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Modified r-matrix and separation of variables for the modified Korteweg-de Vries (MKdV) hierarchy[1]

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Abstract

In this article, it is first given that a Lax representation in terms of 2×2 matrices for the completely integrable finite-dimensional Hamiltonian system (CIFHS) (H) produced through the nonlinearization procedure for the MKdV hierarchy, and then an associated non-dynamical modified r-matrix is constructed. By making use of this r-matrix and matrix trace equality, a set of finite-dimensional involutive functions F_m $(m = 0, 1, \ldots, F_0 = H)$, which guarantees the integrability of Hamiltonian systems (H), and the Lax representations in terms of 2×2 matrices for the whole Hamitonian hierarchies (F_m) $(m = 0, 1, \ldots)$ are obtained. Moreover, the involutive solutions of the MKdV hierarchy are given. Finally, it is found that the Hamilton–Jacobi equation for the Hamiltonian system (H) can be separable under a group of new corrdinates introduced by the 2×2 Lax matrix.

1. Introduction

The modified Korteweg-de Vries (MKdV) equation is widely discussed in the literature. It possesses the Lax pair [1], the soliton solution [1], bi-Hamiltonian structure [2] and other soliton properties such as Darboux transformation, Bäcklund transformation and the Miura transformation between it and the famous KdV equation [3]. In recent years, the appearance of nonlinearization method [4,5] and constrained flows [6] brings out new vitality in soliton theory and integrable system theory, and with the help of them many new completely integrable systems has been successively found [7–15].

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Ref. [16] gave a description that the Lax pair of the MKdV hierarchy is nonlinearized as a set of commutative integrable Hamiltoninan systems under the so-called Bargmann constraint. In Ref. [17], we discussed the Neumann constraint of the MKdV hierarchy and the gauge transformation between the Neumann system and a completely integrable Hamiltonian system.

This article is a continuation of the previous paper [17], i.e, the present work deals with a non-dynamical modified r-matrix, Poisson bracket and separation of variables for the completely integrable finite-dimensional Hamiltonian system (CIFHS) associated with the MKdV hierarchy. The investigation of CIFHS admitting a classical standard r-matrix Poisson structure was developed in the book of Faddeev and Takhtajan [18]. Recently, the study of CIFHS with the dynamical r-matrix (i.e. depending on dynamical canonical variables) and the separation of variables has received some attention [19–23]. It is a celebrated fact that the famous Calogero–Moser system, whose integrability was proved a number of years ago [24], has been found to possess a classical dynamical r-matrix [25]. Many CIFHS generated through the nonlinearization method are developed to have dynamical or non-dynamical r-matrix by several authors [26–28]. The r-matrix algebra is not only used as a main approach to finite-dimensional involutive systems [29,30], but also applied to the separation of variables of the integrable Hamiltonian systems [31,32].

The present article has been written to search for the non-dynamical r-matrix, separation of variables for the MKdV hierarchy, and is organized as follows.

In the next section we briefly recall the generation process of the MKdV hierarchy and Lax representations.

In Section 3, for the nonlinearized system of the MKdV eigenvalue problem under the Bargmann constraint, we derive its Hamiltonian function and Lax representation.

In Section 4, the following typical non-dynamical r-matrix:

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P-\frac{2}{\mu}S\,,$$

$$P=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\qquad S=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

where λ, μ are two parameters, are found. Obvoiusly, this non-dynamical r-matrix is more a term $-(2/\mu)S$ than the standard r-matrix, which is called a modification of the standard r-matrix $(2/(\mu-\lambda))P$. Sometimes this kind of r-matrices structure are also recorded as $R\hat{q}$-matrices [33] or rs-matrices [34] in the literature, but most of these rs-matrices are usually dynamical [33,34]. The r-matrix and rs-matrix methods are the two effective ways to the Lie–Poisson structure [33,35], which provides fundamental commutator relations in the quantum inverse scattering [30]. The rs-bracket induced by the rs-matrices has a close connection with the common hierarchy of Lax equations induced by a Hamiltonian [33]. And the additional term matrix $S(\lambda,\mu)$ in rs-matrix can be usually restricted to a common Poisson subspace for the rs-bracket (see Ref. [34, Theorem 3]).

Additionally, in this section by making use of the modified r-matrix and matrix trace equality, a set of finite-dimensional involutive functions F_m ($m=0,1,\ldots,F_0=H$), which guarantees the complete integrability of the nonlinearized system (i.e., Hamiltonian system (H)) of the MKdV eigenvalue problem, and the Lax representations in terms of 2×2 matrices for the whole Hamiltonian hierarchies (F_m) ($m=0,1,\ldots$) are obtained. Furthermore, the involutive solutions of the well-known MKdV equation and the whole MKdV hierarchies are also given in this section.

Section 5 deals with the aspects of separation of variables in the Hamilton–Jacobi equation for the Hamiltonian system (H).

2. Background: MKdV hierarchy and Lax representations

In this section, let us first recall the generation process of the MKdV hierarchy and Lax representations. Thus, we consider the MKdV spectral problem

$$\psi_x=\tilde U\psi,\quad \tilde U=\begin{pmatrix} u & \lambda \\ -1 & -u\end{pmatrix},\quad \psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{1}$$

where λ is a spectral parameter, $\psi_x=\partial\psi/\partial x$. We can easily calculate the spectral gradient $\delta\lambda/\delta u$ of the eigenparameter λ with respect to the scalar potential u

$$\frac{\delta\lambda}{\delta u}=2\psi_1\psi_2\,. \tag{2}$$

Set

$$\mathscr{L}=-\frac{1}{4}\partial^2+u\partial^{-1}u\partial,\quad \partial=\frac{\partial}{\partial x},\quad \partial\partial^{-1}=\partial^{-1}\partial=1\,, \tag{3}$$

then $\delta\lambda/\delta u$ satisfies

$$\mathscr{L}\frac{\delta\lambda}{\delta u}=\lambda\frac{\delta\lambda}{\delta u}\,. \tag{4}$$

Choosing $J=\partial$, $K=J\mathscr{L}=-\frac{1}{4}\partial^3+\partial u\partial^{-1}u\partial$ as a pair of Lenard's operators, we define the Lenard's recursive sequence $\{G_j\}_{j=-1}^{\infty}$ as follows:

$$\begin{cases} G_{-1}=1\in\operatorname{Ker} J, & G_0=u\,,\\ G_j=\mathscr{L}G_{j-1}, & j=1,2,\ldots\,, \end{cases} \tag{5}$$

which produces the well-known MKdV hierarchy

$$u_{t_m}=JG_m=KG_{m-1}=J\mathscr{L}^m u,\quad m=0,1,2,\ldots\,. \tag{6}$$

As $m=1$, (6) becomes the famous MKdV equation

$$u_{t_1}=-\tfrac{1}{4}u_{xxx}+\tfrac{3}{2}u^2u_x\,. \tag{7}$$

Proposition 1. The MKdV hierarchy (6) possesses the Lax pair

$$\begin{cases} \psi_x = \begin{pmatrix} u & \lambda \\ -1 & -u \end{pmatrix}\psi, \\ \psi_{t_m} = \begin{pmatrix} u & \lambda \\ -1 & -u \end{pmatrix}\lambda^m\psi \\ \qquad + \sum_{j=1}^{m} \begin{pmatrix} G_j & \lambda(\frac{1}{2}G_{j-1,x} + \partial^{-1}u\partial G_{j-1}) \\ \frac{1}{2}G_{j-1,x} - \partial^{-1}u\partial G_{j-1} & -G_j \end{pmatrix}\lambda^{m-j}\psi . \end{cases} \tag{8}$$

In particular, the MKdV equation (7) has the Lax pair

$$\begin{cases} \psi_x = \begin{pmatrix} u & \lambda \\ -1 & -u \end{pmatrix}\psi, \\ \psi_{t_1} = \begin{pmatrix} \lambda u - \frac{1}{4}u_{xx} + \frac{1}{2}u^3 & \lambda^2 + \frac{1}{2}\lambda(u_x + u^2) \\ -\lambda + \frac{1}{2}(u_x - u^2) & -\lambda u + \frac{1}{4}u_{xx} - \frac{1}{2}u^3 \end{pmatrix}\psi . \end{cases} \tag{9}$$

3. Nonlinearized system of (1), its Hamiltonian and Lax representation

Let $\lambda_1, \ldots, \lambda_N$ be N distinct spectral parameters. Consider the Bargmann constraint [5] $G_0 = \frac{1}{2}\sum_{j=1}^{N} \delta\lambda_j/\delta u$, i.e.

$$u = \langle p, q \rangle , \tag{10}$$

where $p = (p_1, \ldots, p_N)^T \equiv (\psi_{21}, \ldots, \psi_{2N})^T, q = (q_1, \ldots, q_N)^T \equiv (\psi_{11}, \ldots, \psi_{1N})^T; \langle \cdot, \cdot \rangle$ stands for the standard inner product in the Euclid space R^{2N}. Denote $\Lambda = diag(\lambda_1, \ldots, \lambda_N)$, then under the Bargmann constraint (10) the nonlinearization of (1) gives a Hamiltonian system with N degrees of freedom

$$\begin{aligned} q_x &= \langle p, q \rangle q + \Lambda p = \frac{\partial H}{\partial p}, \\ p_x &= -q - \langle p, q \rangle p = -\frac{\partial H}{\partial q} \end{aligned} \tag{11}$$

with

$$H = \tfrac{1}{2}\langle \Lambda p, p \rangle + \tfrac{1}{2}\langle q, q \rangle + \tfrac{1}{2}\langle p, q \rangle^2 . \tag{12}$$

Proposition 2. Set

$$U = \begin{pmatrix} \langle p, q \rangle & \lambda \\ -1 & -\langle p, q \rangle \end{pmatrix}, \tag{13}$$

$$M = \begin{pmatrix} 0 & 1 \\ -\lambda^{-1}(1 + \langle p, p \rangle) & 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} p_j q_j & -q_j^2 \\ p_j^2 & -p_j q_j \end{pmatrix}, \tag{14}$$

where $\lambda \neq 0$ is a spectral parameter. Then the Hamiltonian system (11) has the 2×2 matrix Lax representation

$$M_x = [U, M]\,. \tag{15}$$

Proof. Calculate directly. □

The matrix M satisfying (15) is called the Lax matrix of Hamiltonian system (11).

4. A modified non-dynamical *r*-matrix and finite-dimensional involutive systems

It should be pointed out that the following non-dynamical r-matrix $r_{12}(\lambda, \mu)$ is more a term $-(2/\mu)S$ (the expression of S is seen below) than the classical standard r-matrix $(2/(\mu - \lambda))P$. Hence, it can be viewed as a modification of the classical standard r-matrix, and is both different from and inequivalent to the r-matrix, which is dynamical, see Ref. [21, (2.23), (2.24)].

Set

$$M(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}, \tag{16}$$

where

$$A(\lambda) = \sum_{j=1}^{N} \frac{p_j q_j}{\lambda - \lambda_j}, \tag{17a}$$

$$B(\lambda) = 1 - \sum_{j=1}^{N} \frac{q_j^2}{\lambda - \lambda_j}, \tag{17b}$$

$$C(\lambda) = -\lambda^{-1}(1 + \langle p, p\rangle) + \sum_{j=1}^{N} \frac{p_j^2}{\lambda - \lambda_j}\,. \tag{17c}$$

Define the standard Poisson bracket $\{F, G\}$ of two Hamiltonian functions F, G in the symplectic space $(R^{2N}, dp \wedge dq)$ as follows:

$$\{F, G\} = \sum_{j=1}^{N} \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} \right). \tag{18}$$

Then, we have the following proposition.

Proposition 3. For two free parameters λ and μ, the following Poisson brackets always holds:

$$\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0\,, \tag{19}$$

$$\{A(\lambda),B(\mu)\}=\frac{2}{\mu-\lambda}(B(\mu)-B(\lambda))\,, \tag{20}$$

$$\{A(\lambda),C(\mu)\}=\frac{2}{\mu-\lambda}(-C(\mu)+C(\lambda))-\frac{2}{\mu}C(\lambda)\,, \tag{21}$$

$$\{B(\lambda),C(\mu)\}=\frac{4}{\mu-\lambda}(A(\mu)-A(\lambda))+\frac{4}{\mu}A(\lambda)\,. \tag{22}$$

Let $M_1(\lambda)=M(\lambda)\otimes I$, $M_2(\mu)=I\otimes M(\mu)$, here I is the 2×2 unit matrix. Then it follows from (19)–(22) that:

Proposition 4. The Lax matrix M satisfies the following fundamental Poisson bracket:

$$\{M_1(\lambda),M_2(\mu)\}=[r_{12}(\lambda,\mu),M_1(\lambda)]-[r_{21}(\mu,\lambda),M_2(\mu)]\,, \tag{23}$$

where $\{M_1(\lambda),M_2(\mu)\}$ is a 4×4 matrix [18] consisting of various Poisson brackets of the elements of $M(\lambda)=(m_{ij}(\lambda))_{2\times 2}$ and $M(\mu)=(m_{ij}(\mu))_{2\times 2}$

$$\{M_1(\lambda),M_2(\mu)\}_{jk,ln}=\{m_{jl}(\lambda),m_{kn}(\mu)\},\quad jk,ln=11,12,21,22\,, \tag{24}$$

and the r-matrix $r_{12}(\lambda,\mu)$ is given by

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P-\frac{2}{\mu}S,\quad r_{21}(\mu,\lambda)=Pr_{12}(\lambda,\mu)P\,, \tag{25}$$

$$P=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\qquad S=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Evidently, the r-matrix formula (25) depends only on the two constant parameters λ,μ, and has no relation to the dynamical canonical variables p_j,q_j $(j=1,\dots,N)$. So, again from the beginning of this section we call Eq. (25) as a non-dynamical modified version of the classical standard r-matrix.

By (23), we can immediately calculate

$$\{M_1^2(\lambda),M_2^2(\mu)\}=[\bar{r}_{12}(\lambda,\mu),M_1(\lambda)]-[\bar{r}_{21}(\mu,\lambda),M_2(\mu)]\,, \tag{26}$$

where [19]

$$\bar{r}_{ij}(\lambda,\mu)=\sum_{k=0}^{1}\sum_{l=0}^{1}M_1^{1-k}(\lambda)M_2^{1-l}(\mu)\,r_{ij}(\lambda,\mu)\,M_1^k(\lambda)M_2^l(\mu),\quad ij=12,21\,. \tag{27}$$

From (26), we derive that

$$4\{\mathrm{Tr}\,M^2(\lambda),\mathrm{Tr}\,M^2(\mu)\}=\mathrm{Tr}\{M_1^2(\lambda),M_2^2(\mu)\}=0\,. \tag{28}$$

One obtains from (16) that

$$\frac{1}{2}\lambda^2 \operatorname{Tr} M^2(\lambda) = -\lambda + 2H + \sum_{j=1}^{N} \frac{\lambda_j E_j}{\lambda - \lambda_j}, \tag{29}$$

where H is defined by (12) and

$$E_j = \lambda_j p_j^2 + \langle p, p\rangle q_j^2 + q_j^2 - \lambda_j \sum_{k=1,\, k\neq j}^{N} \frac{(p_j q_k - p_k q_j)^2}{\lambda_j - \lambda_k}, \quad j = 1, 2, \ldots, N. \tag{30}$$

Due to (28) and (29), we can easily know

$$\{E_i, E_j\} = 0, \ \{H, E_j\} = 0, \quad \forall i, j = 1, 2, \ldots, N. \tag{31}$$

Hence, from the view point of modified non-dynamical r-matrix we have proved.

Proposition 5. The Hamiltonian system (11) is completely integrable in the Liouville sense and its finite-dimensional involutive systems are $\{E_j\}_{j=1}^N$.

Set

$$F_m = \frac{1}{2}\sum_{j=1}^{N} \lambda_j^m E_j, \quad m = 0, 1, 2, \ldots, \tag{32}$$

then

$$\begin{aligned} F_m &= \tfrac{1}{2}\langle \Lambda^m q, q\rangle + \tfrac{1}{2}\langle p, p\rangle \langle \Lambda^m q, q\rangle + \tfrac{1}{2}\langle \Lambda^{m+1} p, p\rangle \\ &\quad - \frac{1}{2}\sum_{i+j=m} \left(\langle \Lambda^i p, p\rangle \langle \Lambda^j q, q\rangle - \langle \Lambda^i p, q\rangle \langle \Lambda^j p, q\rangle\right). \end{aligned} \tag{33}$$

Apparently, $F_0 = H$, and by virtue of (31), $\{H, F_0\} = 0$ and $\{F_m, F_l\} = 0$, $\forall m, l \in Z^+$. So, the whole Hamiltonian system

$$(F_m): \quad q_{t_m} = \frac{\partial F_m}{\partial p}, \ p_{t_m} = -\frac{\partial F_m}{\partial q}, \quad m = 0, 1, 2, \ldots, \ t_0 = x, \tag{34}$$

are also completely integrable.

Analogous to Proposition 2, we can also derive the Lax representations of the whole Hamiltonian systems (34).

Proposition 6. The Hamiltonian systems (34) admit the 2×2-matrix Lax representations

$$M_{t_m} = [V^{(m)}, M], \quad m = 0, 1, 2, \ldots, \tag{35}$$

where M is the Lax matrix (14), and

$$V^{(m)} = \begin{pmatrix} \lambda^m\langle p,q\rangle & \lambda^{m+1} \\ -\lambda^m & -\lambda^m\langle p,q\rangle \end{pmatrix} + \sum_{j=1}^{m} \begin{pmatrix} \lambda^{m-j}\langle \Lambda^j p,q\rangle & -\lambda^{m-j+1}\langle \Lambda^{j-1}q,q\rangle \\ -\lambda^{m-j}\langle \Lambda^j p,p\rangle & -\lambda^{m-j}\langle \Lambda^j p,q\rangle \end{pmatrix}, \quad m=0,1,2,\ldots, \tag{36}$$

particularly point out $V^{(0)}=U$.

In addition, on the solutions of the MKdV hierarchy and famous MKdV equation we have

Proposition 7. Let $q(x,t_m), p(x,t_m)$ be the solution of commutative flow (H) and (F_m). Then $u(x,t_m)=\langle q(x,t_m), p(x,t_m)\rangle$ satisfies the higher-order MKdV equation (6): $u_{t_m} = J\mathscr{L}^m u$ $(m=0,1,2,\ldots)$. In particular, the well-known MKdV equation $u_{t_1} = -\frac{1}{4}u_{xxx} + \frac{3}{2}u^2u_x$ has the solution $u(x,t_1)=\langle q(x,t_1), p(x,t_1)\rangle$, where $q(x,t_1), p(x,t_1)$ are the solutions of commutative flow (H) and (F_1).

Proof. According to equalities (4), (5), (33), (34) and (11), through a lengthy calculations it is known that Proposition 7 holds. □

5. Separation of variables

Separation of variables in the Hamilton–Jacobi equation

$$H(p_1,\ldots,p_N,q_1,\ldots,q_N)=E, \qquad p_i=\frac{\partial W}{\partial q_i}, \quad i=1,\ldots,N \tag{37}$$

is an important method of solving the Liouville integrable systems of classical mechanics [36]. The separation of variables means the solution of partial differential equation (37) for the action function W in the following additives form:

$$W=\sum_{i=1}^{N} W_i(u_i;H_1,\ldots,H_N), \quad H_N=H, \tag{38}$$

where u_i $(i=1,2,\ldots,N)$ are called separation variables and the functions W_i $(i=1,2,\ldots,N)$ depend only on their separation variables u_i, i.e., the separation variables [37,38] are comprehent in the given hierarchy of Hamiltonian systems as the construction of N pairs of canonical variables u_i, v_i, $i=1,\ldots,N$,

$$\{u_i,u_k\}=\{v_i,v_k\}=0, \quad \{v_i,u_k\}=\delta_{ik}, \tag{39}$$

and N functions W_i such that

$$W_i(u_i, v_i; H_1, \dots, H_N) = 0, \quad i = 1, \dots, N, \tag{40}$$

where H_i are the involutive integrals of motion. Eq. (40) is referred to as the separation equation.

Since the integrable systems (11) and (34) admit the 2×2-matrix Lax representations (15) and (35), in order to separate the finite-dimensional integrable Hamiltonian systems (11) and (34) we need to introduce N pairs of new coordinates u_i, v_i:

$$C(u_i) = 0, \;\; v_i = \tfrac{1}{2} A(u_i), \quad i = 1, \dots, N, \tag{41}$$

as the separation variables.

Set

$$C(\lambda) = -\lambda^{-1}(1 + \langle p, p\rangle) + \sum_{j=1}^{N} \frac{p_j^2}{\lambda - \lambda_j} \equiv \frac{R(\lambda)}{S(\lambda)}, \tag{42}$$

where

$$R(\lambda) = \prod_{k=1}^{N} (\lambda - u_k), \qquad S(\lambda) = \prod_{j=1}^{N} \lambda(\lambda - \lambda_j). \tag{43}$$

Suppose that $\lambda_j \neq 0$ $(j = 1, \dots, N)$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_N$, then we can chose N different zero points u_j of $C(\lambda)$ such that [39] $\lambda_{j-1} \leqslant u_j \leqslant \lambda_j$. From (42) we have

$$-\frac{1}{u_k}(1 + \langle p, p\rangle) + \sum_{j=1}^{N} \frac{p_j^2}{u_k - \lambda_j} = 0, \quad k = 1, 2, \dots N, \tag{44}$$

and by taking a residum at the pole λ_j we obtain

$$p_j^2 = \frac{R(\lambda_j)}{S'(\lambda_j)}, \qquad S'(\lambda_j) = \frac{d}{d\lambda} S(\lambda)\bigg|_{\lambda = \lambda_j}, \tag{45}$$

which reads

$$2 p_j \, dp_j = \frac{R(\lambda_j)}{S'(\lambda_j)} \sum_{k=1}^{N} \frac{du_k}{u_k - \lambda_j} = p_j^2 \sum_{k=1}^{N} \frac{du_k}{u_k - \lambda_j}, \tag{46}$$

i.e.,

$$dp_j = \frac{1}{2}\, p_j \sum_{k=1}^{N} \frac{du_k}{u_k - \lambda_j}. \tag{47}$$

Letting $\sum_{j=1}^{N} q_j \cdot$ upon both sides of (47) and noticing (41), we have

$$\sum_{j=1}^{N} q_j \, dp_j = \tfrac{1}{2} \sum_{j=1}^{N} p_j q_j \sum_{k=1}^{N} \frac{du_k}{u_k - \lambda_j} = \sum_{k=1}^{N} v_k \, du_k, \tag{48}$$

which implies [36] that the transformation from (q_i, p_i) to (v_k, u_k) is a canonical one preserving the Hamiltonian structure. Thus, we have proved

Proposition 8. The N pairs of coordinates u_i, v_i defined by (41) are canonically conjugated, i.e.,

$$\{u_i, u_j\} = \{v_i, v_j\} = 0, \quad \{v_j, u_i\} = \delta_{ij}, \quad i, j = 1, 2, \ldots, N. \tag{49}$$

Because of Proposition 8, in light of the thought of quantization introduced in Ref. [21] we can promptly know that separation of variables has a direct quantum counterpart [31,40]. The corresponding quantization procedure is similar to the one in Ref. [21], and thus is omitted here.

6. Discussion and comparison

We should emphasize that the integrable Hamiltonian systems (11) and (34) studied in the present article cannot be included in the integrable systems considered in Ref. [21] where a large number of completely integrable systems were discussed. The simple reason is that the Lax matrix M contains the term of λ^{-1}. The case of the Lax matrix M including some positive power of λ has been studied by Eilbeck et al. [21], and the associated separation of variables may be realized. But the general case about some negative power of λ in the Lax matrix M is not yet discussed. The present article only deals with the Lax matrix M containing the term of λ^{-1}, and gives an associated modified r-matrix and separation variables. Then, we naturally desire to see that a generalized Lax matrix M, including both negative and positive powers of λ, the associated dynamical or non-dynamical r-matrix and separation variables are investigated in a short time.

It should be mentioned that the modified r-matrix (25) obtained in the present paper depends only on the parameters λ, μ, and has no relation to the canonical (or dynamical) variables p_i, q_i, i.e., (25) is a non-dynamical modified r-matrix, which assures that the Poisson algebra of a model whose structural constants are given by the non-dynamical r-matrix is closed, and there is the closed-form Yang–Baxter equation connected with their r-matrix. But the dynamical r-matrix, generally speaking, has no longer those attributes [22]. Nijhoff and Capel systematically studied the different aspects, such as Miura transformations, integrable mapping, similarity reductions, etc. [41–43], of integrable discretization in space and time of the KdV, MKdV, and other nonlinear evolution equations. Afterwards, Nijhoff and his collaborators present the dynamical r-matrix for the elliptic Ruijsenaars–Schneider system [22]. Eilbeck et al. [21] constructed the dynamical r-matrix of a class of integrable systems in the continuous case. Apparently, the r-matrix (25) cannot be included in the above literature by virtue of its non-dynamical property. Recently, Ragnisco [44] finds a dynamical r-matrix for the Garnier-constrained system of the discrete Toda lattice. But a very interesting thing is that the author looks for a non-dynamical r-matrix for the same system of discrete

Toda lattice through choosing a different Lax matrix, which is being organized for a paper for publication [45]. Just as pointed out in the end of Ref. [44], for the same integrable constrained system, it is important to search for a non-dynamical (or constant) r-matrix as earlier as possible, because that will largely reduce the complicated process of calculations such as Yang–Baxter equation etc.

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