operator solution*

$$V = V\,(G) = G_{xx}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + G_x\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix} L + (-2G)\begin{pmatrix} i & 1-u \\ 1 & -i \end{pmatrix} L^2. \tag{18}$$

Proof. Let

$$W = \begin{pmatrix} -i & u-1 \\ -1 & i \end{pmatrix}, \qquad V_0 = G_{xx}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$V_1 = G_x \begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix}, \qquad V_2 = -2G \begin{pmatrix} i & 1-u \\ 1 & -i \end{pmatrix}. \tag{19}$$

Then the commutator $[V, L]$ of $V = V_0 + V_1 L + V_2 L^2$ and L is ($L = W^{-1}\partial$):

$$[V, L] = -W^{-1}V_{0x} + \left(V_0 - W^{-1}V_0 W - W^{-1}V_{1x}\right) L + \\ \left(V_1 - W^{-1}V_1 W - W^{-1}V_{2x}\right) L^2 + \left(V_2 - W^{-1}V_2 W\right) L^3. \tag{20}$$

Substituting every expressions of (19) into (20), through lengthy calculations we can find that the right-hand side of (20) is equal to $L_*(KG)\, L^{-1} - L_*\,(JG)\, L$.

Thus, the conditions of both Theorem 1 and Theorem 2 hold. So, the Harry-Dym hierarchy of NLEEs $u_t = X_m(u)$ $(m = 0, 1, 2, ...)$ possesses the following commutator representations

$$\begin{cases} L_t = [W_m, L], \quad m = 0, 1, 2, ..., \\ W_m = \sum_{j=0}^{m} \left\{ G_{2(j-1),xx} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + G_{2(j-1),x} \begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix} L - \right. \\ \qquad \left. 2G_{2(j-1)} \begin{pmatrix} i & 1-u \\ 1 & -i \end{pmatrix} L^2 \right\} L^{2(m-j)+1}. \end{cases}$$

Particularly, as $m = 0$, the Harry-Dym equation $u_t = X_0\,(u) = (u)_{xxx}$ has the commutator representation

$$\begin{cases} L_t = [W_0, L], \\ W_0 = (u^{-(1/2)})_{xx} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} L + \\ \qquad (u^{-(1/2)})_x \begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix} L^2 - 2u^{-(1/2)} \begin{pmatrix} i & 1-u \\ 1 & -i \end{pmatrix} L^3. \end{cases}$$

2. Consider the spectral problem proposed by Kaup and Newell [11]

$$\psi_x = U\,(u, v, \lambda)\, \psi, \quad U\,(u, v, \lambda) = \begin{pmatrix} -i\lambda^2 & \lambda u \\ \lambda v & i\lambda^2 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad i^2 = -1. \tag{21}$$

It isn't difficult to get the spectral gradient $\nabla_{(u,v)}\lambda$

$$\nabla_{(u,v)}\lambda = \begin{pmatrix} \delta\lambda \,/\, \delta u \\ \delta\lambda \,/\, \delta v \end{pmatrix} = \begin{pmatrix} \lambda & \psi_2^2 \\ -\lambda & \psi_1^2 \end{pmatrix} \left(\int_\Omega (v\psi_1^2 + 4i\psi_1\psi_2 - u\psi_2^2)\, dx \right)^{-1} \tag{22}$$

which satisfies

$$K\nabla_{(u,v)}\lambda = \lambda^2 \cdot J\nabla_{(u,v)}\lambda, \tag{23}$$

where

$$K = \begin{pmatrix} \frac{1}{2}\,\partial u \partial^{-1} u \partial & \frac{1}{2}\, i\partial^2 + \frac{1}{2}\,\partial u \partial^{-1} v \partial \\ -\frac{1}{2}\, i\partial^2 + \frac{1}{2}\,\partial v \partial^{-1} u \partial & \frac{1}{2}\,\partial v \partial^{-1} v \partial \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}$$

are the pair of Lenard's operators of (21).

Let $G_{-1} = (1,0)^T \in \operatorname{Ker} J$, $G_0 = J^{-1}KG_{-1} = (v,u)^T$. The Lenard recursive sequences G_j $(j = 0,1,2,...)$ are determined by

$$KG_{j-1} = JG_j, \quad j = 0,1,2,\dots, \tag{24}$$

which produces the Kaup-Newell hierarchy of NLEEs

$$(u,v)_t^T = X_j(u,v) \stackrel{\triangle}{=} JG_j, \quad j = 0,1,2,\dots, \tag{25}$$

with the representative equation

$$(u,v)_t^T = X_1(u,v) \equiv \Big(\frac{1}{2}\, iu_{xx} + \frac{1}{2}\,(u^2v)_x, -\frac{1}{2}\, iv_{xx} + \frac{1}{2}\,(v^2u)_x\Big)^T. \tag{26}$$

As $j = 1$ and $v = u^*$, (26) reduces to the famous derivative Schrödinger equation (DSE):

$$u_t = \frac{1}{2}\, iu_{xx} + \frac{1}{2}\,(u\,|\,u\,|^2)_x. \tag{27}$$

(21) is equivalent to

$$L\psi = \lambda^2\psi, \qquad L = \begin{pmatrix} i\partial & -i\lambda u \\ i\lambda v & -i\partial \end{pmatrix}. \tag{28}$$

The Gateaux derivative L_* of L is

$$L_*(\xi) = \begin{pmatrix} 0 & -i\xi_1 \\ i\xi_2 & 0 \end{pmatrix} L^{1/2}, \quad \forall \xi = (\xi_1, \xi_2)^T, \quad L_* \text{ is injective.} \tag{29}$$

Let $G\,(x) \stackrel{\triangle}{=} (G^{(1)}(x), \quad G^{(2)}(x))^T$ be any given smooth vector field. For the spectral problem (28), we construct the related operator equation with $V = V\,(G)$ as follows

$$[V, L] = L_*(KG)\, L^{-(1/2)} - L_*(JG)\, L^{1/2} \tag{30}$$

which is exactly (6) with $\alpha = \dfrac{1}{2}$, $\beta = -\dfrac{1}{2}$.

Theorem 4 *The operator equation (30) possesses the operator solution*

$$V = V(G) = \begin{pmatrix} 0 & \frac{1}{2}\, iG_x^{(2)} + \frac{1}{2}\, u\partial^{-1}(uG_x^{(1)} + vG_x^{(2)}) \\ -\frac{1}{2}\, iG_x^{(1)} + \frac{1}{2}\, v\partial^{-1}(uG_x^{(1)} + vG_x^{(2)}) & 0 \end{pmatrix} +$$
$$\begin{pmatrix} -\frac{1}{2}\, i\partial^{-1}(uG_x^{(1)} + vG_x^{(2)}) & 0 \\ 0 & \frac{1}{2}\, i\partial^{-1}(uG_x^{(1)} + vG_x^{(2)}) \end{pmatrix} L^{1/2}. \tag{31}$$

Proof. The method of prooving this Theorem is similar to that used in Theorem 3. The process is omitted.

So, the Kaup-Newell hierarchy of NLEEs $(u,v)_t^T = X_m(u,v) \quad (m = 0,1,2,...)$ has the commutator representations

$$\begin{cases} L_t = [W_m, L], \quad m = 0,1,2,\dots, \\ W_m = \sum_{j=0}^{m} \left\{ \begin{pmatrix} 0 & \frac{1}{2}\, iG^{(2)}_{j-1,x} + \frac{1}{2}\, u\partial^{-1}(uG^{(1)}_{j-1,x} + vG^{(2)}_{j-1,x}) \\ -\frac{1}{2}\, iG^{(1)}_{j-1,x} + \frac{1}{2}\, v\partial^{-1}(uG^{(1)}_{j-1,x} + vG^{(2)}_{j-1,x}) & 0 \end{pmatrix} + \right. \\ \left. \begin{pmatrix} -\frac{1}{2}\, i\partial^{-1}(uG^{(1)}_{j-1,x} + vG^{(2)}_{j-1,x}) & 0 \\ 0 & \frac{1}{2}\, i\partial^{-1}(uG^{(1)}_{j-1,x} + vG^{(2)}_{j-1,x}) \end{pmatrix} L^{1/2} \right\} L^{m-j+(1/2)}. \end{cases}$$

Especially, if one lets $m = 1, \quad v = u^*$, then the DSE (27) has the commutator representation

$$\begin{cases} L_t = [W_1, L], \\ W_1 = \frac{1}{2} \begin{pmatrix} 0 & iu_x + u\,|u|^2 \\ -iu_x^* + u^*\,|u|^2 & 0 \end{pmatrix} L^{1/2} + \frac{1}{2} \begin{pmatrix} -i\,|u|^2 & 0 \\ 0 & i\,|u|^2 \end{pmatrix} L + \\ \qquad \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix} L^{3/2} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} L^2. \end{cases}$$

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