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The Neumann Type Systems and Algebro-Geometric Solutions of a System of Coupled Integrable Equations

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Abstract A system of (1+1)-dimensional coupled integrable equations is decomposed into a pair of new Neumann type systems that separate the spatial and temporal variables for this system over a symplectic submanifold. Then, the Neumann type flows associated with the coupled integrable equations are integrated on the complex tour of a Riemann surface. Finally, the algebro-geometric solutions expressed by Riemann theta functions of the system of coupled integrable equations are obtained by means of the Jacobi inversion.

Keywords Integrable equations · Neumann type systems · Algebro-geometric solutions

Mathematics Subject Classifications (2010) 37K10 · 37J35 · 70H06

1 Introduction

The Neumann system of harmonic oscillator constrained on the unit sphere is a prototype of finite dimensional integrable system (FDIS) with rich mathematical natures in the area of classical mechanics [22]. Based on the Flaschka's idea, Moser's, Veselov's and Knoerrer's work [14, 19, 23, 24, 35], a number of new

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FDISs of both Neumann and Bargmann types were found under a symmetric constraint between spectral potentials and eigenfunctions in the framework of the nonlinearization of Lax pair [4, 5]. The FDISs of Bargmann type are the canonical Hamiltonian systems produced under a Bargmann constraint from the Lax pair of an integrable equation; while the FDISs of Neumann type are generated under a Neumann constraint on the symplectic submanifold [6, 9, 11, 27, 28, 33, 37, 38]. Those resultant FDISs not only enrich the content of integrable systems itself, but also pave an effective way to solve integrable equations via the separation of spatial and temporal variables. It is already noticed that finite dimensional integrable Hamiltonian systems have been used to get algebro-geometric solutions through the finite parametric (or involutive) solutions of integrable equations with the help of the theory of algebraic curves [1, 7, 16, 17, 28, 30, 31, 36, 37]. In particular, a Neumann type system was already applied by Qiao to obtain the algebro-geometric solution of the Camassa–Holm (CH) equation on a symplectic submanifold [33], where the Lax matrix, dynamical r-matrix and Jacobi inversion were involved in.

To understand deeply the physical applications of integrable dynamical systems , one has to derive all kinds of explicit solutions for nonlinear evolution equations from different standpoints. After the breakthrough discovery of inverse scattering transformation [15], many interesting explicit solutions have been found, including the classical soliton solutions, the algebro-geometric (or finite-gap, quasi-periodic) solutions, and the polar expansion solutions. One can easily see that all explicit solutions of physical interests have a finite number of parameters. A deeper insight indicates that they may satisfy certain solvable ordinary differential equations and can be obtained through tackling the associated FDISs, which are reduced from integrable equations. Apart from the fruitful application of finite dimensional integrable Hamiltonian systems [1, 7, 16, 17, 28, 30, 31, 36, 37] and the work of the CH Neumann system with algebro-geometric solution [33], we also found that the Neumann type flow is in essential the Hamiltonian flow in the sense of Dirac–Poisson bracket over a symplectic submanifold, and the Neumann constraint under the scheme of nonlinearization of Lax pair directly cast in a finite dimensional invariant submanifold in quite a few cases [11, 28, 33]. In particular, the generating function of integrals of motion of Neumann type system determines a Riemann surface of hyperelliptic curve that pave a bridge to construct Abel–Jacobi (or angel) variables for integrable equations [12, 33]. Following the above-mentioned analysis, in this paper we present a distinct way by using the Neumann type systems to derive new algebro-geometric solutions for more integrable equations of physical and mathematical interests.

To illustrate our scheme, we study the algebro-geometric solutions of the following (1+1)-dimensional nonlinear evolution equations [34]

$$\begin{cases} u_t = v^{-2}v_x v_{xx} - v^{-1}v_{xxx} - 2uu_x - 4vv_x, \\ v_t = -2uv_x - u_x v. \end{cases} \tag{1}$$

In fact, the system (1) is the coupled integrable equations from the TD hierarchy, which allows the zero-curvature representation in the sense of Lax

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compatibility [20], the Hamiltonian structure in view of the trace identity [34], and the one- and two-soliton solutions by the Darboux transformation [10]. In the following, we will provide a feasible relation between two Neumann type systems stemmed from the Lax pair of (1) and algebro-geometric solutions of the integrable system (1). To see this, the integrable system (1) is reduced to two FDISs of Neumann type, whose compatible solutions yield solutions of (1) through a direct algebraic operation [8]. An interesting thing is that two Neumann type systems share the common Lax matrix and a dynamical r-matrix structure in the Dirac–Poisson bracket [28, 32, 37, 39], instead of the standard Poisson bracket since we construct Neumann type systems on a symplectic submanifold.

The Lax matrix and the dynamical r-matrix guarantee that the two Neumann type systems are completely integrable in the Liouville sense. Referring to the approach for getting algebro-geometric solutions for (1+1)- and (2+1)-dimensional integrable equations [3, 7, 16, 17, 21, 28, 30, 31, 36, 37], two sets of elliptic variables are singled out from the entries of Lax matrix, and solutions of the integrable system (1) are expressed by the symmetric functions with respect to these elliptic variables. Furthermore, through discussing the Jacobi inversion, we attain the algebro-geometric solutions of integrable system (1) in terms of Riemann theta functions.

The whole paper is organized as follows. In the next section, we decompose the integrable system (1) into two FDISs of Neumann type. In Section 3, the Neumann type flows are linearized/straightened out on the complex tour of a Riemann surface, and in Section 4 we derive the algebro-geometric solutions of integrable system (1) through the Jacobi inversion.

2 Decomposition of Integrable Equations

To describe our results, we first collect some necessary notations and formulas. Let us begin with the spectral problem [34]

$$\varphi_x = U\varphi, \qquad U = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}u & -v \\ v & \frac{1}{2}\lambda - \frac{1}{2}u \end{pmatrix}, \qquad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{2}$$

where λ is a spectral parameter, and u and v are two spectral potentials. In order to derive the integrable hierarchy associated with (2), we define the Lenard sequence $\{g_j\}$ $(-1 \leqslant j \in \mathbb{Z})$ by

$$Kg_{j-1} = Jg_j, \qquad Jg_{-1} = 0, \qquad j \geqslant 0, \tag{3}$$

with

$$K = \begin{pmatrix} -\frac{1}{2}\partial v^{-1}\partial v^{-1}\partial - 2\partial & -\frac{1}{2}\partial v^{-1}u \\ -\frac{1}{2}uv^{-1}\partial & -\frac{1}{2}\partial \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -\frac{1}{2}\partial v^{-1} \\ -\frac{1}{2}v^{-1}\partial & 0 \end{pmatrix}, \tag{4}$$

where $\partial = \partial/\partial x$ and ∂^{-1} is the inverse of ∂: $\partial^{-1}\partial = \partial\partial^{-1} = 1$. Noticing that the kernel of J is of dimension 2 with two generators $g_{-1} = (0, 2v)^T$ and $g_{-2} = \left(\frac{1}{2}, 0\right)^T$, one can easily get

$$\ker J = \{\varrho_1 g_{-1} + \varrho_2 g_{-2} | \forall \varrho_1, \varrho_2 \in \mathbb{R}\}.$$

Each g_j can be determined by the recursion formula (3). In particular, we have

$$g_0 = \left(v^2, 2uv\right)^T, \qquad g_1 = \left(2uv^2, 2v_{xx} + 2u^2 v + 4v^3\right)^T. \tag{5}$$

Let us consider an auxiliary spectral problem that is the time-dependent part of (2)

$$\varphi_{t_n} = V^{(n)}\varphi, \qquad V^{(n)} = \begin{pmatrix} V_{11}^{(n)} & V_{12}^{(n)} \\ V_{21}^{(n)} & -V_{11}^{(n)} \end{pmatrix}, \qquad n \geqslant 1, \tag{6}$$

where

$$V_{11}^{(n)} = -\frac{1}{4}v^{-1}\partial v^{-1}\partial g^{(1)} + \frac{1}{4}(\lambda - u)v^{-1}g^{(2)}, \quad V_{12}^{(n)} = -\frac{1}{2}v^{-1}\partial g^{(1)} + \frac{1}{2}g^{(2)},$$

$$V_{21}^{(n)} = -\frac{1}{2}v^{-1}\partial g^{(1)} - \frac{1}{2}g^{(2)}, \qquad g = \left(g^{(1)}, g^{(2)}\right)^T = \sum_{j=0}^{n} g_{j-2}\lambda^{n-j}.$$

Then the compatibility condition of (2) and (6) gives the integrable hierarchy [34]

$$(u, v)_{t_n}^T = J g_{n-1}, \qquad n \geqslant 1. \tag{7}$$

Apparently, the first nontrivial member of (7) is the integrable system (1) with $t = t_2$, which is the compatibility condition of Lax pair (2) and

$$\varphi_t = V^{(2)}\varphi, \qquad V^{(2)} = \begin{pmatrix} \frac{1}{2}\lambda^2 - \frac{1}{2}u^2 - \frac{1}{2}v^{-1}v_{xx} & \lambda v - v_x + uv \\ -\lambda v - v_x - uv & -\frac{1}{2}\lambda^2 + \frac{1}{2}u^2 + \frac{1}{2}v^{-1}v_{xx} \end{pmatrix}. \tag{8}$$

In what follows, we want to decompose (1) into two Neumann type systems on a symplectic submanifold. Let us consider N copies of the spectral problem (2) with N distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$ and their corresponding eigenfunctions $\varphi = (p_j, q_j)^T$,

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{2}\lambda_j + \frac{1}{2}u & -v \\ v & \frac{1}{2}\lambda_j - \frac{1}{2}u \end{pmatrix} \begin{pmatrix} p_j \\ q_j \end{pmatrix}, \qquad 1 \leqslant j \leqslant N. \tag{9}$$

One can readily calculate the functional gradient of each eigenvalue λ_j with respect to the spectral potentials u and v [9]

$$\nabla\lambda_j = \left(\delta\lambda_j/\delta u, \delta\lambda_j/\delta v\right)^T = \left(p_j q_j, -\left(p_j^2 + q_j^2\right)\right)^T. \tag{10}$$

Taking into account the Neumann constraint [4, 5, 9]

$$g_{-1} = \sum_{j=1}^{N} \nabla \lambda_j, \tag{11}$$

leads to

$$\begin{aligned} &\langle p, q\rangle = 0, \qquad \langle p, p\rangle - \langle q, q\rangle = 0, \\ &u = \frac{\langle \Lambda p, p\rangle + \langle \Lambda q, q\rangle}{\langle p, p\rangle + \langle q, q\rangle} = \frac{1}{2}\left(\frac{\langle \Lambda p, p\rangle}{\langle p, p\rangle} + \frac{\langle \Lambda q, q\rangle}{\langle q, q\rangle}\right), \\ &v = -\frac{\langle p, p\rangle + \langle q, q\rangle}{2} = -\langle p, p\rangle, \end{aligned} \tag{12}$$

where $p = (p_1, \cdots, p_N)^T$, $q = (q_1, \cdots, q_N)^T$, $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, and $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $\mathbb{R}^N$. In accordance with the rule of the nonlinearization of Lax pair, substituting (12) into (9) gives rise to the first nonlinear dynamical system of Neumann type,

$$\begin{cases} p_x = -\dfrac{1}{2}\Lambda p + \dfrac{1}{4}\left(\dfrac{\langle \Lambda p, p\rangle}{\langle p, p\rangle} + \dfrac{\langle \Lambda q, q\rangle}{\langle q, q\rangle}\right) p + \langle p, p\rangle q, \\ q_x = \dfrac{1}{2}\Lambda q - \dfrac{1}{4}\left(\dfrac{\langle \Lambda p, p\rangle}{\langle p, p\rangle} + \dfrac{\langle \Lambda q, q\rangle}{\langle q, q\rangle}\right) q - \langle q, q\rangle p, \\ \qquad \langle p, q\rangle = 0, \quad \langle p, p\rangle - \langle q, q\rangle = 0. \end{cases} \tag{13}$$

On condition that the independent temporal variable t is regarded as the equivalence to the spatial variable x in the view point of mathematics, imposing the Neumann constraint (12) onto the time-dependent part (8) leads to another new Neumann type system

$$\begin{cases} p_t = \dfrac{1}{2}\Lambda^2 p + \langle \Lambda p, q\rangle p - \dfrac{1}{4}\left(\dfrac{\langle \Lambda^2 p, p\rangle}{\langle p, p\rangle} + \dfrac{\langle \Lambda^2 q, q\rangle}{\langle q, q\rangle}\right) p - \langle p, p\rangle \Lambda q - \langle \Lambda p, p\rangle q, \\ q_t = \langle q, q\rangle \Lambda p + \langle \Lambda q, q\rangle p - \dfrac{1}{2}\Lambda^2 q - \langle \Lambda p, q\rangle q + \dfrac{1}{4}\left(\dfrac{\langle \Lambda^2 p, p\rangle}{\langle p, p\rangle} + \dfrac{\langle \Lambda^2 q, q\rangle}{\langle q, q\rangle}\right) q, \\ \langle p, q\rangle = 0, \quad \langle p, p\rangle - \langle q, q\rangle = 0. \end{cases} \tag{14}$$

A direct but lengthy computation yields the following proposition

Proposition 1 *Let* $(p(x,t), q(x,t))^T$ *be the compatible solution of the two Neumann type systems* (13) *and* (14), *then*

$$u(x,t) = \frac{1}{2}\left(\frac{\langle \Lambda p, p\rangle}{\langle p, p\rangle} + \frac{\langle \Lambda q, q\rangle}{\langle q, q\rangle}\right), \qquad v(x,t) = -\langle p, p\rangle, \tag{15}$$

are solutions of the integrable equations (1).

So, by this proposition, the integrable equations (1) can be solved with a finite parametric solution (15) through solving a pair of (finite dimensional) nonlinear dynamical systems of ordinary differential equations (13) and (14).

By using the procedure shown in [9, 28, 31, 32, 37, 39], we know that the Neumann type system (13) admits the Lax representation

$$L_x(\lambda) = [\bar{U}, L(\lambda)], \qquad L_x(\lambda) = \partial L(\lambda)/\partial x, \tag{16}$$

where

$$L(\lambda) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} q_j p_j & -p_j^2 \\ q_j^2 & -q_j p_j \end{pmatrix} \triangleq \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}, \tag{17}$$

and

$$\bar{U} = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{4}\left(\frac{\langle \Lambda p, p\rangle}{\langle p, p\rangle} + \frac{\langle \Lambda q, q\rangle}{\langle q, q\rangle}\right) & \langle p, p\rangle \\ -\langle p, p\rangle & \frac{1}{2}\lambda - \frac{1}{4}\left(\frac{\langle \Lambda p, p\rangle}{\langle p, p\rangle} + \frac{\langle \Lambda q, q\rangle}{\langle q, q\rangle}\right) \end{pmatrix}. \tag{18}$$

Actually, the Lax matrix (17) was first discussed in [28, 32, 39] to classify the FDISs. A very interesting fact is that the Neumann type system (14), i.e. the nonlinearization of the time-dependent part (8) under the Neumann constraint, admits the Lax representation with the same Lax matrix $L(\lambda)$ defined by (17)

$$L_t(\lambda) = [\bar{V}^{(2)}, L(\lambda)], \qquad L_t(\lambda) = \partial L(\lambda)/\partial t, \tag{19}$$

where

$$\bar{V}^{(2)} = \begin{pmatrix} \bar{V}_{11}^{(2)} & -\lambda\langle p, p\rangle - \langle \Lambda p, p\rangle \\ \lambda\langle q, q\rangle + \langle \Lambda q, q\rangle & -\bar{V}_{11}^{(2)} \end{pmatrix}, \tag{20}$$

with

$$\bar{V}_{11}^{(2)} = \frac{1}{2}\lambda^2 + \langle \Lambda p, q\rangle - \frac{1}{4}\left(\frac{\langle \Lambda^2 p, p\rangle}{\langle p, p\rangle} + \frac{\langle \Lambda^2 q, q\rangle}{\langle q, q\rangle}\right).$$

The Neuamnn type systems (13) and (14) are completely integrable in the Liouville sense since $L(\lambda)$ satisfies a dynamical r-matrix structure in the Dirac–Poisson bracket [9, 32, 38, 39]. Consequently, this assures the compatibility of the two Neumann type systems (13) and (14), which implies that the Neumann type flows mutually commute [2].

3 Straightening Out of the Neumann Type Flows

To get explicit solutions of integrable system (1), we adopt the procedure of straightening out Neumann type flows that are restricted on a symplectic submanifold. To do this, we select two sets of elliptic variables $\mu_1, \mu_2, \cdots, \mu_{N-1}$ and $\nu_1, \nu_2, \cdots, \nu_{N-1}$ from the entries of $L(\lambda)$,

$$\begin{aligned} B(\lambda) &= -\sum_{j=1}^{N} \frac{p_j^2}{\lambda - \lambda_j} = -\langle p, p\rangle \frac{m(\lambda)}{a(\lambda)}, \\ C(\lambda) &= \sum_{j=1}^{N} \frac{q_j^2}{\lambda - \lambda_j} = \langle q, q\rangle \frac{n(\lambda)}{a(\lambda)}, \end{aligned} \tag{21}$$

where

$$a(\lambda) = \prod_{k=1}^{N}(\lambda - \lambda_k), \quad m(\lambda) = \prod_{k=1}^{N-1}(\lambda - \mu_k), \quad n(\lambda) = \prod_{k=1}^{N-1}(\lambda - \nu_k). \tag{22}$$

The combination of (21) and (22) gives

$$\begin{aligned} \frac{\langle \Lambda p, p\rangle}{\langle p, p\rangle} &= \sum_{j=1}^{N} \lambda_j - \sum_{j=1}^{N-1} \mu_j \triangleq \sigma - \sigma_1, \\ \frac{\langle \Lambda q, q\rangle}{\langle q, q\rangle} &= \sum_{j=1}^{N} \lambda_j - \sum_{j=1}^{N-1} \nu_j \triangleq \sigma - \sigma_2. \end{aligned} \tag{23}$$

By (12) and (20), one obtains

$$u = \sigma - \frac{1}{2}(\sigma_1 + \sigma_2), \qquad \partial_x \ln v = \frac{1}{2}(\sigma_1 - \sigma_2), \tag{24}$$

and

$$\begin{cases} \bar{V}_{12}^{(2)} = -\langle p, p\rangle(\lambda + \sigma - \sigma_1), \\ \bar{V}_{21}^{(2)} = \langle q, q\rangle(\lambda + \sigma - \sigma_2). \end{cases} \tag{25}$$

Define

$$\det L(\lambda) = -A(\lambda)^2 - B(\lambda)C(\lambda) = -\frac{b(\lambda)}{4a(\lambda)} = -\frac{R(\lambda)}{4a^2(\lambda)}, \tag{26}$$

where

$$b(\lambda) = \prod_{k=1}^{N}(\lambda - \lambda_{N+k}), \qquad R(\lambda) = a(\lambda)b(\lambda) = \prod_{k=1}^{2N}(\lambda - \lambda_k).$$

It follows from (21), (22) and (26) that

$$A(\mu_k) = \frac{\sqrt{R(\mu_k)}}{2a(\mu_k)}, \qquad A(\nu_k) = \frac{\sqrt{R(\nu_k)}}{2a(\nu_k)}, \qquad 1 \leqslant k \leqslant N-1. \tag{27}$$

By (21), (16) and (19), we arrive at the evolution equation of all μ_k and ν_k regarding x and t,

$$\frac{d\mu_k}{dx} = -\frac{\sqrt{R(\mu_k)}}{\prod_{i=1,i\neq k}^{N-1}(\mu_k-\mu_i)}, \qquad \frac{d\nu_k}{dx} = \frac{\sqrt{R(\nu_k)}}{\prod_{i=1,i\neq k}^{N-1}(\nu_k-\nu_i)}, \qquad 1 \leqslant k \leqslant N-1, \tag{28}$$

and

$$\begin{cases} \dfrac{d\mu_k}{dt} = \dfrac{(\mu_k-\sigma_1+\sigma)\sqrt{R(\mu_k)}}{\prod_{i=1,i\neq k}^{N-1}(\mu_k-\mu_i)}, \\ \dfrac{d\nu_k}{dt} = \dfrac{(-\nu_k+\sigma_2-\sigma)\sqrt{R(\nu_k)}}{\prod_{i=1,i\neq k}^{N-1}(\nu_k-\nu_i)}, \end{cases} \qquad 1 \leqslant k \leqslant N-1. \tag{29}$$

These formulas naturally lead to the consideration of the Riemann surface Γ of hyperelliptic curve given by the equation $\xi^2 = R(\lambda)$, whose genus is $N-1$. For the same λ, there exist two points $(\lambda, \sqrt{R(\lambda)})$ and $(\lambda, -\sqrt{R(\lambda)})$ on the upper and lower sheets of Γ, and there are two points at infinity that are not the branch points because $\deg R(\lambda) = 2N$. Under an alternative local coordinate $z = \lambda^{-1}$, they are marked as $\infty_1 = (0, 1)$ and $\infty_2 = (0, -1)$.

Let $a_1, a_2, \cdots, a_{N-1}; b_1, b_2, \cdots, b_{N-1}$ be a set of regular cycle paths on Γ, which are automatically independent if they have the intersection numbers

$$a_i \circ a_j = b_i \circ b_j = 0, \qquad a_i \circ b_j = \delta_{ij}, \qquad i, j = 1, 2, \cdots, N-1.$$

It is well known that

$$\tilde{\omega}_l = \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \qquad 1 \leqslant l \leqslant N-1,$$

are $N-1$ linearly independent holomorphic differentials of Γ. Let

$$A_{ij} = \int_{a_j} \tilde{\omega}_i, \qquad C = (A_{ij})^{-1}, \qquad 1 \leqslant i, j \leqslant N-1, \tag{30}$$

then $\tilde{\omega}_l$ can be normalized into a new basis ω_j,

$$\omega_j = \sum_{l=1}^{N-1} C_{jl}\tilde{\omega}_l, \qquad \int_{a_i} \omega_j = \sum_{l=1}^{N-1} C_{jl} \int_{a_i} \tilde{\omega}_l = \sum_{l=1}^{N-1} C_{jl} A_{li} = \delta_{ji},$$

and each

$$B_{ij} = \int_{b_j} \omega_i, \qquad 1 \leqslant i, j \leqslant N-1,$$

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is an entry of $(N-1)\times(N-1)$ matrix $B=(B_{ij})$ that characterizes the Riemann surface Γ and applies to construct Riemann theta functions of Γ. Let p_0 be a fixed point, then the Abel–Jacobi variables can be given by

$$\begin{aligned} \rho_j^{(1)}(x,t) &= \sum_{k=1}^{N-1}\int_{p_0}^{\mu_k(x,t)}\omega_j = \sum_{k=1}^{N-1}\sum_{l=1}^{N-1}C_{jl}\int_{p_0}^{\mu_k}\frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}}, \\ \rho_j^{(2)}(x,t) &= \sum_{k=1}^{N-1}\int_{p_0}^{\nu_k(x,t)}\omega_j = \sum_{k=1}^{N-1}\sum_{l=1}^{N-1}C_{jl}\int_{p_0}^{\nu_k}\frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}}, \end{aligned} \qquad 1\leqslant j\leqslant N-1. \tag{31}$$

Taking derivative with respect to x on both sides of $(31)_1$ leads to

$$\partial_x\rho_j^{(1)} = \sum_{l=1}^{N-1}\sum_{k=1}^{N-1}C_{jl}\frac{\mu_k^{l-1}\mu_{k,x}}{\sqrt{R(\mu_k)}} = \sum_{l=1}^{N-1}\sum_{k=1}^{N-1}C_{jl}\frac{-\mu_k^{l-1}}{\prod_{i=1,i\neq k}^{N-1}(\mu_k-\mu_i)}. \tag{32}$$

With the help of the formulae [26],

$$I_s=\sum_{k=1}^{N-1}\frac{\mu_k^s}{\prod_{i=1,i\neq k}^{N-1}(\mu_k-\mu_i)}=\delta_{s,N-2},\quad I_{N-1}=\sigma_1 I_{N-2},\qquad 1\leqslant s\leqslant N-2, \tag{33}$$

we obtain

$$\partial_x\rho_j^{(1)}=\Omega_j^{(0)},\quad \Omega_j^{(0)}=-C_{jN-1},\qquad 1\leqslant j\leqslant N-1. \tag{34}$$

A similar calculation directly yields

$$\partial_t\rho_j^{(1)}=\Omega_j^{(1)},\qquad \partial_x\rho_j^{(2)}=-\Omega_j^{(0)},\qquad \partial_t\rho_j^{(2)}=-\Omega_j^{(1)}, \tag{35}$$

where $\Omega_j^{(1)}=C_{jN-2}+\sigma C_{jN-1}$. Clearly, $\rho_j^{(1)}$ and $\rho_j^{(2)}$ can be integrated and written as linear superpositions in the flow variables x and t,

$$\begin{aligned} \rho_j^{(1)} &= \Omega_j^{(0)}x+\Omega_j^{(1)}t+\gamma_j^{(1)}, \\ \rho_j^{(2)} &= -\Omega_j^{(0)}x-\Omega_j^{(1)}t+\gamma_j^{(2)}, \end{aligned}\qquad 1\leqslant j\leqslant N-1, \tag{36}$$

where

$$\gamma_j^{(1)}=\sum_{k=1}^{N-1}\int_{p_0}^{\mu_k(0,0)}\omega_j,\qquad \gamma_j^{(2)}=\sum_{k=1}^{N-1}\int_{p_0}^{\nu_k(0,0)}\omega_j,$$

are two integral constants.

4 Algebro-Geometric Solutions of the Integrable Equations

Since the Abel–Jacobi solutions $(\rho^{(1)},\rho^{(2)})$ (see (36)) are solved explicitly, the remaining steps are to write down the explicit expression of u and v of integrable system (1). For this purpose, we turn to the procedure of Jacobi inversion

$$(\rho^{(1)}, \rho^{(2)}) \Longrightarrow (\mu_k, \nu_k).$$

Let T be the lattice in $\mathbb{C}^{N-1}$, which is generated by $2(N-1)$ periodic vectors $\{\delta_i, B_j\}$. Then we have the following complex tour—called Jacobian $J(\Gamma) = \mathbb{C}^{N-1}/T$ of Γ. The Abel map is defined by

$$\mathcal{A}: \quad \mathrm{Div}(\Gamma) \to \mathrm{J}(\Gamma), \quad \mathcal{A}(\tilde{\mathrm{p}}) = \left(\int_{\mathrm{p}_0}^{\tilde{\mathrm{p}}} \omega_1, \cdots, \int_{\mathrm{p}_0}^{\tilde{\mathrm{p}}} \omega_{\mathrm{N}-1}\right),$$

where $\tilde{p}$ is an arbitrary point on Γ. Moreover, $\mathcal{A}$ can linearly be extended to the factor group

$$\mathrm{Div}(\Gamma): \quad \mathcal{A}\left(\sum n_k \tilde{p}_k\right) = \sum n_k \mathcal{A}(\tilde{p}_k).$$

From [18, 25], the Riemann theta function is defined by

$$\theta(\zeta) = \sum_{z \in \mathbb{Z}^{N-1}} \exp\left(\pi i \langle Bz, z\rangle + 2\pi i \langle \zeta, z\rangle\right), \quad \zeta \in \mathbb{C}^{N-1},$$
$$\langle Bz, z\rangle = \sum_{i,j=1}^{N-1} B_{ij} z_i z_j, \quad \langle \zeta, z\rangle = \sum_{i=1}^{N-1} z_i \zeta_i.$$

Let us consider two special divisors $\sum_{k=1}^{N-1} \tilde{p}_k^{(m)}$,

$$\mathcal{A}\left(\sum_{k=1}^{N-1} \tilde{p}_k^{(m)}\right) = \sum_{k=1}^{N-1} \mathcal{A}\left(\tilde{p}_k^{(m)}\right) = \sum_{k=1}^{N-1} \int_{p_0}^{\tilde{p}_k^{(m)}} \omega = \rho^{(m)}, \qquad m = 1, 2,$$

where $\tilde{p}_k^{(1)} = (\mu_k, \zeta(\mu_k))$ and $\tilde{p}_k^{(2)} = (\nu_k, \zeta(\nu_k))$. Conforming to the Riemann theorem [18], there exist two constant vectors (called Riemann constants) $M^{(1)}, M^{(2)} \in \mathbb{C}^{N-1}$ determined by Γ such that

- $f^{(1)}(\lambda) \triangleq \theta(\mathcal{A}(\zeta(\lambda)) - \rho^{(1)} - M^{(1)})$ has $N-1$ simple zeros at $\mu_1, \cdots, \mu_{N-1}$,
- $f^{(2)}(\lambda) \triangleq \theta(\mathcal{A}(\zeta(\lambda)) - \rho^{(2)} - M^{(2)})$ has $N-1$ simple zeros at $\nu_1, \cdots, \nu_{N-1}$.

To make the functions single valued, Γ is cut by all paths a_k, b_k to form a simply connected region whose boundary is denoted by γ. By the residue formulas, one gets

$$\sum_{j=1}^{N-1} \mu_j = I(\Gamma) - \sum_{s=1}^{2} \operatorname*{Res}_{\lambda=\infty_s} \lambda d \ln f^{(1)}(\lambda),$$
$$\sum_{j=1}^{N-1} \nu_j = I(\Gamma) - \sum_{s=1}^{2} \operatorname*{Res}_{\lambda=\infty_s} \lambda d \ln f^{(2)}(\lambda), \tag{37}$$

where

$$I(\Gamma)=\frac{1}{2\pi i}\oint_{\gamma}\lambda d\ln f^{(m)}(\lambda)=\sum_{j=1}^{N-1}\int_{a_j}\lambda\omega_j,\qquad m=1,2,$$

is a constant independent of $\rho^{(m)}$ [13, 36]. The only requirement is to calculate the residues at both infinities:

$$\begin{aligned}f^{(m)}(\lambda)|_{\lambda=\infty_s}&=\theta\left(\int_{p_0}^{\tilde{p}}\omega-\rho^{(m)}-M^{(m)}\right)=\theta\left(\int_{\infty_s}^{\tilde{p}}\omega-\pi_s-\rho^{(m)}-M^{(m)}\right)\\&=\theta\left(\cdots,\int_{\infty_s}^{\tilde{p}}\omega_j-\pi_{sj}-\rho_j^{(m)}-M_j^{(m)},\cdots\right)\\&=\theta\left(\cdots,\rho_j^{(m)}+M_j^{(m)}+\pi_{sj}+(-1)^s\right.\\&\qquad\left.\times\left(C_{jN-1}z+\frac{1}{2}\left(C_{jN-2}+\sigma C_{jN-1}\right)z^2+\cdots\right),\cdots\right)\\&=\theta_s^{(m)}\left(\rho^{(m)}+M^{(m)}+\pi_s\right)+(-1)^{s+m}\theta_{s,x}^{(m)}z+\cdots,\end{aligned}$$

where $\pi_{sj}=\int_{\infty_s}^{p_0}\omega_j\ (s,\ m=1,2)$. Therefore, we arrive at

$$\operatorname*{Res}_{\lambda=\infty_s}\lambda d\ln f^{(m)}(\lambda)=(-1)^{s+m}\partial_x\ln\theta_s^{(m)},\tag{38}$$

where

$$\theta_s^{(1)}=\theta(\Omega^{(0)}x+\Omega^{(1)}t+\Upsilon_s),\quad \theta_s^{(2)}=\theta(-\Omega^{(0)}x-\Omega^{(1)}t+\Lambda_s),$$

with

$$\Upsilon_{sj}=\gamma_j^{(1)}+M_j^{(1)}+\pi_{sj},\qquad \Lambda_{sj}=\gamma_j^{(2)}+M_j^{(2)}+\pi_{sj},\qquad 1\leqslant j\leqslant N-1.$$

From (37) and (38), we have

$$\sum_{l=1}^{N-1}\mu_l=I(\Gamma)+\partial_x\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}},\quad \sum_{l=1}^{N-1}\nu_l=I(\Gamma)+\partial_x\ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}}.\tag{39}$$

Substituting (39) into (24), we get the algebro-geometric solutions of integrable system (1),

$$u=-\frac{1}{2}\partial_x\ln\frac{\theta(\Omega^{(0)}x+\Omega^{(1)}t+\Upsilon_2)\,\theta(-\Omega^{(0)}x-\Omega^{(1)}t+\Lambda_1)}{\theta(\Omega^{(0)}x+\Omega^{(1)}t+\Upsilon_1)\,\theta(-\Omega^{(0)}x-\Omega^{(1)}t+\Lambda_2)}-I(\Gamma)+\sigma,$$

$$v^2=\frac{\theta(\Omega^{(0)}x+\Omega^{(1)}t+\Upsilon_2)}{\theta(\Omega^{(0)}x+\Omega^{(1)}t+\Upsilon_1)}\frac{\theta(-\Omega^{(0)}x-\Omega^{(1)}t+\Lambda_2)}{\theta(-\Omega^{(0)}x-\Omega^{(1)}t+\Lambda_1)}\frac{\theta(\Omega^{(1)}t+\Upsilon_1)}{\theta(\Omega^{(1)}t+\Upsilon_2)}\frac{\theta(-\Omega^{(1)}t+\Lambda_1)}{\theta(-\Omega^{(1)}t+\Lambda_2)}v^2(0,t).$$

In conclusion, the algebro-geometric solutions of integrable system (1) are attained, which implies that the two Neumann type systems in this paper are successfully used to derive algebro-geometric solutions of integrable equations in (1+1)-dimensional just like the procedure shown in [33]. This procedure is different from the utilization of finite dimensional integrable Hamiltonian systems in the case of Bargmann constraint [19, 24, 35] that corresponds to the whole symplectic space. We will try to solve some other integrable equations under the Neumann constraint.

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