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Two new integrable systems in Liouville's sense

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Under a constraint on potentials and eigenfunctions, two spectral problems (Kaup–Newell spectral problem and Levi spectral problem) are nonlinearized to be new finite-dimensional completely integrable Hamiltonian systems in Liouville's sense.

It is quite an important task to look for new completely integrable Hamiltonian systems in soliton theory. The key lies in looking for involutive functional systems, which are obtained in general by the spectral technique associated with nonlinear evolution equations, and thus new completely integrable systems are produced [1,2]. Flaschka [3] pointed out an important principle to obtain finite-dimensional integrable systems by constraining infinite-dimensional integrable systems on a finite-dimensional invariant subset. Recently, Cao Cewen has presented the thought [4] of generating finite-dimensional integrable systems through nonlinearization of a Lax system for isospectral evolution equations and has successfully found many finite-dimensional completely integrable systems [5]. The basis for the nonlinearization of a Lax system is that the soliton equation $U_t = X_m$ is first expressed as a Lax form, $L_t = [V_m, L]$; then under the Bargmann constraint or the Neumann constraint the time part of the Lax pair is reduced to a finite-dimensional Hamiltonian system, whose Hamiltonian F_m constitutes the involutive system of the nonlinearized system of the space part of the Lax pair under the above constraint. Another important application of the nonlinearization method is that finding the solution of the soliton equation associated with an eigenvalue problem is reduced to solving the compatible system of nonlinear ordinary differential equations [6–10].

In this Letter, by the use of the "nonlinearization method" [4,5,11,12], we obtain two new completely integrable Hamiltonian systems in Liouville's sense which are generated through nonlinearization of the Kaup–Newell eigenvalue problem and the Levi eigenvalue problem, respectively. So, the scope of the finite-dimensional integrable systems proposed in ref. [5] is enlarged furthermore.

We describe briefly here the procedure for the nonlinearization of the eigenvalue problem. Consider the eigenvalue problem

$$y_x = M(u, \lambda) y, \qquad y = (y_1, y_2)^{\mathrm{T}} . \tag{1}$$

The functional gradient $\delta\lambda_j/\delta u$ of the eigenvalue λ_j, with regard to the potential u satisfies

$$K\delta\lambda_j/\delta u = \lambda_j J \delta\lambda_j/\delta u . \tag{2}$$

Here K and J are called the Lenard operator pair, and their Lenard gradient sequence G_j can be determined recursively:

$$KG_{j-1} = JG_j, \qquad JG_{-1} = 0, \qquad j = 0, 1, 2, \ldots . \tag{3}$$

The soliton hierarchy $u_t = JG_m$ has the Lax pair, the eigenvalue problem (1) and the auxiliary problem

$$y_{t_m} = V_m y . \tag{4}$$

The following two constraints,

$$G_0=\sum_{j=1}^{N}\gamma_j\,\delta\lambda_j/\delta u\,,\qquad G_{-1}=\sum_{j=1}^{N}\gamma_j\,\delta\lambda_j/\delta u\,, \tag{5}$$

which are called the Bargmann and Neumann constraint, respectively, play a central role in the process of nonlinearization of the eigenvalue problem (1). From (5) we can obtain the relations

$$u=f(q,p)\qquad \text{and } g(q,p)=0,\qquad u=f(q,p)\,, \tag{6}$$

where $q=(q_1,...q_N)^{\mathrm{T}}$, $p=(p_1,...p_N)^{\mathrm{T}}$, $(q_j,p_j)=(y_1(\lambda_j),y_2(\lambda_j))$. Under the two constraints, the eigenvalue problem (1) is nonlinearized into the two finite-dimensional systems

$$\begin{pmatrix}q_x\\ p_x\end{pmatrix}=M(f(q,p),\Lambda)\begin{pmatrix}q\\ p\end{pmatrix},\qquad \Lambda=\mathrm{diag}(\lambda_1,...,\lambda_N)\,, \tag{7}$$

$$\begin{pmatrix}q_x\\ p_x\end{pmatrix}=M(f(q,p),\Lambda)\begin{pmatrix}q\\ p\end{pmatrix},\qquad g(q,p)=0\,, \tag{8}$$

which are called the Bargmann and Neumann systems, respectively. The time part of the Lax pair (4) is reduced to two finite-dimensional Hamiltonian systems satisfying the above constraints, equalities (2) and (3), whose Hamiltonian, F_m, constitutes the involutive system of the Bargmann system (7) or that of the Neumann system (8) through the Moser constraint procedure. Sometimes some modifications are made, especially for the Neumann system.

Following the thought of ref. [5], we continue to use the concerned sign of ref. [5] in this Letter. Consider the following spectral problems:

(1) The Kaup–Newell spectral problem [13]

$$y_x=My\equiv\begin{pmatrix}-\mathrm{i}\lambda^2 & \lambda u\\ \lambda v & \mathrm{i}\lambda^2\end{pmatrix}y\,, \tag{9}$$

where $\mathrm{i}=\sqrt{-1}$, λ is the eigenparameter and the functions $u=u(x,t)$, $v=v(x,t)$ are called potentials of (9). The underlying interval Ω is $(-\infty,+\infty)$ or $(0,T)$ for the decaying conditions at infinity or for periodic conditions separately.

Let $\lambda_1,...,\lambda_N$ be N different eigenvalues of (9), then the functional gradient $\nabla\lambda_j$ of λ_j is

$$\nabla\lambda_j\triangleq\begin{pmatrix}\delta\lambda_j/\delta u\\ \delta\lambda_j/\delta v\end{pmatrix}=\begin{pmatrix}\lambda_j p_j^2\\ -\lambda_j q_j^2\end{pmatrix},\qquad \int\limits_{\Omega}(vp_j^2+4\mathrm{i}\lambda_j p_j q_j-uq_j^2)\,\mathrm{d}x=1\,.$$

where $y=(p_j(x),q_j(x))^{\mathrm{T}}$ is the eigenfunction corresponding to the eigenvalue λ_j, i.e.

$$q_{j,x}=-\mathrm{i}\lambda_j^2 q_j+\lambda_j u p_j\,,\qquad p_{j,x}=\lambda_j v q_j+\mathrm{i}\lambda_j^2 p_j\,. \tag{10}$$

The Lenard operator pair associated with the Kaup–Newell hierarchy is

$$K=\begin{bmatrix}\tfrac12\partial u\partial^{-1}u\partial & -\dfrac{1}{2\mathrm{i}}\partial^2+\tfrac12\partial u\partial^{-1}v\partial\\ \dfrac{1}{2\mathrm{i}}\partial^2+\tfrac12\partial v\partial^{-1}u\partial & \tfrac12\partial v\partial^{-1}v\partial\end{bmatrix},\qquad J=\begin{pmatrix}0 & \partial\\ \partial & 0\end{pmatrix},$$

where $\partial=\partial/\partial x$, $\partial^{-1}\partial=\partial\partial^{-1}=1$. $\nabla\lambda_j$ satisfies the linear equation $K\nabla\lambda_j=\lambda_j J\nabla\lambda_j$.

The Lenard recursive sequence G_m and the Kaup–Newell vector field $X_m=JG_m$ can be calculated recursively with $X_m=KG_{m-1}=JG_m$, $m=0,1,2,...$, $G_{-1}=(1,0)^{\mathrm{T}}$, the first few being

$$G_0=(v,u)^{\mathrm{T}},\qquad G_1=((1/2\mathrm{i})v_x+\tfrac{1}{2}v^2u,\ -(1/2\mathrm{i})u_x+\tfrac{1}{2}u^2v)^{\mathrm{T}},$$

$$X_0=(u_x,v_x)^{\mathrm{T}},\qquad X_1=(-(1/2\mathrm{i})u_{xx}+\tfrac{1}{2}(u^2v)_x,\ (1/2\mathrm{i})v_{xx}+\tfrac{1}{2}(v^2u)_x)^{\mathrm{T}}.$$

The soliton equation $(u,v)_t^{\mathrm{T}}=X_1=JG_1$ is reduced to the well-known derivative Schrödinger equation $u_t=\frac{1}{2}\mathrm{i}u_{xx}+\frac{1}{2}(u|u|^2)_x$ as $u=v^*$. The Kaup–Newell hierarchy $(u,v)_t^{\mathrm{T}}=X_m(u,v)$ is the compatible condition of eigenvalue problem (9) and

$$y_{t_m}=\sum_{j=0}^{m}V(G_{j-1})\lambda^{2(m-j)}y,\qquad m=0,1,2,\ldots. \tag{11}$$

$$V(G_{j-1})=\begin{pmatrix}-\frac{1}{2}\mathrm{i}\lambda\partial^{-1}(uG^{(1)}_{j-1,x}+vG^{(2)}_{j-1,x}) & \frac{1}{2}\mathrm{i}G^{(2)}_{j-1,x}+\frac{1}{2}u\partial^{-1}(uG^{(1)}_{j-1,x}+vG^{(2)}_{j-1,x})\\ -\frac{1}{2}\mathrm{i}G^{(1)}_{j-1,x}+\frac{1}{2}v\partial^{-1}(uG^{(1)}_{j-1,x}+vG^{(2)}_{j-1,x}) & \frac{1}{2}\mathrm{i}\lambda\partial^{-1}(uG^{(1)}_{j-1,x}+vG^{(2)}_{j-1,x})\end{pmatrix},$$

$$j=0,1,2,\ldots. \tag{12}$$

$G_{j-1}=(G^{(1)}_{j-1},G^{(2)}_{j-1})^{\mathrm{T}}$ is the Lenard recursive sequence; $G^{(i)}_{j-1,x}=\partial G^{(i)}_{j-1}/\partial x$, $i=1,2$. The Bargmann constraint is given by $G_0=\sum_{j=1}^{N}\nabla\lambda_j$, which is equivalent to

$$u=-\langle\Lambda q,q\rangle,\qquad v=\langle\Lambda p,p\rangle, \tag{13}$$

where $q=(q_1,\ldots,q_N)^{\mathrm{T}}$, $p=(p_1,\ldots p_N)^{\mathrm{T}}$; $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$; $\langle\ ,\ \rangle$ is the standard inner product in $\mathbb{R}^N$.

Under the Bargmann constraint (13), the Kaup–Newell spectral problem (9) is nonlinearized as

$$q_x=-\mathrm{i}\Lambda^2q-\langle\Lambda q,q\rangle\Lambda p=-\partial H/\partial p,\qquad p_x\mathrm{i}\Lambda^2p+\langle\Lambda p,p\rangle\Lambda q=\partial H/\partial q, \tag{14}$$

which is an integrable Hamiltonian system $(\mathbb{R}^{2N},\mathrm{d}p\wedge\mathrm{d}q,H)$ with

$$H=\mathrm{i}\langle\Lambda^2p,q\rangle+\tfrac{1}{2}\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle, \tag{15}$$

whose involutive system of conserved integrals is

$$F_m=\mathrm{i}\langle\Lambda^{2m+2}p,q\rangle+\tfrac{1}{2}\langle\Lambda p,p\rangle\langle\Lambda^{2m+1}q,q\rangle+\tfrac{1}{2}\sum_{j=1}^{m}\begin{vmatrix}\langle\Lambda^{2(m-j)+1}q,q\rangle & \langle\Lambda^{2(m-j)+2}p,q\rangle\\ \langle\Lambda^{2j}p,q\rangle & \langle\Lambda^{2j+1}p,p\rangle\end{vmatrix}.$$

Remark. $H=F_0$. $(F_k,F_l)=0$, $\forall k,l\in\mathbb{Z}^+$ is easily proved by using properties of the Poisson bracket.

(2) The Levi spectral problem [14]

$$\psi_x=U\psi\equiv\begin{pmatrix}0 & u\\ v & \lambda-u+v\end{pmatrix}\psi. \tag{16}$$

By making the transformation $\psi=y\exp\{\frac{1}{2}[\lambda x+\partial^{-1}(v-u)]\}$ and its inverse $y=\psi\exp\{\frac{1}{2}[-\lambda x+\partial^{-1}(u-v)]\}$, (16) is equivalent to the eigenvalue problem

$$y_x=My\equiv\begin{pmatrix}-\frac{1}{2}\lambda+\frac{1}{2}(u-v) & u\\ v & \frac{1}{2}\lambda+\frac{1}{2}(v-u)\end{pmatrix}y,$$

$$\nabla\lambda_j\triangleq(\delta\lambda_j/\delta u,\delta\lambda_j/\delta v)^{\mathrm{T}}=(p_j(p_j+q_j),\ -(p_j+q_j)q_j)^{\mathrm{T}},\qquad j=1,\ldots,N. \tag{17}$$

We choose

$$K=\begin{pmatrix}-\partial u-u\partial & -\partial^2-v\partial+\partial u\\ \partial^2-\partial v+u\partial & v\partial+\partial v\end{pmatrix},\qquad J=\begin{pmatrix}0 & \partial\\ \partial & 0\end{pmatrix}.$$

Then $\nabla\lambda_j$ satisfies $K\nabla\lambda_j=\lambda_jJ\nabla\lambda_j$.

The Lenard recursive sequence G_j of (17) is determined by the formula $KG_{j-1}=JG_j$, $j=0,1,\ldots$, $G_{-1}=(0,1)^{\mathrm{T}}$, the first few being

$$G_0=(v,u)^{\mathrm{T}},\qquad G_1=(v_x+2uv-v^2,\ -u_x-2uv+u^2)^{\mathrm{T}},$$

$$G_2=(v_{xx}+3uv_x-3vv_x+v^3+3vu^2-6v^2u,\ u_{xx}-3uu_x+3u_xv+u^3+3uv^2-6u^2v)^{\mathrm{T}}.$$

The representative equations of the Levi hierarchy have $(u,v)_t^{\mathrm{T}}=JG_1$ and $(u,v)_t^{\mathrm{T}}=JG_2$, which is reduced to the well-known Burgers equation, $u_t=-u_{xx}+2uu_x$, as $v=0$ and to the well-known mKdV equation, $u_t=u_{xxx}-6u^2u_x$, as $u=v$, respectively. The Levi hierarchy of equations $(u,v)_t^{\mathrm{T}}=JG_m$ has the Lax representation, the eigenvalue problem (17) and the auxiliary problem

$$y_{t_m}=\sum_{j=0}^{m}V(G_{j-1})\lambda^{m-j}y\,, \tag{18}$$

where

$$V(G_{j-1})=\begin{pmatrix} -\frac{1}{2}(G^{(1)}_{j-1,x}+G^{(2)}_{j-1,x})+\frac{1}{2}(u-v)(G^{(2)}_{j-1}-G^{(1)}_{j-1})-\frac{1}{2}\lambda(G^{(2)}_{j-1}-G^{(1)}_{j-1}) & -G^{(2)}_{j-1,x}+u(G^{(2)}_{j-1}-G^{(1)}_{j-1}) \\ G^{(1)}_{j-1,x}+v(G^{(2)}_{j-1}-G^{(1)}_{j-1}) & \frac{1}{2}(G^{(1)}_{j-1,x}+G^{(2)}_{j-1,x})+\frac{1}{2}(v-u)(G^{(2)}_{j-1}-G^{(1)}_{j-1})+\frac{1}{2}\lambda(G^{(2)}_{j-1}-G^{(1)}_{j-1}) \end{pmatrix}, \tag{19}$$

$G_{j-1}=(G^{(1)}_{j-1},G^{(2)}_{j-1})^{\mathrm{T}}$ is the Lenard recursive sequence of (17), $j=0,1,\ldots$; $G^{(i)}_{j-1,x}=\partial G^{(i)}_{j-1}/\partial x$, $i=1,2$.

Consider the Bargmann constraint $G_0=\sum_{j=1}^{N}\nabla\lambda_j$, which is equivalent to

$$u=-\langle p+q,q\rangle,\qquad v=\langle p+q,p\rangle\,. \tag{20}$$

Under the Bargmann constraint (20), the nonlinearized equation (17),

$$q_x=-\tfrac{1}{2}\Lambda q-\tfrac{1}{2}\langle p+q,p+q\rangle q-\langle p+q,q\rangle p,\qquad p_x=\tfrac{1}{2}\Lambda p+\tfrac{1}{2}\langle p+q,p+q\rangle p+\langle p+q,p\rangle q\,,$$

has the Hamiltonian $H=F_0$ and the conserved integrals involutive systems in pairs F_m,

$$F_0=H=\tfrac{1}{2}\langle\Lambda p,q\rangle+\tfrac{1}{2}\langle p+q,p+q\rangle\langle p,q\rangle+\tfrac{1}{2}\begin{vmatrix}\langle q,q\rangle & \langle p,q\rangle\\ \langle p,q\rangle & \langle p,p\rangle\end{vmatrix},$$

$$F_m=\tfrac{1}{2}\langle\Lambda^{m+1}p,q\rangle+\tfrac{1}{2}\langle\Lambda^m(p+q),p+q\rangle\langle p,q\rangle+\tfrac{1}{2}\sum_{j=0}^{m}\begin{vmatrix}\langle\Lambda^j q,q\rangle & \langle\Lambda^j p,q\rangle\\ \langle\Lambda^{m-j}p,q\rangle & \langle\Lambda^{m-j}p,p\rangle\end{vmatrix}. \tag{21}$$

It is not difficult to obtain the Poisson bracket $(F_k,F_l)=0$, $\forall k,l\in\mathbb{Z}^+$, through a series of careful calculations and using the properties of the Poisson bracket.

Remark. In this Letter the Poisson bracket of two functions $E(p,q)$ and $F(p,q)$ in the symplectic manifoild $(\mathbb{R}^{2N},\mathrm{d}p\wedge\mathrm{d}q)$ is defined as

$$(E,F)=\sum_{j=1}^{N}\frac{\partial E}{\partial q_j}\frac{\partial F}{\partial p_j}-\frac{\partial E}{\partial p_j}\frac{\partial F}{\partial q_j}=\langle E_q,F_p\rangle-\langle E_p,F_q\rangle\,,$$

which is skew-symmetric, bilinear, satisfies the Jacobi identity and the Leibniz rule: $(EF,H)=E(F,H)+F(E,H)$. A system of functions $\{f_j\}$ is called involutive if $(f_i,f_j)=0$.

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