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# A completely integrable system associated with the Harry-Dym hierarchy

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**Abstract**

By use of nonlinearization method about spectral problem, a classical completely integrable system associated with the Harry-Dym ($HD$) hierarchy is obtained. Furthermore, the involutive solution of each equation in the $HD$ hierarchy is presented, in particular, the involutive solution of the well-known $HD$ equation $u_t = (u^{-\frac{1}{2}})_{xxx}$ is given.

## 1 Introduction

The Harry-Dym ($HD$) equation $u_t = (u^{-\frac{1}{2}})_{xxx}$ is celebrated for its cuspidal soliton equation. The isospectral property the $HD$ hierarchy has been discussed in [1, 2]. Recently, Cao Cewen [3] has studied the nonlinearization of the Lax pair for the $HD$ equation $u_t = (u^{-\frac{1}{2}})_{xxx}$, and considered the stationary $HD$ equation and its relation with geodesics on ellipsoid. In this paper, we shall study the $HD$ hierarchy of nonlinear evolution equations, which contains the $HD$ equation $u_t = (u^{-\frac{1}{2}})_{xxx}$. The whole paper is divided into four sections. In the next section, the commutator (or Lax) representation of each $HD$ hierarchy is secured. In Sec. 3, using the "nonlinearization" [4–8] of spectral problem by which many completely integrable systems in the Liouville's sense have been found [7–19] in recent years, we present a classical completely integrable system in the Liouville's sense and an involutive functional system. Section 4 gives a discription about the involutive solutions of the $HD$ hierarchy, particularly, the involutive solution of the well-known $HD$ equation $u_t = (u^{-\frac{1}{2}})_{xxx}$ is obtained.

## 2 Commutator (or Lax) representations of the $HD$ hierarchy

Consider the spectral problem

$$y_x = My, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad M = \begin{pmatrix} -i\lambda & (u-1)\lambda \\ -\lambda & i\lambda \end{pmatrix} \tag{2.1}$$

which is the special case of the WKI spectral problem [20]

$$y_x = My, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad M = \begin{pmatrix} -i\lambda & q\lambda \\ r\lambda & i\lambda \end{pmatrix}$$

as $q = u-1,\ r = -1$. Here, $i^2 = -1,\ y_x = \partial y/\partial x$, $u$ is a scalar potential, $\lambda$ is a spectral parameter, $x \in \Omega\ (\Omega = (-\infty, +\infty)$ or $(0, T))$. It is easy to know that (2.1) is equivalent to the well-know Sturm-Liouville equation $-\psi_{xx} = \mu u \psi$ via the transformation $y_1 = i\psi - \lambda^{-1}\psi_x,\ y_2 = \psi,\ \mu = \lambda^2$ and its inverse.

**Proposition 2.1** Let $\lambda$ be a spectral parameter of (2.1). Then the spectral gradient $\bigtriangledown\lambda$ of spectral $\lambda$ with regart to the potential $u$ is

$$\bigtriangledown\lambda = \frac{\delta\lambda}{\delta u} = \lambda y_2^2 \cdot \Big(\int_\Omega (2iy_1y_2 - uy_2^2 - y_1^2)dx\Big)^{-1}, \tag{2.2}$$

where $(y_1, y_2)^T$ is the spectral function of (2.1) corresponding to $\lambda$.

**Proof.** See Ref. 4 Sec. II.

Choosing the operator $K$ and $J$ : $K = \partial^3,\ J = 2(\partial u + u\partial)$, (here $\partial = \partial/\partial x$), we immediately have

**Proposition 2.2** Let $\lambda$ be a spectral parameter of (2.1). Then the spectral gradient $\bigtriangledown\lambda$ defined by (2.2) satisfies the linear relation

$$K \bigtriangledown \lambda = \lambda^2 \cdot J \bigtriangledown \lambda. \tag{2.3}$$

**Proof.** In virtue of (2.1) and $\partial^{-1}u\partial y_2^2 = 2iy_1y_2 + y_2^2 - y_1^2$, directly calculate.

The operators $K$ and $J$ which satisfy (2.3) are called the pair of Lenard's operators of (2.1). Now, recursively define the Lenard's gradient sequence $\{G_{2j}\}$:

$$\begin{aligned} KG_{2(j-1)} &= JG_{2j}, \quad j = 0, 1, 2, \dots, \\ G_{-2} &= u^{-1/2} \in \mathrm{Ker}J. \end{aligned} \tag{2.4}$$

$X_m(u) = JG_{2m}\ (m = 0, 1, 2, ...)$ are called the $HD$ vector fields which produces the isospectral hierarchy of equations of (2.1)

$$u_{t_m} = X_m(u), \quad m = 0, 1, 2, ..., \tag{2.5}$$

with the representative equation

$$u_t = X_0(u) = JG_0 = KG_{-2} = (u^{-1/2})_{xxx}, \quad t_0 = t$$

which is exactly the well-known Harry-Dym ($HD$) equation. Thus, the isospectral hierarchy of equations (2.5) of (2.1) yields the $HD$ hierarchy. (2.1) can be written as

$$Ly \equiv L(u)y = \lambda y, \quad L \equiv L(u) = \frac{1}{u}\begin{pmatrix} i & 1-u \\ 1 & -i \end{pmatrix}\partial. \tag{2.6}$$

The Gateaux derivative of $L$ in the direction $\xi$ is

$$L_*(\xi) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(u+\varepsilon\xi) = \frac{\xi}{u^2}\begin{pmatrix} -i & -1 \\ -1 & i \end{pmatrix}\partial = \frac{\xi}{u}\begin{pmatrix} 0 & -i \\ 0 & -1 \end{pmatrix} L \tag{2.7}$$

and $L_*$ is an injective homomorphism.

Assume $G(x)$ is an arbitary smooth function. For the spectral problem (2.6), we construct an operator equation of operator $V = V(G)$

$$[V, L] = L_*(KG)L^{-1} - L_*(JG)L, \tag{2.8}$$

where $[\cdot,\cdot]$ is the Lie bracket, $K$ and $J$ are the pair of Lenard's operators.
**Theorem 2.1** *The operator equation (2.8) posesses the operator solution*

$$V = V(G) = G_{xx}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + G_x\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix} L + (-2G)\begin{pmatrix} i & 1-u \\ 1 & -i \end{pmatrix} L^2. \tag{2.9}$$

**Proof.** Let

$$W = \begin{pmatrix} -i & u-1 \\ -1 & i \end{pmatrix}, \quad V_0 = G_{xx}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$V_1 = G_x\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix}, \quad V_2 = -2G\begin{pmatrix} i & 1-u \\ 1 & -i \end{pmatrix}. \tag{2.10}$$

It is not difficult to calculate the commutator $[V, L]$ of $V = V_0 + V_1L + V_2L^2$ and $L$ (note $L = W^{-1}\partial$) :

$$[V, L] = VL - LV = -W^{-1}V_{0x} + (V_0 - W^{-1}V_0W - W^{-1}V_{1x})L +$$
$$(V_1 - W^{-1}V_1W - W^{-1}V_{2x})L^2 + (V_2 - W^{-1}V_2W)L^3. \tag{2.11}$$

Substituting (2.10) into the right-hand side of (2.11) and carefully calculating, it is found to be equivalent to the right-hand side of (2.8).
**Theorem 2.2** *Let $G_{2j}$ be the Lenard's recursive sequence, and $V_j = V(G_{2j})$. Then the operator $W_m = \sum_{j=0}^{m} V_{j-1}L^{2(m-j)+1}$ satisfies*

$$[W_m, L] = L_*(X_m), \quad m = 0, 1, 2, ..., \tag{2.12}$$

*i.e., $W_m$ is the Lax operator [21] of the $HD$ vector field $X_m(u)$.*

**Proof.**

$$
\begin{aligned}
[W_m, L] &= \sum_{j=0}^{m}[V_{j-1}, L]L^{2(m-j)+1} = \\
&\sum_{j=0}^{m}(L_*(KG_{2(j-1)})L^{2(m-j)} - L_*(JG_{2(j-1)})L^{2(m-j)+2}) = \\
&L_*(X_m).
\end{aligned}
$$

**Theorem 2.3** *The HD Hierarchy of equation* $u_{t_m} = X_m(u)$ *have the commutator representations*

$$L_{t_m} = [W_m, L], \ \ m = 0, 1, 2, ..., \tag{2.13}$$

*i.e.,* $u_{t_m} = X_m(u)$ *is the natural compatible condition of* $Ly = \lambda y$ and $y_{t_m} = W_m y$.

**Proof.** $L_{t_m} = L_*(u_{t_m}) \Longrightarrow L_{t_m} - [W_m, L] = L_*(u_{t_m}) - L_*(X_m) = L_*(u_{t_m} - X_m)$. $L_*$ is injective, so $L_{t_m} = [W_m, L] \Longleftrightarrow u_{t_m} = X_m(u)$.

**Corollary 2.1** The potential $u$ satisfies a stationary $HD$ system

$$X_{N+c_1}X_{N-1} + \cdots + c_N X_0 = 0, \quad N = 0, 1, 2, ..., \tag{2.14}$$

iff

$$[W_{N+c_1}W_{N-1} + \cdots + c_N W_0, L] = 0, \tag{2.15}$$

where $c_1, ..., c_N$ are constants.

## 3 An integrable system and involutive functional system

Let $\lambda_j$ $(j = 1, ..., N)$ be $N$ different spectral values of (2.1), and $y = (p_j, \ \ q_j)^T$ be the associated spectral functions. Introduce the Bargmann constraint [5] as follows

$$G_{-2} = \sum_{j=1}^{N} \bigtriangledown \lambda_j \tag{3.1}$$

which is equivalent to

$$u = \langle \Lambda p, p \rangle^{-2}, \tag{3.2}$$

where $p = (p_1, ..., p_N)^T$, $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_N)$, $\langle \cdot, \cdot \rangle$ stands for the standard inner-product in $R^N$.

Under the Bargmann constraint (3.2), (2.1) is nonlinearized as a Hamiltonian system (here $q = (q_1, ..., q_N)^T$)

$$(H): \ \begin{cases} q_x = -i\Lambda q + (\langle \Lambda p, p \rangle^{-2} - 1)\Lambda p = \frac{\partial H}{\partial p} \\ p_x = -\Lambda q + i\Lambda p = -\frac{\partial H}{\partial q} \end{cases} \tag{3.3}$$

with

$$H = -i\langle \Lambda p, q\rangle + \frac{1}{2}\langle \Lambda q, q\rangle - \frac{1}{2}\langle \Lambda p, p\rangle - \frac{1}{2}\langle \Lambda p, p\rangle^{-1}. \tag{3.4}$$

A natural problem is whether $(H)$ is completely integrable in the Liouville sense or not? In order to answer this question, we consider a functional system $\{F_m\}$ :

$$F_m = \langle \Lambda^{2m+3}p, p\rangle \, \langle \Lambda p, p\rangle^{-1} +$$

$$\sum_{j=0}^{m} \begin{vmatrix} \langle \Lambda^{2j+2}p, p\rangle & \langle \Lambda^{2j+1}p, p\rangle \\ \langle \Lambda^{2(m-j)+3}p, p\rangle & \langle \Lambda^{2(m-j)+2}p, p\rangle \end{vmatrix} + \begin{vmatrix} \langle \Lambda^{2j+3}q, q\rangle & \langle \Lambda^{2j+2}p, q\rangle \\ \langle \Lambda^{2(m-j)+2}p, q\rangle & \langle \Lambda^{2(m-j)+1}p, p\rangle \end{vmatrix} +$$

$$2i\sum_{j=0}^{m} \begin{vmatrix} \langle \Lambda^{2j+2}p, q\rangle & \langle \Lambda^{2j+3}p, q\rangle \\ \langle \Lambda^{2(m-j)+1}p, p\rangle & \langle \Lambda^{2(m-j)+2}p, p\rangle \end{vmatrix}, \quad m = 0, 1, 2, \dots . \tag{3.5}$$

The Poisson bracket of two functions $E,\ F$ in the symplectic space $(R^{2N}, dp \wedge dq)$ is defined by [22]

$$(E, F) = \sum_{j=1}^{N}\Big(\frac{\partial E}{\partial q_j}\frac{\partial F}{\partial p_j} - \frac{\partial E}{\partial p_j}\frac{\partial F}{\partial q_j}\Big) = \Big\langle \frac{\partial E}{\partial q}, \frac{\partial F}{\partial p}\Big\rangle - \Big\langle \frac{\partial E}{\partial p}, \frac{\partial F}{\partial q}\Big\rangle \tag{3.6}$$

which is skew-symmetric, bilinear, satisfies the Jacobi identity and Leibniz rule: $(EF, H) = F(E, H) + E(F, H)$. $E, F$ are called involutive [22], if $(E, F) = 0$.

Write $F_m$ as $F_m = U_m + S_m + T_m + 2iR_m$ where

$$U_m = \langle \Lambda^{2m+3}p, p\rangle \, \langle \Lambda p, p\rangle^{-1}, \tag*{(3.7)$_1$}$$

$$S_m = \sum_{j=0}^{m}\Big(\langle \Lambda^{2j+2}p, p\rangle \, \langle \Lambda^{2(m-j)+2}p, p\rangle - \langle \Lambda^{2j+1}p, p\rangle \, \langle \Lambda^{2(m-j)+3}p, p\rangle\Big), \tag*{(3.7)$_2$}$$

$$T_m = \sum_{j=0}^{m}\Big(\langle \Lambda^{2j+3}q, q\rangle \, \langle \Lambda^{2(m-j)+1}p, p\rangle - \langle \Lambda^{2j+2}p, q\rangle \, \langle \Lambda^{2(m-j)+2}p, q\rangle\Big), \tag*{(3.7)$_3$}$$

$$R_m = \sum_{j=0}^{m}\Big(\langle \Lambda^{2j+2}p, q\rangle \, \langle \Lambda^{2(m-j)+2}p, p\rangle - \langle \Lambda^{2j+3}p, q\rangle \, \langle \Lambda^{2(m-j)+1}p, p\rangle\Big). \tag*{(3.7)$_4$}$$

**Lemma 3.1**

$$(U_m, U_n) = (S_m, S_n) = (U_m, S_n) = (T_m, T_n) = (R_m, R_n) = 0, \quad \forall m, n \in Z^+. \tag{3.8}$$

**Proof.** $(U_m, U_n) = (S_m, S_n) = (U_m, S_n) = 0$ is obvious.

$$\frac{\partial T_m}{\partial q} = 2\sum_{j=0}^{m}\Big(\langle \Lambda^{2(m-j)+1}p, p\rangle \, \Lambda^{2j+3}q - \langle \Lambda^{2(m-j)+2}p, q\rangle \, \Lambda^{2j+2}p\Big), \tag*{(3.9)$_1$}$$

$$\frac{\partial T_n}{\partial p} = 2\sum_{j=0}^{n}\Big(\langle \Lambda^{2k+3}q, q\rangle\, \Lambda^{2(n-k)+1}p - \langle \Lambda^{2k+2}p, q\rangle\, \Lambda^{2(n-k)+2}q\Big), \qquad (3.9)_2$$

$$\frac{\partial R_m}{\partial q} = \sum_{j=0}^{m}\Big(\langle \Lambda^{2(m-j)+2}p, p\rangle\, \Lambda^{2j+2}p - \langle \Lambda^{2(m-j)+1}p, p\rangle\, \Lambda^{2j+3}p\Big), \qquad (3.10)_1$$

$$\frac{\partial R_n}{\partial q} = 2\sum_{j=0}^{n}\Big(\langle \Lambda^{2k+2}p, q\rangle\, \Lambda^{2(n-k)+2}p - \langle \Lambda^{2k+3}p, q\rangle\, \Lambda^{2(n-k)+1}p\Big) +$$

$$\sum_{j=0}^{n}\Big(\langle \Lambda^{2(n-k)+2}p, p\rangle\, \Lambda^{2k+2}q - \langle \Lambda^{2(n-k)+1}p, p\rangle\, \Lambda^{2k+3}q\Big). \qquad (3.10)_2$$

Substituting $(3.9)_1$, $(3.9)_2$ and $(3.10)_1$, $(3.10)_2$ into the inner-product $\langle \frac{\partial T_m}{\partial q}, \frac{\partial T_n}{\partial p}\rangle$ and $\langle \frac{\partial R_m}{\partial q}, \frac{\partial R_n}{\partial p}\rangle$, respectively, through a lengthy calculation we may know that $\langle \frac{\partial T_m}{\partial q}, \frac{\partial T_n}{\partial p}\rangle$ and $\langle \frac{\partial R_m}{\partial q}, \frac{\partial R_n}{\partial p}\rangle$ are symmetrical about $m.n.\ s_0$,

$$(T_m, T_n) = \Big\langle \frac{\partial T_m}{\partial q}, \frac{\partial T_n}{\partial p}\Big\rangle - \Big\langle \frac{\partial T_m}{\partial p}, \frac{\partial T_n}{\partial q}\Big\rangle = 0,$$

$$(R_m, R_n) = \Big\langle \frac{\partial R_m}{\partial q}, \frac{\partial R_n}{\partial p}\Big\rangle - \Big\langle \frac{\partial R_m}{\partial p}, \frac{\partial R_n}{\partial q}\Big\rangle = 0.$$

**Lemma 3.2** $(U_m, T_n)$, $(U_m, R_n)$, $(S_m, T_n)$, $(S_m, R_n)$ and $(T_m, R_n)$ are symmetrical about $m$, $n \in Z^+$, i.e.,

$$(U_m, T_n) = (U_n, T_m), \quad (U_m, R_n) = (U_n, R_m), \quad (S_m, T_n) = (S_n, T_m),$$

$$(S_m, R_n) = (S_n, R_m), \quad (T_m, R_n) = (T_n, R_m), \quad \forall m, n \in Z^+. \qquad (3.11)$$

**Proof.** Here we prove $(U_m, T_n) = (U_n, T_m)$. Other equalities can be proved in the same way.

$$\frac{\partial U_m}{\partial p} = -2\langle \Lambda^{2m+3}p, p\rangle \langle \Lambda p, p\rangle^{-2}\Lambda p + 2\langle \Lambda p, p\rangle^{-1}\Lambda^{2m+3}p, \qquad (3.12)$$

$$\frac{\partial T_n}{\partial q} = 2\sum_{j=0}^{n}\Big(\langle \Lambda^{2(n-j)+1}p, p\rangle \Lambda^{2j+3}q - \langle \Lambda^{2(n-j)+2}p, q\rangle \Lambda^{2j+2}p\Big). \qquad (3.13)$$

According to (3.6), we have

$$(U_m, T_n) = -\Big\langle \frac{\partial U_m}{\partial p}, \frac{\partial T_n}{\partial q}\Big\rangle. \qquad (3.14)$$

Substituting (3.12) and (3.13) into the right-hand side of (3.14) and carefully calculating the inner-product $\langle \frac{\partial U_m}{\partial p}, \frac{\partial T_n}{\partial q}\rangle$, we find that $\langle \frac{\partial U_m}{\partial p}, \frac{\partial T_n}{\partial q}\rangle$ is symmetrical about $m, n$. Thus $(U_m, T_n) = (U_n, T_m)$.

**Theorem 3.1**

$$(F_m, F_n) = 0, \quad \forall m, n \in Z_+. \tag{3.15}$$

**Proof.** In virtue of Lemma 3.1, Lemma 3.2 and the property of Poisson bracket, we get

$$\begin{aligned}(F_m, F_n) &= (U_m, T_n) + 2i(U_m, R_n) + (S_m, T_n) + \\ &\quad 2i(S_m, R_n) + (T_m, U_n) + (T_m, S_n) + 2i(T_m, R_n) + \\ &\quad 2i(R_m, U_n) + 2i(R_m, S_n) + 2i(R_m, T_n) = 0. \end{aligned} \tag{3.16}$$

**Theorem 3.2** $(H, F_m) = 0, \quad \forall m \in Z^+$.

**Proof.**

$$\begin{aligned}\frac{\partial F_m}{\partial q} &= 2\sum_{j=0}^{m}\Big(\langle\Lambda^{2(m-j)+1}p, p\rangle\Lambda^{2j+3}q - \langle\Lambda^{2(m-j)+2}p, q\rangle\Lambda^{2j+2}p\Big) + \\ &\quad 2i\sum_{j=0}^{m}\Big(\langle\Lambda^{2(m-j)+2}p, p\rangle\Lambda^{2j+2}p - \langle\Lambda^{2(m-j)+1}p, p\rangle\Lambda^{2j+3}p\Big),\end{aligned} \tag*{$(3.17)_1$}$$

$$\begin{aligned}\frac{\partial F_m}{\partial p} &= -2\langle\Lambda^{2m+3}p, p\rangle\langle\Lambda p, p\rangle^{-2}\Lambda p + 2\langle\Lambda p, p\rangle^{-1}\Lambda^{2m+3}p + \\ &\quad 4\sum_{j=0}^{m}\langle\Lambda^{2(m-j)+2}p, p\rangle\Lambda^{2j+2}p - \\ &\quad 2\sum_{j=0}^{m}\Big(\langle\Lambda^{2j+1}p, p\rangle\Lambda^{2(m-j)+3}\langle\Lambda^{2(m-j)+3}p, p\rangle\Lambda^{2j+1}p\Big) + \\ &\quad 2\sum_{j=0}^{m}\Big(\langle\Lambda^{2j+3}q, q\rangle\Lambda^{2(m-j)+1}p - \langle\Lambda^{2j+2}p, q\rangle\Lambda^{2(m-j)+2}q\Big) + \\ &\quad 4i\sum_{j=0}^{m}\Big(\langle\Lambda^{2j+2}p, q\rangle\Lambda^{2(m-j)+2}p - \langle\Lambda^{2j+3}p, q\rangle\Lambda^{2(m-j)+1}p\Big) + \\ &\quad 2i\sum_{j=0}^{m}\Big(\langle\Lambda^{2(m-j)+2}p, p\rangle\Lambda^{2j+2}q - \langle\Lambda^{2(m-j)+1}p, p\rangle\Lambda^{2j+3}q\Big).\end{aligned} \tag*{$(3.17)_2$}$$

Substitute (3.3), $(3.17)_1$ and $(3.17)_2$ into the following formula

$$(H, F_m) = \Big\langle\frac{\partial H}{\partial q}, \frac{\partial F_m}{\partial p}\Big\rangle - \Big\langle\frac{\partial H}{\partial p}, \frac{\partial F_m}{\partial q}\Big\rangle. \tag{3.18}$$

Through a series of careful calculations, we can obtain (3.16).

**Theorem 3.3** *i) The Hamiltonian system $(H)$ (or (3.3)) is completely integrable in the Liouville sense, and its involutive functional system is $F_m$.*

*ii) The Hamiltonian systems*

$$(F_m): \quad q_{t_m} = \frac{\partial F_m}{\partial p}, \quad p_{t_m} = -\frac{\partial F_m}{\partial q}, \quad m = 0, 1, 2, ... \tag{3.19}$$

*are completely integrable, too.*

## 4 The involutive solutions of the $HD$ hierarchy

Since $(H, F_m) = 0, \ \forall m \in Z^+$, the Hamiltonian systems $(H)$ and $(F_m)$ are compatible [22]. Hence, the solution operators $g^X$ and $g_m^{t_m}$ of initial problem of $(H)$ and $(F_m)$ commute [22]. Define

$$\begin{pmatrix} q(x,t_m) \\ p(x,t_m) \end{pmatrix} = g^x g_m^{t_m} \begin{pmatrix} q(0,0) \\ p(0,0) \end{pmatrix}, \quad m = 0, 1, 2, ..., \tag{4.1}$$

which are called the involutive solutions of compatible systems $(H)$ and $(F_m)$.

**Theorem 4.1** *Let $(q(x,t_m), p(x,t_m))^T$ be an involutive solution of compatible systems $(H)$ and $(F_m)$. Then*

$$u(x, t_m) = \langle \Lambda p, p \rangle^{-2} \tag{4.2}$$

*satisfies the higher-order $HD$ equation*

$$u_{t_m} = X_m(u) = J(J^{-1}K)^{m+1} G_{-2}, \quad G_{-2} = u^{-1/2}, \quad m = 0, 1, 2, .... \tag{4.3}$$

**Proof.** First note that

$$\begin{aligned} u_{t_m} &= -2\langle \Lambda p, p\rangle^{-3} \cdot 2\langle \Lambda p, p_{t_m}\rangle = \\ &\quad 4\langle \Lambda p, p\rangle^{-3} \langle \Lambda p, \frac{F_m}{q}\rangle = \\ &\quad 8\langle \Lambda p, p\rangle^{-3} (\langle \Lambda p, p\rangle \langle \Lambda^{2m+4} p, q\rangle - \langle \Lambda^2 q, p\rangle \langle \Lambda^{2m+3} p, p\rangle + \\ &\quad i\langle \Lambda^2 p, p\rangle \langle \Lambda^{2m+3} p, p\rangle - i\langle \Lambda p, p\rangle \langle \Lambda^{2m+4} p, p\rangle\Big). \end{aligned} \tag{4.4}$$

Acting with the operator $(J^{-1}K)^{m+1}$ upon $G_{-2} = \sum_{j=1}^{N} \bigtriangledown \lambda_j$ and noticing (2.3), we have

$$(J^{-1}K)^{m+1} G_{-2} = \sum_{j=1}^{N} \lambda_J^{2(m+1)} \bigtriangledown \lambda_j = \langle \Lambda^{2m+3} p, p\rangle. \tag{4.5}$$

Note

$$\begin{aligned} u_x &= -4\langle \Lambda p, p\rangle^{-3} \langle \Lambda p, -\Lambda q + i\Lambda p\rangle = \\ &\quad 4(\langle \Lambda p, p\rangle^{-3} \langle \Lambda^2 p, q\rangle - i\langle \Lambda p, p\rangle^{-3} \langle \Lambda^2 p, p\rangle), \end{aligned} \tag{4.6}$$

$$\begin{aligned}(\langle\Lambda^{2m+3}p,p\rangle)_x &= 2\langle\Lambda^{2m+3}p,p_x\rangle = \\ &2\langle\Lambda^{2m+3}p,-\Lambda q+i\Lambda p\rangle = \\ &2(i\langle\Lambda^{2m+4}p,p\rangle-\langle\Lambda^{2m+4}p,q\rangle),\end{aligned} \tag{4.7}$$

hence

$$\begin{aligned}(J^{-1}K)^{m+1}G_{-2} &= -2(\partial u+u\partial)\langle\Lambda^{2m+3}p,p\rangle = \\ &-2(u_x+2u\partial)\langle\Lambda^{2m+3}p,p\rangle = \\ &-8(\langle\Lambda p,p\rangle^{-3}\langle\Lambda^2 p,q\rangle\langle\Lambda^{2m+3}p,p\rangle - \\ &i\langle\Lambda p,p\rangle^{-3}\langle\Lambda^2 p,p\rangle\langle\Lambda^{2m+3}p,p\rangle) - \\ &8(i\langle\Lambda p,p\rangle^{-2}\langle\Lambda^{2m+4}p,p\rangle-\langle\Lambda p,p\rangle^{-2}\langle\Lambda^{2m+4}p,q\rangle).\end{aligned} \tag{4.8}$$

$S_0$, $u(x,t_m)=\langle\Lambda p,p\rangle^{-2}$ satisfies $u_{t_m}=J(J^{-1}K)^{m+1}G_{-2}$.

In Theorem 4.1, letting $m=0$, we can obtain the involutive solution of the $HD$ equation $u_t=(u^{-1/2})_{xxx}$, $t_0=t$.

**Corollary 4.1** Let $(q(x,t),p(x,t))^T$ be an involutive solution of the compatible systems $(H)$ and $(F_0)$. Then $u(x,t)=\langle\Lambda p,p\rangle^{-2}$ is a solution of the $HD$ equation $u_t=(u^{-1/2})_{xxx}$.

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