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GENERALIZED STRUCTURE OF LAX REPRESENTATIONS FOR NONLINEAR EVOLUTION EQUATION*

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Abstract

A new production form for a hierarchy of nonlinear evolution equations (NLEEs) is given in this paper. The form contains productions of isospectral and non-isospectral hierarchy. Under this form a generalized structure of Lax representations for the hierarchy of NLEEs is this presented. As a concrete example, the Levi-hierarchy of evolution equations are discussed at the end of this paper.

Key words production form, generalized structure, Levi hierarchy

I. Introduction

In soliton theory, it is very interesting for us to find the Lax representations of nonlinear evolution equations (NLEEs) and to discuss the algebraic structure of Lax operator and other properties. Ma Wenxiu studied the Lax representation structure for the hierarchies of isospectral and non-isospectral evolution equations in Refs. [1, 2]. For the convinience of discussion, the related results in Refs. [1, 2] are unified as follows:

Let $u=(u_1, \cdots, u_m)^T$ be the potential vector function. Then $N\times N$ spectral problem

$$L(u)y=\lambda y,\ \lambda_t=a\lambda^m \qquad (m\geqslant 0,\ a=\text{const}) \tag{1.1}$$

is connected with its hierarchy of NLEEs (isospectral case: $a=0$; non-isospectral case: $a\neq 0$)

$$u_t=J\mathscr{L}^m G_0 \qquad (m\geqslant 0) \tag{1.2}$$

which admits the Lax representation

$$L_t=[W_m,\ L]+aL^m,\ W_m=\sum_{j=0}^{m} V_{j-1}L^{m-j} \qquad (m\geqslant 0) \tag{1.3}$$

under the following two conditions:

(i) The function G_0 in (1.2) and the operator V_{-1} in (1.3) are determined by the operator equation

$$[V_{-1},\ L]=L_*(JG_0)-aI \quad (I \text{ is the } N\times N \text{ identity matrix operator }) \tag{1.4}$$

(ii) The operator $V_j=V(G_j)=V(\mathscr{L}^jG_0)$ in (1.3) is given by the operator solution $V=V(G)$ with $G=\mathscr{L}^jG_0$ of operator equation

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$$[V(G),\ L]=L_*(KG)-L_*(JG)L \qquad (\forall G=(G_1,\ \cdots,\ G_N)^T) \tag{1.5}$$

where

$$K=J\mathscr{L},\ L_*(\xi)\triangleq\frac{d}{d\varepsilon}|_{\varepsilon=0}L(u+\varepsilon\xi)$$

According to the above procedure, some isospectral ($\alpha=0$) and non-isospectral ($\alpha\neq0$) hierarchies of NLEEs and their corresponding Lax representations are derived.

For the spectral problem (1.1), if the hierarchy (1.2) is needed to produce, then its key lies in solving both the function G_0 and the operator V_{-1} satisfying (1.4), and if the hierarchy (1.2) is wanted to possess the Lax representation (1.3), then its key lies in solving the operator solution $V=V(G)$ of (1.5). On the latter we have had some discussions[4]. On the former we have found that for somewhat spectral problems such as Levi spectral problem, Kaup-Newell spectral problem etc., (1.4) is not easily solved or even if it can be solved, the solution expression is complicated, which is not available to produce the hierarchy (1.2). To solve this problem, a new production form for the hierarchy (1.2) is presented in this paper, and the applicable range and operation process of (1.4) are also separately expanded and reduced. Consequently, a generalized structure of Lax representations for the hierarchy of NLEEs is given under this new production form.

II. Production for the Hierarchy of NLEEs and Generalized Structure of Lax Representation

Let us consider a general $N\times N$ spectral problem

$$Ly=L(u)y=\lambda y \tag{2.1}$$

where $L=L(u)$ is spectral operator, $u=(u_1,\ \cdots,\ u_l)^T$ is potential vector-value function, λ is spectral parameter, and $y=(y_1,\ \cdots,\ y_N)^T$.

By virtue of the spectral gradient method[6], we can always find a pair of operators $K=K(u,\ \partial,\ \partial^{-1})$, $J=J(u,\ \partial,\ \partial^{-1})$ $(\partial=\partial/\partial x,\ \partial\partial^{-1}=\partial^{-1}\partial=1)$ called the pair of Lenard's operators such that

$$K\nabla_u\lambda=\lambda^c\cdot J\nabla_u\lambda \qquad c=\text{some fixed constant} \tag{2.2}$$

where $\nabla_u\lambda\triangleq\frac{\delta\lambda}{\delta u}$ stands for the spectral gradient of (2.1), which can be calculated according to some certain methods[6, 7]. The operator $\mathscr{L}\triangleq J^{-1}K$ is actually ordinary recursion operator or strong symmetric operator. Generally, K and J are skew-symmetric and at least one of them is Hamiltonian operator.

Now, we will explain the production procedure for the hierarchy of NLEEs of (2.1). Still denote $L_*(\xi)\triangleq\frac{d}{d\varepsilon}|_{\varepsilon=0}L(u+\varepsilon\xi)$. For an arbitary given $N\times N$ matrix operator $M=(m_{ij})_{N\times N}$, we construct the following operator equation yielded by the operators J and L_* with regard to the vector function $G_{-1}\triangleq(G_{-1}^{(1)},\ \cdots,\ G_{-1}^{(l)})^T$

$$L_*(JG_{-1})=M \tag{2.3}$$

Denote the solution set of (2.3) by $\mathscr{B}$ (generally $\mathscr{B}\neq\phi$). Suppose $\mathscr{B}\neq\phi$, chose $G_{-1}\in\mathscr{B}$ and define the generalized Lenard's recursive sequence as follows:

$$G_j=J^{-1}KG_{j-1}=\mathscr{L}^{j+1}G_{-1} \qquad (j=0,\ 1,\ 2,\ \cdots) \tag{2.4}$$

The NLEEs

$$u_{tm}=X_m(u) \qquad (m=0,\ 1,\ 2,\ \cdots) \tag{2.5}$$

which are produced by the generalized vector fields (GVF) $X_m(u)\triangleq JG_m=J\mathscr{L}^{m+1}G_{-1}$, are called the hierarchy of generalized nonlinear evolution equations (GNLEEs) of (2.1).

By virtue of the new form (2.3), the GNLEEs (2.5) of (2.1) is generated, and (2.3) is simpler than (1.4). From the following Theorem, we can know that (2.3) includes (1.4).

Theorem Let M be an arbitary given $N\times N$ matrix operator. For the spectral problem (2.1), suppose

(i) $L_*(\xi)=0\Longleftrightarrow\xi=0$,

(ii) $\mathscr{B}\neq\phi$,

(iii) for an arbitary $G=(G^{(1)},\ \cdots,\ G^{(l)})^T$, the following operator equation

$$[V(G),\ L]=L_*(KG)-L_*(JG)L \tag{2.6}$$

possesses the operator solution $V=V(G)$. Then the hierarchy of GNLEEs (2.5) has the following form of Lax representations

$$\left.\begin{aligned}L_{t_m}&=[W_m,\ L]+ML^{m+1}\\ W_m&=\sum_{j=0}^{m}V(G_{j-1})L^{m-j}\end{aligned}\right\}\ (m\geqslant 0) \tag{2.7}$$

where G_{j-1} are determined by (2.4).

Proof

$$[W_m,\ L]=\sum_{j=0}^{m}[V(G_{-1}),\ L]L^{m-j}\overset{\text{(iii)}}{=\!=}\sum_{j=0}^{m}(L_*(KG_{j-1})-L_*(JG_{j-1})L)L^{m-j}\overset{(2.4)}{=\!=}$$

$$\sum_{j=0}^{m}(L_*(JG_j)L^{m-j}-L_*(JG_{j-1})L^{m-j+1})$$

$$=L_*(JG_m)-L_*(JG_{-1})L^{m+1}\overset{(2.3)}{=\!=}L_*(JG_m)-ML^{m+1}=L_*(X_m)-ML^{m+1}$$

Thus,

$$[W_m,\ L]+ML^{m+1}=L_*(X_m) \tag{2.8}$$

From $L_*(u_{t_m})=L_{t_m}$ and (2.8), we have

$$L_*(u_{t_m}-X_m(u))=L_{t_m}-([W_m,\ L]+ML^{m+1})$$

Because L_* is injective, (2.7) and (2.5) are equivalent.

By (2.8) and (i), we easily obtain

Corollary The vector potential function u satisfies the stationary system

$$C_0X_N(u)+C_1X_{N-1}(u)+\cdots+C_NX_0(u)=0 \qquad (C_0,\cdots,C_N=\text{const}) \tag{2.9}$$

if and only if

$$\left[\sum_{k=0}^{N}C_{N-k}W_k,\ L\right]=-M\left(\sum_{k=0}^{N}C_{N-k}L^{k+1}\right) \tag{2.10}$$

Remark The operator equation (2.3) and representations (2.7) contain all the desired information concerning the GNLEEs (2.5). Some special cases display as follows:

1. As $M=0$, then (2.3) becomes $L_*(JG_{-1})=0$, i. e, $JG_{-1}=0$, and the corresponding hierarchy (2.5) reads the isospectral ($\lambda_t=0$) hierarchy of spectral problem (2.1), whose Lax representations are exactly the structure presented in Ref. [8].

2. As $M=aI$ (I is the $N\times N$ unit matrix operator, $a=$const.), then from the theorem the Lax representation structure for the hierarchy of (2.5) reads: $Lt_m=[W_m,L]+aL^{m+1}$, $W_m=\sum_{j=0}^{m} V(G_{j-1})L^{m-j}$, which are actually the Lax representations[9] for the non-isospectral ($\lambda_t=a\lambda^{m+1}$) hierarchy of spectral problem (2.1). But the form of (2.3) changes as $L_*(JG_{-1})=aI$, which is obviously simpler than (1.4) and easily calculated.

3. In (2.7), let $m=0$, then (2.7) reads $Lt_0=[W_0, L]+ML$, which is exactly the so-called L-A-B representations presented by Manakov[10]. The equation $u_{t_0}=X_0(u)$ possesses this kind of representation. In the present paper, by (2.7) and the arbitarity of M the Manakov operators pair of the associated evolution equation $u_{t_0}=X_0(u)$ have been constructed. Thus, the representation (2.7) is naturally a generalization of L-A-B representation of integrable system. Certainly, we shall consider the operator algebraic structure yielded by the representation (2.7), which is discussed in detail in another paper.

4. For an arbitary spectral problem given (2.1), the conditions (i), (ii) in the theorem are easily checked. Hence, in order to obtain the representations (2.7) of the hierarchy (2.5), its key lies in solving the operator solution $V=V(G)$ of operator equation (2.6), which has had a definite answer for some examples in previous papers[12-14].

5. From (2.10) in the corollary, we may still construct the Lie-algebraic structure of operator for the stationary systems[15].

In view of remark 1, 2 and the arbitarity of M, we call (2.7) as the generalized structure of Lax representation (GSLR) for the hierarchy of GNLEEs (2.5).

In the following, we shall give an example (i. e. Levi spectral problem) to explain the production procedure of GNLEEs (2.5), and present the corresponding GSLR. The following calculating method can be applied to other spectral problems like (2.1).

III. Concrete Examples

Consider the Levi spectral problem[16]

$$Ly=\frac{\lambda}{2}y,\quad L=L(u,\ v)=\begin{pmatrix} -\partial+\frac{u-v}{2} & u \\ -v & \partial+\frac{u-v}{2} \end{pmatrix},\quad \partial=\partial/\partial x \tag{3.1}$$

whose Lenard's operators K, J are[17]

$$K=\begin{pmatrix} -u\partial-\partial u & -\partial^2-v\partial+\partial u \\ \partial^2-\partial v+u\partial & v\partial+\partial v \end{pmatrix},\quad J=\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \tag{3.2}$$

Apparently, $L_*(\xi)=\begin{pmatrix} (\xi_1-\xi_2)/2 & \xi_1 \\ -\xi_2 & (\xi_1-\xi_2)/2 \end{pmatrix}$, $\xi=(\xi_1,\ \xi_2)^T$, and L_* is injective. From

Ref. [13], we can know that for the Levi spectral problem (3.1) the corresponding operator equation $[V, L]=L_*(KG)-L_*(JG)L$ possesses the operator solution

$$V=V(G)=\begin{pmatrix} -\frac{1}{2}(G^{(1)}+G^{(2)})_x+(G^{(2)}-G^{(1)})\partial & -G_x^{(1)} \\ G_x^{(1)} & \frac{1}{2}(G^{(1)}+G^{(2)})_x+(G^{(2)}-G^{(1)})\partial \end{pmatrix}$$

$$(\forall G=(G^{(1)}, G^{(2)})^T) \qquad (3.3)$$

In (2.3), if let $M=0$ then (2.3) becomes $JG_{-1}=0$, i. e, $G_{-1}\in\mathrm{Ker}J$, and the hierarchy of NLEEs given by (2.5) is actually the isospectral ($\lambda_t=0$) hierarchy of (3.1) (see Ref. [17], sec. 2), whose Lax representations (2.7) are accorded with the results obtained in Ref. [13].

In (2.3), if let $M=aI$ ($a\neq 0$, a=const., I is the 2×2 unit matrix operator), then (2.3) has no solutions, which matches the fact that (1.4) can't easily solved. But, if set

$$M=\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad a=\text{const.} \qquad (3.4)$$

then (2.3) reads

$$\begin{pmatrix} (G_{-1,x}^{(2)}-G_{-1,x}^{(1)})/2 & G_{-1,x}^{(2)} \\ -G_{-1,x}^{(1)} & (G_{-1,x}^{(2)}-G_{-1,x}^{(1)})/2 \end{pmatrix}=\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad G_{-1}=\begin{pmatrix} G_{-1}^{(1)} \\ G_{-1}^{(2)} \end{pmatrix} \qquad (3.5)$$

which is solved without difficulty

$$G_{-1}=\begin{pmatrix} ax+d_1 \\ ax+d_2 \end{pmatrix}, \qquad \forall\ d_1,\ d_2=\text{const.}$$

Hence, the corresponding hierarchy of GNLEEs (2.5) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_m}=J\mathscr{L}^{m+1}\begin{pmatrix} ax+d_1 \\ ax+d_2 \end{pmatrix}=J(J^{-1}K)^{m+1}\begin{pmatrix} ax+d_1 \\ ax+d_2 \end{pmatrix} \qquad (m=0,\ 1,\ 2,\ \cdots) \qquad (3.6)$$

where the two operators K, J are given by (3.2). According to the theorem, the hierarchy (3.6) has the following GSLR.

$$\left.\begin{aligned} &L_{t_m}=[W_m,\ L]+\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\cdot 2^m L^{m+1} \\ &W_m=\sum_{j=0}^{m}\begin{pmatrix} -\frac{1}{2}(G_{j-1}^{(1)}+G_{j-1}^{(2)})_x+(G_{j-1}^{(2)}-G_{j-1}^{(1)})\partial & -G_{j-1,x}^{(2)} \\ G_{j-1,x}^{(1)} & \frac{1}{2}(G_{j-1}^{(1)}+G_{j-1}^{(2)})_x+(G_{j-1}^{(2)}-G_{j-1}^{(1)})\partial \end{pmatrix}(2L)^{m-j} \end{aligned}\right\} \qquad (3.7)$$

where $G_{j-1}=(G_{j-1}^{(1)}, G_{j-1}^{(2)})^T$ are determined by $G_{j-1}=J^{-1}KG_{j-2}=(J^{-1}K)^j\begin{pmatrix} ax+d_1 \\ ax+d_2 \end{pmatrix}$, $j=0, 1, 2, \cdots$. It should be pointed out that (3.6) and (3.7) are two completely new results.

Now, we consider the general case $M=(m_{ij})_{2\times 2}$ as follows. Let $A=A(x, t, u, v)$.

$B=B(x,t,u,v)$ (u,v are the two potentials in (3.1) as two arbitary smooth functions). Then for the Levi spectral problem (3.1), if and only if

$$M=\begin{pmatrix}\frac{A+B}{2} & B\\ A & \frac{A+B}{2}\end{pmatrix}$$

(2.3) has the solution

$$G_{-1}=(-\partial^{-1}A+d_1,\ \partial^{-1}B+d_2)^T,\ \forall d_1,d_2=\text{const},\ \partial=\partial/\partial x\ \ \partial\partial^{-1}=\partial^{-1}\partial=1$$

which produces the Levi hierarchy of GNLEEs

$$\begin{pmatrix}u\\ v\end{pmatrix}_{t_m}=J\mathscr{L}^{m+1}\begin{pmatrix}-\partial^{-1}A+d_1\\ \partial^{-1}B+d_2\end{pmatrix}=J(J^{-1}K)^{m+1}\begin{pmatrix}-\partial^{-1}A+d_1\\ \partial^{-1}B+d_2\end{pmatrix}\quad(m=0,1,2,\cdots)\tag{3.8}$$

whose GSLR are

$$L_{t_m}=[W_m,\ L]+\begin{pmatrix}\frac{A+B}{A} & B\\ A & \frac{A+B}{2}\end{pmatrix}\cdot 2^m L^{m+1}\qquad m=0,\ 1,\ 2,\ \cdots\tag{3.9}$$

where the form of W_m is defined by $(3.7)_2$, but each $G_{j-1}=(G_{j-1}^{(1)},\ G_{j-1}^{(2)})^T$ in W_m is given by $G_{j-1}=(J^{-1}K)^j\begin{pmatrix}-\partial^{-1}A+d_1\\ \partial^{-1}B+d_2\end{pmatrix}$ $(j=0,\ 1,\ 2,\ \cdots)$.

Obviously, as $A=B=0$, (3.8) and (3.9) give the isospectral ($\lambda_t=0$) hierarchy and its Lax representation, respectively. As A, $B=$const., and $A+B=0$, i. e, $A=-B$, (3.8) and (3.9) exactly read the non-isospectral ($\lambda_t=\pm a_t\lambda^{m+1}$, $m=0,1,2,\ldots$, $i^2=-1$) hierarchy (3.6) and its GSLR (3.7), respectively. So the Levi hierarchy of GNLEEs (3.8) includes all possible hierarchies of NLEEs and all possible equations in every hierarchy generated through the production element G_{-1} for the Levi spectral problem (3.1), and the GSLR (3.9) gives the Lax representation for all possible NLEEs in every hierarchy. For two pairs of different $(A_1,\ B_1)$ and $(A_2,\ B_2)^T$, the corresponding Levi hierarchies (3.8) are also different. Then what is the relation between the two different Levi hierarchies? This problem still needs further discussion.

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