

# Filtered Back Projection Inversion of Turntable ISAR Data

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## ABSTRACT

In this paper, an inversion scheme for turntable inverse synthetic aperture radar (ISAR) data is derived for both two and three dimensions from a scalar wave equation model. The proposed data inversion scheme motivates the use of a filtered back projection (FBP) imaging algorithm. The paper provides a derivation of the the general imaging filter needed for FBP, which will be shown to reduce to a familiar result for near-field ISAR imaging.

**Keywords:** Synthetic Aperture Radar, ISAR, Near-Field Imaging, Fourier Diffraction Theorem, Turntable Radar, Filtered Back-Projection, Radar Tomography

## 1. INTRODUCTION

Inverse Synthetic Aperture Radar (ISAR) is a high-resolution imaging system that consists of both a real-aperture radar system and a moving target scene. An ISAR system uses a set of stationary antennas to emit a sequence of short energy bursts, and to take scattering measurements in the presence of a moving target scene. There are many works on the fundamental principles of ISAR imaging,<sup>1,5,12,16</sup> in which it has been shown that both two-dimensional and three-dimensional image formation is possible for targets undergoing either rotational or translational motion. Efficient near-field algorithms<sup>3,10,22</sup> for processing ISAR data have also been proposed for two and three-dimensional imaging, which can be derived via the filtered adjoint inversion approach proposed in this paper.

In section 2, the general ISAR received signal for start-stop turntable setups is first derived from a scalar wave equation model. The scalar wave model used in the paper is derived from Maxwell's equations by using a first-order Born approximation.<sup>2</sup> A short review of far-field radar imaging is included beginning with the derivation of the Fourier diffraction theorem in radar imaging, which relates the ISAR received signal  $S$  to the Fourier transform of the target reflectivity function  $V$ . The Fourier Diffraction theorem (FDT)<sup>9</sup> provides the basis for most FFT radar imaging algorithms. The derivation of the FDT is followed by a quick discussion of the forward model for turntable ISAR data in two and three dimensions.

In section 3 the general ISAR received signal,  $S$ , is first written in terms of a Fourier integral operator (FIO),  $S = FV$ , in which it will be noted that the ISAR signal in the frequency domain is a function on the unit cylinder  $C^m$ , where  $m = 2, 3$  depending on system configuration. The formal adjoint of the forward ISAR FIO on  $L^2(C^m)$  is then derived for both two and three dimensions on the basis of a filtered back projection ISAR imaging algorithm via  $V = F^\dagger Q F V$ . In the case of turntable ISAR, applying the back projection operator does not result in an exact inversion of the data, which actually requires evaluating an appropriate data filter  $Q$ . The filter  $Q$  is determined by using the stationary phase method.

## 2. THE FORWARD PROBLEM

In order to begin a discussion of the ISAR FBP imaging operator, we will first introduce the mathematical notion of the radar received signal. In radar imaging applications the received signal is a recorded measurement of a scattered electromagnetic field. For this reason, the ideal mathematical model for radar imaging would be

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that found in Maxwell's equations for the electromagnetic field.<sup>20</sup> We will use the following scalar-wave equation model\* for radar imaging:

$$(\nabla^2 + k^2) U^{\text{in}}(f, \mathbf{x}) = -J(f, \mathbf{x}), \quad (1a)$$

$$(\nabla^2 + k^2) U^{\text{sc}}(f, \mathbf{x}) = -V(\mathbf{x})U^{\text{in}}(f, \mathbf{x}), \quad (1b)$$

where  $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ ,  $k = 2\pi f/c$  and  $c$  is the speed of light in a vacuum. The scalar quantities  $U^{\text{in}}$  and  $U^{\text{sc}}$  denote the incident and scattered fields, respectively. The quantity  $J^\dagger$  denotes the source due to signal transmission, and  $V$  is the target reflectivity function. We will assume that the quantities  $U^{\text{in}}$ ,  $U^{\text{sc}}$  and  $J$  are square-integrable in the frequency variable,  $f$ , so that their corresponding inverse Fourier transforms,  $u^{\text{in}}$ ,  $u^{\text{sc}}$  and  $j$ , exist in the time variable  $t$ . The time domain version of (1) is given by:

$$\left(\nabla^2 - \frac{1}{c_0^2} \partial_t^2\right) u^{\text{in}}(t, \mathbf{x}) = -j(t, \mathbf{x}), \quad (2a)$$

$$\left(\nabla^2 - \frac{1}{c_0^2} \partial_t^2\right) u^{\text{sc}}(t, \mathbf{x}) = -V(\mathbf{x})u^{\text{in}}(t, \mathbf{x}), \quad (2b)$$

According to this scalar model (2), the scattered field depends on the incident field, which depends on the source due to transmission. Thus, in order to come up with an explicit expression for the received signal, we can begin by re-writing the source term  $j$  in (2a) in terms of the transmitted signal  $p(t)$ . Here we will assume that the transmitter radiates isotropically and is centered at  $\mathbf{x}_0 \in \mathbb{R}^3$  so that we can write  $j(t, \mathbf{x}) = p(t)\delta(\mathbf{x} - \mathbf{x}_0)$ . With knowledge of the Green's function for the 3D scalar wave operator, we find that the incident field measured at a point  $\mathbf{z}$  is given by:

$$u^{\text{in}}(t, \mathbf{z}) = \frac{p(t - |\mathbf{z} - \mathbf{x}_0|/c_0)}{4\pi |\mathbf{z} - \mathbf{x}_0|} \quad (3)$$

Using the same procedure leads to the scattered field measured at a point  $\mathbf{y}$ :

$$u^{\text{sc}}(t, \mathbf{x}) = \int_{\mathbb{R}^3} V(\mathbf{z}) \frac{p\left(t - \frac{|\mathbf{z} - \mathbf{x}_0| + |\mathbf{z} - \mathbf{y}|}{c_0}\right)}{(4\pi)^2 |\mathbf{z} - \mathbf{x}_0|^2} d\mathbf{z}. \quad (4)$$

Thus the received signal, assuming a mono-static setup, is given by:

$$s(t) \equiv u^{\text{sc}}(t, \mathbf{x}_0) = \int_{\mathbb{R}^3} V(\mathbf{z}) \frac{p\left(t - \frac{2|\mathbf{z} - \mathbf{x}_0|}{c_0}\right)}{(4\pi)^2 |\mathbf{z} - \mathbf{x}_0|^2} d\mathbf{z}, \quad (5)$$

or in the frequency domain:

$$S(f) = P(f) \int_{\mathbb{R}^3} V(\mathbf{z}) \frac{e^{-i2k|\mathbf{z} - \mathbf{x}_0|}}{(4\pi)^2 |\mathbf{z} - \mathbf{x}_0|^2} d\mathbf{z}, \quad (6)$$

## 2.1 The Fourier Diffraction Theorem in ISAR Imaging

Many ISAR imaging systems are set up so that distances to targets are considerably large. In these cases it is quite useful to employ some sort of far-field approximation to aid in the analysis of the imaging problem. Note that the range from the antenna located at  $\mathbf{x}_0$  on the transmitter to that target located at  $\mathbf{z}$  is given by:

$$|\mathbf{z} - \mathbf{x}_0| = |\mathbf{x}_0| - \widehat{\mathbf{x}}_0 \cdot \mathbf{z} + O\left(\frac{|\mathbf{z}|^2}{|\mathbf{x}_0|}\right) \quad (7)$$

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\*One can derive this scalar wave model from Maxwell's equations by making three assumptions: 1) The scattering medium described by  $V$  has no internal sources. 2) The inductive capacities  $\mu$  and  $\epsilon$  of the medium vary "slowly" throughout the medium with regards to the variable  $\mathbf{x}$ . 3) The total fields  $\mathbf{E}$  and  $\mathbf{H}$  can be approximated by the incident fields  $\mathbf{E}^{\text{inc}}$  and  $\mathbf{H}^{\text{inc}}$  (i.e. the Born Approximation). Consequently each scalar component of the fields  $\mathbf{E}^{\text{inc}}$ ,  $\mathbf{H}^{\text{inc}}$ ,  $\mathbf{E}^{\text{sc}}$ ,  $\mathbf{H}^{\text{sc}}$  satisfy (1) with  $V(f, \mathbf{x}) = -k^2 [n^2(\mathbf{x}) - 1]$ , where the quantity  $n(\mathbf{x}) = \sqrt{\frac{\mu(\mathbf{x})\epsilon(\mathbf{x})}{\mu_0\epsilon_0}}$  is the refractive index of the medium.

†The source term  $J$  does not directly correspond to the current density function  $\mathbf{J}$ , but is related to both  $\mathbf{J}$  and the charge density function  $\rho$  by  $[J_i] = \frac{1}{c_0} \nabla(\rho) + i\omega\mu_0\mathbf{J}$  for the total field  $\mathbf{E}$  field and  $[J_i] = -\nabla \times \mathbf{J}$  for the  $\mathbf{H}$  field.

We find that we can rewrite the integrand of the integral in (6) as:

$$V(\mathbf{z}) \frac{e^{-i2k|\mathbf{z}-\mathbf{x}_0|}}{(4\pi)^2 |\mathbf{z}-\mathbf{x}_0|^2} = V(\mathbf{z}) \frac{e^{-i2k|\mathbf{x}_0|}}{(4\pi)^2 |\mathbf{x}_0|^2} e^{i2k\widehat{\mathbf{x}}_0 \cdot \mathbf{z}} \left(1 + O\left(\frac{|\mathbf{z}|}{|\mathbf{x}_0|}\right)\right) \left(1 + O\left(\frac{k|\mathbf{z}|^2}{|\mathbf{x}_0|}\right)\right) \quad (8)$$

Thus, if we make the assumption that  $|\mathbf{z}|, k|\mathbf{z}|^2 \ll |\mathbf{x}_0|$ , then our far-field version of the received signal is given by:

$$\begin{aligned} S(f) &= P(f) \frac{e^{-i2k|\mathbf{x}_0|}}{(4\pi)^2 |\mathbf{x}_0|^2} \int_{\mathbb{R}^3} V(\mathbf{z}) e^{i2k\widehat{\mathbf{x}}_0 \cdot \mathbf{z}} d\mathbf{z} \\ &= P(f) \frac{e^{-i2k|\mathbf{x}_0|}}{(4\pi)^2 |\mathbf{x}_0|^2} \mathcal{F}\{V\}(-2k\widehat{\mathbf{x}}_0). \end{aligned} \quad (9)$$

Many radar imaging algorithms employ a matched-filtering step to recover the set of range profiles,  $d$ , whose spectrum,  $D$ , is given by:

$$\begin{aligned} D(f) &= (4\pi)^2 |\mathbf{x}_0|^2 e^{i2k|\mathbf{x}_0|} \overline{P(f)} S(f) \\ &= |P(f)|^2 \mathcal{F}\{V\}(-2k\widehat{\mathbf{x}}_0) \end{aligned} \quad (10)$$

where  $\overline{P}$  denotes the complex conjugate of  $P$ . The actual matched-filtering step is implemented by multiplying each individual ISAR signal  $S$  by  $\overline{P}$  in the frequency domain. In equation (10), the  $(4\pi)^2 |\mathbf{x}_0|^2$  factor compensates for power attenuation while the complex exponential compensates for what is called ‘‘range offset’’. Equations (9) and equation (10), alike, are known as the Fourier Diffraction theorem (FDT) for radar imaging, which states that the radar received signal scattered from a target scene is proportional to the three-dimensional Fourier transform of its reflectivity function evaluated at  $-2k\widehat{\mathbf{x}}_0$ .

The transmitted signal  $P$  is band-limited with a support band given by the set  $\{f_c - \frac{B}{2}, f_c + \frac{B}{2}\}$ , where  $B$  is the signal bandwidth, and is usually chosen so that its power spectrum is given by  $|P(f)|^2 \approx \text{rect}\left(\frac{f-f_c}{B}\right)$ . Thus the FDT reveals that each radar signal gives the three-dimensional Fourier transform of the target reflectivity evaluated on the finite line-segment determined by  $\mathbf{K} = -2k\widehat{\mathbf{x}}_0$ ,  $f \in [f_c - \frac{B}{2}, f_c + \frac{B}{2}]$ . Consequently, the object in radar imaging is to populate the wave number domain with as much data as needed to reconstruct the reflectivity function  $V$ .

## 2.2 Two-Dimensional Turntable ISAR Imaging

In Synthetic Aperture Radar (SAR) imaging, a stationary target scene is imaged via numerous radar returns measured along some sort of flight path. A changing sensor position corresponds to a changing  $\widehat{\mathbf{x}}_0$ , which results in a means of populating the wave number domain according to the FDT. In ISAR, a stationary radar sensor measures numerous returns scattered from a moving target, which we will see is also mathematically equivalent to a changing sensor location  $\widehat{\mathbf{x}}_0$ , if we stick the coordinate system origin at the center of the moving target scene. In a turntable ISAR imaging setup<sup>‡</sup> the target scene is rotated counter-clockwise about the  $x_3$ -axis by  $\phi$  radians. If we were to let  $q$  denote a stationary scattering density, then we have  $V(\mathbf{x}) = q(\mathcal{R}_1^{-1}(\phi)\mathbf{x})^\S$ , where it can be found that the rotational matrix  $\mathcal{R}_1$  is given by

$$\mathcal{R}_1(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (11)$$

<sup>‡</sup>Many turntable radar setups have a bi-static configuration. That is the transmitter and receiver are generally located at two different positions in the system geometry. It is usually the case that system configuration can be modified so that the antenna range to any target on the turntable is approximately the same for both the transmitter and receiver.

<sup>§</sup>Note that the use of  $\mathcal{R}_1^{-1}$ , rather than  $\mathcal{R}_1$ , corresponds to counter clockwise turntable rotation.

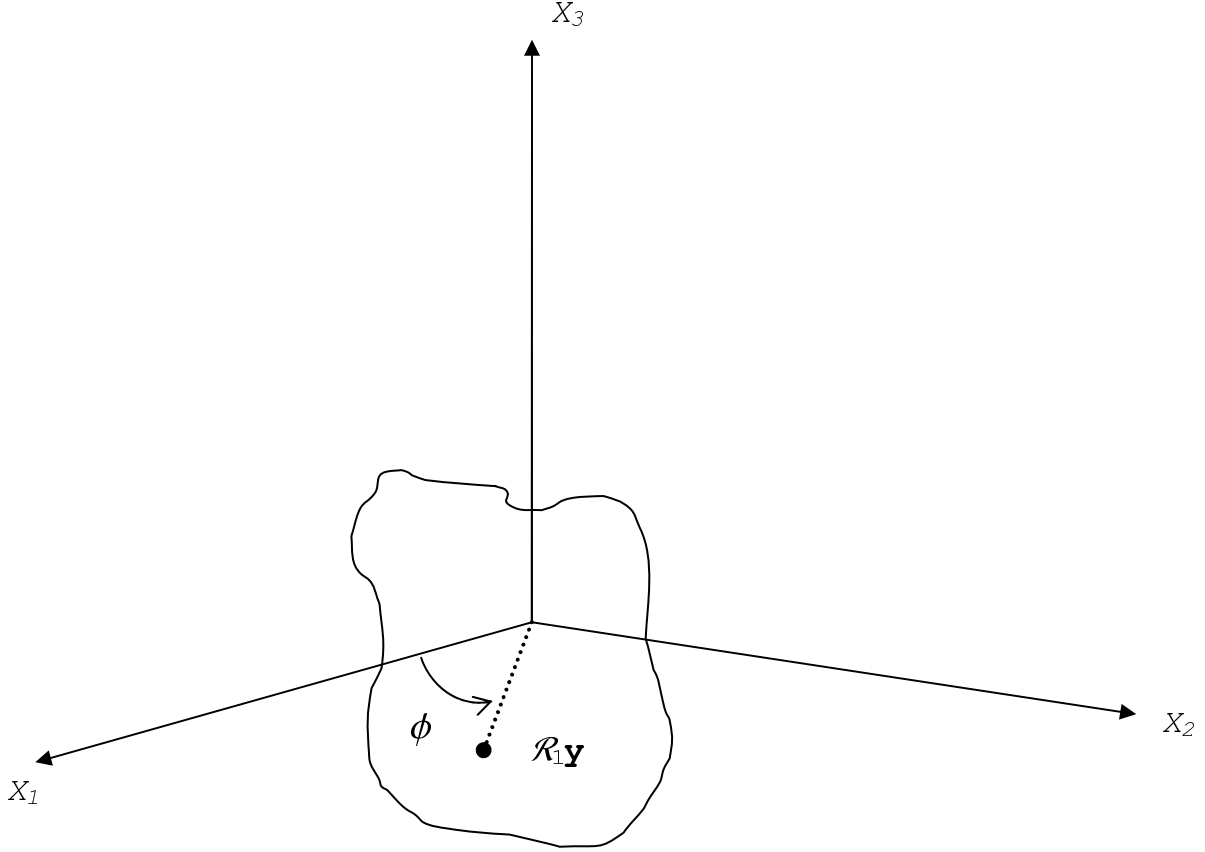


Figure 1: Target rotated by  $\phi$  radians

One can check that for every  $\phi \in [0, 2\pi]$ ,  $\mathcal{R}_1$  is an orthogonal transformation (i.e.  $\mathcal{R}_1^{-1}$  exists and  $\mathcal{R}_1^{-1} = \mathcal{R}_1^T$ ). For each value of  $\phi$  the data set recorded by the ISAR system is given by

$$\begin{aligned}
 D(f) &= |P(f)|^2 \int_{\mathbb{R}^3} q(\mathcal{R}_1^{-1}\mathbf{z}) e^{i2k\widehat{\mathbf{x}}_0 \cdot \mathbf{z}} d\mathbf{z} \\
 &= |P(f)|^2 \int_{\mathbb{R}^3} q(\mathbf{y}) e^{i2k\widehat{\mathbf{x}}_0 \cdot \mathcal{R}_1\mathbf{y}} d\mathbf{y} \\
 &= |P(f)|^2 \int_{\mathbb{R}^3} q(\mathbf{y}) e^{i2k\mathcal{R}_1^{-1}\widehat{\mathbf{x}}_0 \cdot \mathbf{y}} d\mathbf{y} \\
 &= |P(f)|^2 \mathcal{F}_3 \{q\}(\mathbf{K}),
 \end{aligned} \tag{12}$$

where  $\mathbf{K} = -2k\mathcal{R}_1^{-1}\widehat{\mathbf{x}}_0$ . To understand what is happening in the wave number domain, we assume that the target scene is to be rotated clock-wise, and take  $\widehat{\mathbf{x}}_0 = (1, 0, 0)^T$ . Thus, if  $\tilde{k} = -2k$ , then  $\mathbf{K} = (K_1, K_2, K_3)^T = (\tilde{k} \cos \phi, \tilde{k} \sin \phi, 0)^T$ , and the signal in the  $(f, \phi)$  space is determined by:

$$\tilde{D}(f, \phi) = |P(f)|^2 \mathcal{F}_2 \{\tilde{q}\}(\tilde{k}\boldsymbol{\mu}), \tag{13}$$

where  $\mathcal{F}_2 \{h\}(\Omega)$  denotes the two-dimensional Fourier transform of  $h$  evaluated at the point  $\Omega$  in 2D Fourier space,  $\tilde{q}(y_1, y_2) = \int q(y_1, y_2, y_3) dy_3$  is the projection of  $q$  onto the  $x_1 - x_2$  axis, and  $\boldsymbol{\mu} = (\cos \phi, \sin \phi)^T$ . Note that the vector  $\boldsymbol{\mu}$  is an element of the unit circle  $S^1$  (the set of all unit vectors in  $\mathbb{R}^2$ ). The data  $\tilde{D}$  can be

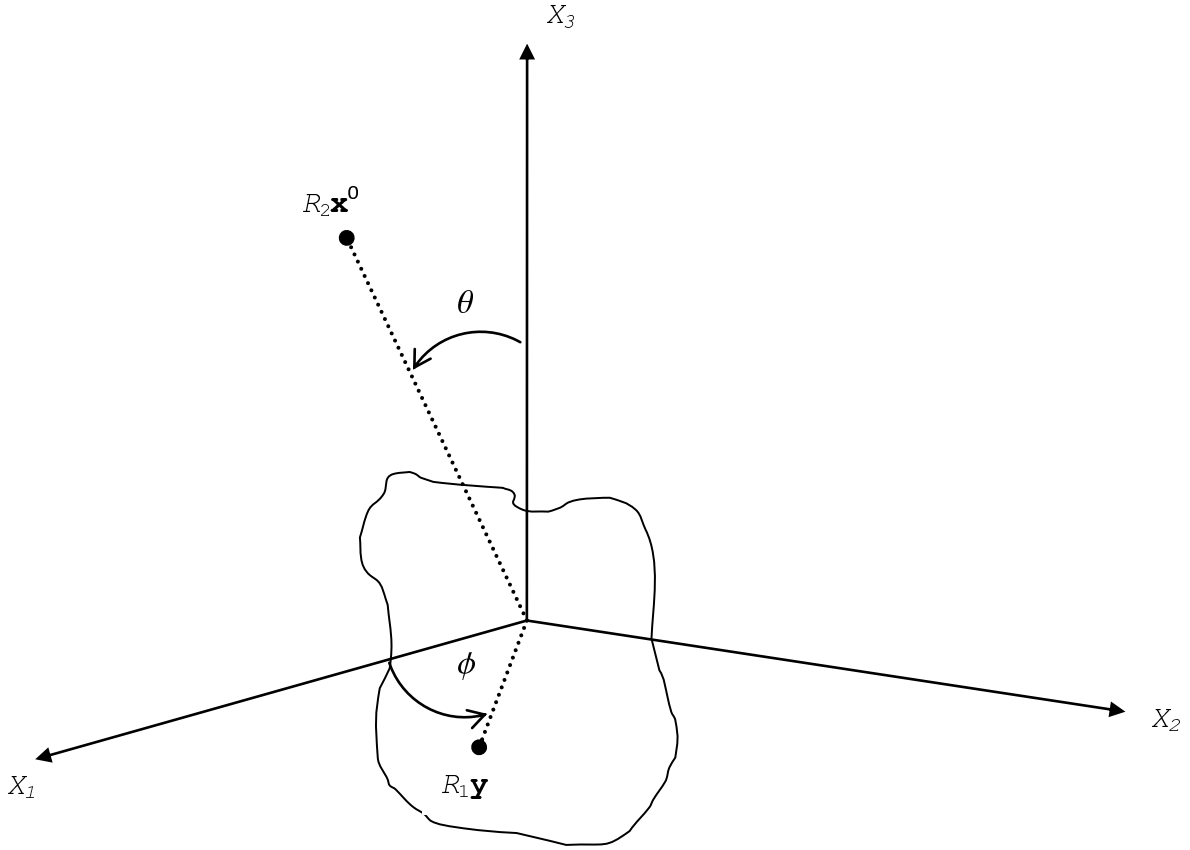


Figure 2: Target rotated by  $\phi$  radians with antenna rotated by  $\theta$  radians.

sampled over a two-dimensional Cartesian grid, allowing us to form a 2D image via an inverse 2D spatial Fourier transform:

$$I(\mathbf{x}) = \frac{1}{(2\pi)^2} \int E(K_1, K_2) e^{iK_1x_1} e^{iK_2x_2} dK_1 dK_2, \quad (14)$$

where  $E$  is the wave number data sampled over a Cartesian data grid.

### 2.3 Three-Dimensional Turntable ISAR Imaging

If in the previous turntable setup we also rotate the radar antenna about the  $x_2$  axis by  $\theta$  radians, we have that the new antenna position is given by  $\mathcal{R}_2(\theta)\mathbf{x}_0$ , where:

$$\mathcal{R}_2(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}. \quad (15)$$

This implies that the data set recorded by the ISAR system is then determined by

$$\begin{aligned} D(f) &= |P(f)|^2 \int_{\mathbb{R}^3} q(\mathcal{R}_1^{-1}(\phi)\mathbf{z}) e^{i2k\widehat{\mathcal{R}_2(\theta)\mathbf{x}_0} \cdot \mathbf{z}} d\mathbf{z} \\ &= |P(f)|^2 \mathcal{F}_3\{q\}(\mathbf{K}), \end{aligned} \quad (16)$$

where  $\mathbf{K} = -2k\mathcal{O}(\phi, \theta)\widehat{\mathbf{x}}_0$ , and  $\mathcal{O}(\phi, \theta) = (\mathcal{R}_1^{-1}\mathcal{R}_2)(\phi, \theta)$ . If we let  $\widehat{\mathbf{x}}_0 = (0, 0, 1)^\top$  (that is the initial antenna position is located on the  $x_3$  axis) we see that

$$\mathbf{K} = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} \tilde{k} \cos \phi \sin \theta \\ \tilde{k} \sin \phi \sin \theta \\ \tilde{k} \cos \theta \end{pmatrix} = \tilde{k}\boldsymbol{\mu}, \quad (17)$$

where  $\boldsymbol{\mu} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)^\top$  is an element of the unit sphere  $S^2$  (the set of all unit vectors in  $\mathbb{R}^3$ ). Thus the  $\mathbf{K}$ -domain data takes up a three-dimensional space, allowing us to form a 3D image via an inverse 3D Fourier transform:

$$I(\mathbf{x}) = \frac{1}{(2\pi)^3} \int E(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{x}}d\mathbf{K}. \quad (18)$$

### 3. FILTERED-ADJOINT ISAR IMAGING

The framework for a filtered back projection ISAR imaging algorithm is given in this section. Recall that the received ISAR signal,  $S$ , is a function of both frequency,  $f$ , and angle,  $\boldsymbol{\mu}$ . We will further assume that  $S$  is a member of  $L^2(C^m)$ , the space of square-integrable functions on the unit cylinder  $C^m = \mathbb{R} \times S^{m-1}$ . Here  $m = 2$  or  $m = 3$ , corresponding to the number of imaging dimensions. In either case we can write the reflectivity function in terms of a rotating scattering density function:

$$V(\mathbf{x}) = q(\mathcal{O}_m\mathbf{x}). \quad (19)$$

When  $m = 2$ , we have  $\mathcal{O}_m = \mathcal{R}_1^{-1}$ , and when  $m = 3$ , we have  $\mathcal{O}_m = \mathcal{R}_1^{-1}\mathcal{R}_2$ , where  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are the orthogonal linear transformations given in the previous section. According to equation (6), this implies that the ISAR signal is given by:

$$S(f) = P(f) \int_{\mathbb{R}^3} q(\mathbf{y}) \frac{e^{-i2k|\mathbf{y}-\mathcal{O}_m\mathbf{x}_0|}}{(4\pi)^2 |\mathbf{y}-\mathcal{O}_m\mathbf{x}_0|^2} d\mathbf{y} \quad (20)$$

#### 3.1 The ISAR Back Projection Operator

If we let  $r_a$  denote the distance of the radar sensor to the scene center, then for  $m = 2$  we let  $\mathbf{x}_0 = (r_a, 0, 0)^\top$ . Thus we can write  $\mathcal{O}_m\mathbf{x}_0 = r_a\boldsymbol{\mu}$ , where  $\boldsymbol{\mu} \in S^1$ . Likewise, for  $m = 3$  we let  $\mathbf{x}_0 = (0, 0, r_a)^\top$  so that we can write  $\mathcal{O}_m\mathbf{x}_0 = R_a\boldsymbol{\mu}$ , where  $\boldsymbol{\mu} \in S^2$ . In either case equation (20) can be written as:

$$S(f, \boldsymbol{\mu}) = F_m \{q\}(f, \boldsymbol{\mu}), \quad (21)$$

where

$$F_m \{q\}(f, \boldsymbol{\mu}) = \int_{\mathbb{R}^3} q(\mathbf{y})e^{-i2kR_m(\mathbf{y}, \boldsymbol{\mu})} A_m(f, \boldsymbol{\mu}, \mathbf{y}) d\mathbf{y}, \quad (22)$$

$$\mathbf{R}_m(\mathbf{z}, \boldsymbol{\mu}) = \begin{cases} \mathbf{z} - R_a(\boldsymbol{\mu}, 0)^\top, & \text{if } m = 2 \\ \mathbf{z} - R_a\boldsymbol{\mu}, & \text{if } m = 3, \end{cases} \quad (23)$$

$$A_m(f, \boldsymbol{\mu}, \mathbf{y}) = \frac{P(f)}{(4\pi)^2 (R_m(\mathbf{y}, \boldsymbol{\mu}))^2}, \quad (24)$$

and  $R_m(\mathbf{z}, \boldsymbol{\mu}) = |\mathbf{R}_m(\mathbf{z}, \boldsymbol{\mu})|$ . Note that we have used the symbol  $F$  to denote the forward ISAR FIO. A derivation of the formal adjoint of this FIO is now in order. For functions  $G, H \in L^2(C^m)$ , define the inner product  $(G, H)$ , by:

$$(G, H)_m = \int_{S^{m-1}} \int_{\mathbb{R}} G(f, \boldsymbol{\mu})\overline{H(f, \boldsymbol{\mu})} df d\boldsymbol{\mu}, \quad (25)$$

and recall the Hermitian inner product on  $L^2(\mathbb{R}^3)$ :

$$\langle g, h \rangle = \int_{\mathbb{R}^3} g(\mathbf{x})\overline{h(\mathbf{x})} d\mathbf{x}, \quad g, h \in L^2(\mathbb{R}^3) \quad (26)$$

Define the operator  $\mathcal{B}_m$  on  $L^2(C^m)$ , by:

$$\mathcal{B}_m \{H\}(\mathbf{x}) = \int_{S^{m-1}} \int_{\mathbb{R}} H(f, \boldsymbol{\mu}) \overline{A_m(f, \boldsymbol{\mu}, \mathbf{x})} e^{i2kR_m(\mathbf{x}, \boldsymbol{\mu})} df d\boldsymbol{\mu}. \quad (27)$$

A quick application of Fubini's theorem shows that  $\mathcal{B}_m = F_m^\dagger$  on  $L^2(C^m)$ . That is for  $g \in L^2(\mathbb{R}^3)$  and  $H \in L^2(C^m)$ , then  $(F_m g, H)_m = \langle g, \mathcal{B}_m H \rangle$ , thus making  $\mathcal{B}_m$  our ISAR back projection operator. We now seek to form an image  $I$  via:

$$I(\mathbf{x}) = (\mathcal{B}_m \mathcal{Q}_m) \{S\}(\mathbf{x}), \quad (28)$$

where  $\mathcal{Q}_m = \mathcal{Q}_m(f, \boldsymbol{\mu}, \mathbf{x})$  is a filter acting on  $S$ .

### 3.2 Point Spread Function Analysis using the Stationary Phase Method

In order to lighten the processing load of an imaging algorithm most radar systems make use of quadrature sampling which has the useful property of converting a real signal into a complex signal. Since the carrier frequency of a radar signal is usually chosen to be larger than half of the transmitted signal bandwidth, the signal's support band is a subset of the positive half plane. This means that we can assume that the ISAR system is operating on only positive frequencies. Thus we can write:

$$I(\mathbf{x}) = \int_{\mathbb{R}^3} q(\mathbf{y}) K(\mathbf{x}, \mathbf{y}) d\mathbf{y}, \quad (29)$$

where the point spread function (PSF),  $K$  is given by:

$$K(\mathbf{x}, \mathbf{y}) = \int_{S^{m-1}} \int_0^\infty e^{i\varphi(f, \boldsymbol{\mu}, \mathbf{x}, \mathbf{y})} a(f, \boldsymbol{\mu}, \mathbf{x}, \mathbf{y}) df d\boldsymbol{\mu}, \quad (30)$$

where  $\varphi(f, \boldsymbol{\mu}, \mathbf{x}, \mathbf{y}) = 2k [R_m(\mathbf{x}, \boldsymbol{\mu}) - R_m(\mathbf{y}, \boldsymbol{\mu})]$  and  $a(f, \boldsymbol{\mu}, \mathbf{x}, \mathbf{y}) = (\overline{A_m} \mathcal{Q}_m A_m)(f, \boldsymbol{\mu}, \mathbf{x}, \mathbf{y})$ . In order to obtain an exact inversion formula, we must have:

$$K(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i(\mathbf{x}-\mathbf{y}) \cdot \boldsymbol{\xi}} d\boldsymbol{\xi} \quad (31)$$

Note that this would require the phase function  $\varphi$  to have a single non-degenerate critical point at  $\mathbf{y} = \mathbf{x}$ . That is,  $\nabla_m \varphi(f, \boldsymbol{\mu}, \mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}} = \mathbf{0}$  such that the Hessian matrix  $D_m^2 \varphi$  is nonsingular, according to the stationary phase theorem.<sup>19</sup> Here, the gradient operator in polar/spherical coordinates is given by  $\nabla_m = \frac{\partial}{\partial f} \hat{\mathbf{f}} + \frac{\partial_m}{\partial \boldsymbol{\mu}}$ , and

$$\frac{\partial}{\partial \boldsymbol{\mu}} = \begin{cases} \frac{1}{f} \frac{\partial}{\partial \phi} \hat{\boldsymbol{\phi}}, & \text{if } m = 2 \\ \frac{1}{f} \frac{\partial_m}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{f \sin \theta} \frac{\partial}{\partial \phi} \hat{\boldsymbol{\phi}}, & \text{if } m = 3 \end{cases} \quad (32)$$

We first make a change of variables,  $f = \beta f'$ , where  $\beta > 0$ :

$$K(\mathbf{x}, \mathbf{y}) = \beta \int_{S^{m-1}} \int_0^\infty e^{i\beta \varphi'(f', \boldsymbol{\mu}, \mathbf{x}, \mathbf{y})} a'(f', \boldsymbol{\mu}, \mathbf{x}, \mathbf{y}) df' d\boldsymbol{\mu}, \quad (33)$$

where  $\varphi'(f', \boldsymbol{\mu}, \mathbf{x}, \mathbf{y}) = \frac{4\pi f'}{c} [R_m(\mathbf{x}, \boldsymbol{\mu}) - R_m(\mathbf{y}, \boldsymbol{\mu})]$  and  $a'(f', \boldsymbol{\mu}, \mathbf{x}, \mathbf{y}) = a(\beta f', \boldsymbol{\mu}, \mathbf{x}, \mathbf{y})$ . We then apply the modified gradient operator,  $\nabla_m = \frac{\partial}{\partial f'} \hat{\mathbf{f}} + \frac{\partial_m}{\partial \boldsymbol{\mu}}$ , to the phase  $\varphi$ . In either case for  $m$ , we have that the major contribution for large  $\beta$  comes from the critical points described by

$$\frac{\partial}{\partial f'} \varphi'(f', \boldsymbol{\mu}, \mathbf{x}, \mathbf{y}) = \frac{4\pi}{c} [R_m(\mathbf{x}, \boldsymbol{\mu}) - R_m(\mathbf{y}, \boldsymbol{\mu})] = 0 \quad (34)$$

$$\frac{\partial_m}{\partial \boldsymbol{\mu}} \varphi'(f', \boldsymbol{\mu}, \mathbf{x}, \mathbf{y}) = \frac{4\pi f'}{c} [\hat{\mathbf{R}}_m(\mathbf{x}, \boldsymbol{\mu}) - \hat{\mathbf{R}}_m(\mathbf{y}, \boldsymbol{\mu})] \cdot \frac{\partial_m}{\partial \boldsymbol{\mu}} (\mathcal{O}_m \mathbf{x}_0) = 0 \quad (35)$$

The first equation tells us that the distance from the antenna to the target located at  $\mathbf{x}$  must be the same as the distance to the target located at  $\mathbf{y}$ . The second equation tells us that the range rate (giving rise to a Doppler shift) must be the same for  $\mathbf{x}$  and  $\mathbf{y}$ . Thus we have the familiar idea of locating a target from its range and Doppler shift.<sup>7</sup>

### 3.3 Evaluating the Filter $\mathcal{Q}$

In Cartesian coordinates (i.e. making the change of variables:  $\boldsymbol{\eta} = f\boldsymbol{\mu}$ ), we have that equation (30):

$$K(\mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^m} e^{i\varphi''(\boldsymbol{\eta}, \mathbf{x}, \mathbf{y})} a''(\boldsymbol{\eta}, \mathbf{x}, \mathbf{y}) d\boldsymbol{\eta}, \quad (36)$$

where  $\varphi''(\boldsymbol{\eta}, \mathbf{x}, \mathbf{y}) = \varphi(\eta, \hat{\boldsymbol{\eta}}, \mathbf{x}, \mathbf{y})$  and  $a''(\boldsymbol{\eta}, \mathbf{x}, \mathbf{y}) = \eta^{1-m} a(\eta, \hat{\boldsymbol{\eta}}, \mathbf{x}, \mathbf{y})$ , where  $\eta = |\boldsymbol{\eta}|$  and  $\hat{\boldsymbol{\eta}} = \frac{\boldsymbol{\eta}}{\eta}$ . Note that:

$$\begin{aligned} \varphi''(\boldsymbol{\eta}, \mathbf{x}, \mathbf{y}) &= \frac{4\pi\eta}{c} \int_0^1 \frac{d}{d\lambda} R_m(\mathbf{x}, \hat{\boldsymbol{\eta}})|_{\mathbf{x}=\mathbf{y}+\lambda(\mathbf{x}-\mathbf{y})} d\lambda \\ &= (\mathbf{x} - \mathbf{y}) \cdot \Xi(\boldsymbol{\eta}, \mathbf{x}, \mathbf{y}), \quad \Xi(\boldsymbol{\eta}, \mathbf{x}, \mathbf{y}) = \frac{4\pi\eta}{c} \int_0^1 \nabla_{\mathbf{x}} R_m(\mathbf{x}, \hat{\boldsymbol{\eta}})|_{\mathbf{x}=\mathbf{y}+\lambda(\mathbf{x}-\mathbf{y})} d\lambda \end{aligned} \quad (37)$$

Introducing the change of variables,  $\boldsymbol{\xi} = \Xi(\boldsymbol{\eta}, \mathbf{x}, \mathbf{y})$ , then we have:

$$K(\mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^m} e^{i(\mathbf{x}-\mathbf{y}) \cdot \boldsymbol{\xi}} a''(\boldsymbol{\eta}(\boldsymbol{\xi}), \mathbf{x}, \mathbf{y}) \left| \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}} \right| d\boldsymbol{\xi}, \quad (38)$$

where  $\left| \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}} \right|$  is the Jacobian determinant resulting from the change of variables. If we note that

$$a''(\boldsymbol{\eta}(\boldsymbol{\xi}), \mathbf{x}, \mathbf{y}) = (\eta(\boldsymbol{\xi}))^{1-m} (\bar{A}_m \mathcal{Q}_m A_m)(f(\boldsymbol{\xi}), \boldsymbol{\mu}(\boldsymbol{\xi}), \mathbf{x}, \mathbf{y}), \quad (39)$$

we see that we can approximate the PSF for exact inversion in equation (31) by letting

$$\mathcal{Q}_m(f, \boldsymbol{\mu}, \mathbf{x}) = \frac{(4\pi)^4 (R_m(\mathbf{x}, \boldsymbol{\mu}))^4 f^{m-1}}{|P(f)|^2 \left| \left( \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}} \right) (f, \boldsymbol{\mu}) \right|} \chi(f, \boldsymbol{\mu}), \quad (40)$$

where  $\chi(\mathbf{x}, f\boldsymbol{\mu})$  is a smooth cutoff function that prevents from dividing by zero.

Given the ISAR data set  $S(f, \boldsymbol{\mu})$ , note that applying the proposed FBP operation to the data set is mathematically equivalent to the following imaging operation:

$$I(\mathbf{x}) = \int_{S^{m-1}} \int_0^\infty S(f, \boldsymbol{\mu}) \Psi(f, \boldsymbol{\mu}, \mathbf{x}) df d\boldsymbol{\mu}, \quad (41)$$

where

$$\Psi(f, \boldsymbol{\mu}, \mathbf{x}) = \frac{(4\pi)^2 (R_m(\mathbf{x}, \boldsymbol{\mu}))^2 f^{m-1}}{P(f) \left| \left( \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}} \right) (f, \boldsymbol{\mu}) \right|} \chi(f, \boldsymbol{\mu}) e^{i2kR_m(\mathbf{x}, \boldsymbol{\mu})} \quad (42)$$

takes on the form of a general ‘‘focusing’’ operator for Spherical Wave ISAR (SWISAR).<sup>3,10,22</sup> Thus, the proposed inversion scheme has immediate applications for near-field configurations. The  $(P(f))^{-1}$  term in equation (42) refers to what is known as a source deconvolution operation for the range imaging problem,<sup>17</sup> and can be replaced by  $\overline{P(f)}$  for imaging systems that make use of matched filtering.

## 4. CONCLUSIONS

In this paper, we have provided a filtered back projection inversion method for mono-static ISAR radar imaging. In order to achieve this we have derived a forward integral operator model for the forward ISAR problem, and evaluated its formal adjoint (back projection operator) in the signal function space. Applying the stationary phase method to our inversion procedure, we have shown that the filtered back projection data approximates the desired target reflectivity function with large system bandwidth. One of the advantages in the filtered back projection approach is that there is no need of employing a far-field approximation for the radar data. Thus our approach makes this filtered back projection method ideal for near-field imaging. Although no numerical results are provided in the paper to demonstrate this, it is shown that the FBP scheme reduces to similar results for near-field turntable radar imaging which have been proven to be efficient in ISAR image formation. We will realize numerical results in both 2D and 3D very soon.



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