Abstract. Following Cao's idea, we present commutator representations for two hierarchies of nonlinear isospectral evolution equations associated with two isospectral problems studied by Hu Xingbiao.

Key Words and Phrases. Isospectral Problem; The Pair of Lenard's Operators; Commutator Representation

It is an important topic to search for the commutator representations for nonlinear isospectral evolution equations in soliton theory. In recent years, a lot of results on commutator representations have been successively obtained (see [1-7]). In this paper, following Cao Cewen's idea about commutator representation theory (see [1]), we study two isospectral problems presented by Hu Xingbiao[8] and give commutator representations for the corresponding hierarchies of nonlinear isospectral evolution equations.

The two spectral problems (see [8])

\[ \psi = \begin{pmatrix} \tau & 1 + q\lambda^{-1} \\ q & - r \end{pmatrix} \psi \]

and

\[ \psi = \begin{pmatrix} q & 1 + r\lambda^{-1} \\ \lambda - r & -q \end{pmatrix} \psi \]

can be rewritten in a unified form

\[ \psi = U\gamma \equiv \begin{pmatrix} \tau & 1 + q\lambda^{-1} \\ \lambda + eq & -r \end{pmatrix} \psi, \quad \epsilon = \pm 1, \quad (1) \]

where \( \gamma^{\epsilon} = (\gamma_1, \gamma_2)^T \), \( \lambda \) is an eigenparameter, and the vector-valued function \( u(x) = (q(x), r(x))^T \) is called the potential of (1). The underlying interval \( \Omega \) is \( (-\infty, +\infty) \) or \( (0, T) \) under the decaying condition at infinity or periodic condition respectively. Let \( \epsilon = u + \epsilon \mu \).
Proposition 1 Let $\lambda$ be an eigenvalue of (1), and $(\psi_1, \psi_2)^T$ be the corresponding eigenfunction:

$$
\begin{align*}
\psi_1 &= r\phi + (1 + q\lambda^{-1})\phi_1, \\
\psi_2 &= (\lambda + eq)\phi - r\phi_1.
\end{align*}
$$

(2)

Then the functional gradient $\nabla \lambda$ of the eigenvalue $\lambda$ with regard to the potential $u$ is

$$
\nabla \lambda = \left( \begin{array}{c}
\frac{\delta \lambda}{\delta u} \\
\frac{\delta \lambda}{\delta \phi}
\end{array} \right) = \left( -e\phi_1 + \lambda^{-1}\phi \right) \cdot \left( \int_\Omega (\psi_1 + q\lambda^{-2}\phi) dx \right)^{-1}.
$$

(3)

Proof In Section II of [9], we choose $m_1 = r$, $m_2 = 1 + q\lambda^{-1}$, $m_2 = \lambda + eq$. Then we have

$$
\int_\Omega \left[ (-e\phi_1 + \lambda^{-1}\phi) \phi_1 + 2\phi_1 \phi_2 \phi \right] dx = \delta \lambda \int_\Omega (\psi_1 + q\lambda^{-2}\phi) dx
$$

which implies (3).

Proposition 2 Let $\lambda$ be an eigenvalue of (1). Then for $\varepsilon = 1$ and $\varepsilon = -1$, $\nabla \lambda$ satisfies the linear relations

$$
K \nabla \lambda = \lambda \nabla \lambda
$$

and

$$
K \nabla \lambda = \lambda J \nabla \lambda
$$

(4)

(5)

respectively, where $K$, $J$ and $K$, $J$ are two pairs of skew-symmetric operators having the forms $(\partial = \partial / \partial x$, $\partial^{-1} = \partial^{-1} = \partial = 1)$

$$
K = \begin{bmatrix}
\frac{1}{8} \partial_x q_x \partial_x - \frac{1}{2} \partial_x q_y \partial_x - \frac{1}{2} \partial_x q_y \partial_x - \frac{1}{2} \partial_x q_y \partial_x & 1 \partial_x q_x - \frac{1}{2} \partial_x q_y \partial_x - \frac{1}{2} \partial_x q_y \partial_x \\
\frac{1}{8} \partial_x q_x \partial_x - \frac{1}{2} \partial_x q_y \partial_x - \frac{1}{2} \partial_x q_y \partial_x - \frac{1}{2} \partial_x q_y \partial_x & 1 \partial_x q_x - \frac{1}{2} \partial_x q_y \partial_x - \frac{1}{2} \partial_x q_y \partial_x
\end{bmatrix},
$$

(6)

$$
J = \begin{bmatrix}
0 & \frac{1}{2} \partial_x q_x \\
\frac{1}{2} \partial_x q_x & \frac{1}{2} \partial_x q_x
\end{bmatrix},
$$

(7)

which are called the pair of Lenard's operators of (1) corresponding to $\varepsilon = 1$ and $\varepsilon = -1$, respectively.

Proof For $\varepsilon = 1$,

$$
J^{-1} K = \begin{bmatrix}
- \partial^{-1} \phi \partial & - \partial^{-1} \phi \\
\frac{1}{4} \partial_x \phi_x - r \partial^{-1} \phi & \frac{1}{4} \partial_x \phi_x - r \partial^{-1} \phi - q
\end{bmatrix}.
$$
Thus, in order to obtain (4) it suffices to prove
\[ J^{-1} K \triangledown \lambda = \lambda \triangledown \lambda. \]

From (2) we get
\[
(- \psi^{\dagger} + \lambda^{-1} \psi^{\dagger}) = -2r(\psi^{\dagger} + \lambda^{-1} \psi^{\dagger}),
\]
\[
(2\psi_{\lambda}^{\psi}) = 2(1 + q\lambda^{-1})\psi^{\dagger} + 2(\lambda + q)\psi^{\dagger}.
\]
So,
\[
-\partial^r q\partial(- \psi^{\dagger} + \lambda^{-1} \psi^{\dagger}) - \partial^r r\partial(2\psi_{\lambda}^{\psi})
\]
\[
= -\partial^r 2r(\psi^{\dagger} + \lambda \psi^{\dagger}) = \lambda \cdot (- \psi^{\dagger} + \lambda^{-1} \psi^{\dagger}),
\]
\[
\left(\frac{1}{4} \frac{\partial q}{\partial \tau} \partial - r \partial^{-1} \partial \partial \right) (- \psi^{\dagger} + \lambda^{-1} \psi^{\dagger}) + \left(\frac{1}{4} \partial \partial - r \partial^{-1} \partial - q \right) (2\psi_{\lambda}^{\psi})
\]
\[
= \lambda \cdot (2\psi_{\lambda}^{\psi}).
\]

which yield (8).

For \( \varepsilon = -1 \),
\[
J^{-1} K = \left[ \begin{array}{cc}
\frac{1}{2} & \frac{1}{2} - \frac{q}{r} \\
\frac{1}{4} \frac{\partial q}{\partial \tau} \partial - r \partial^{-1} \partial \partial & \frac{1}{4} \partial \partial - r \partial^{-1} \partial - \frac{1}{2} \frac{q}{r} \end{array} \right].
\]

Similarly, we can prove
\[ J^{-1} K \triangledown \lambda = \lambda \triangledown \lambda. \] (9)

(8), (9) imply (4), (5), respectively.

Proposition 3 The spectral problem (1) is equivalent to
\[ L \phi = \lambda \phi, \quad L = L(u, \varepsilon) = \left( \begin{array}{cc}
-q & \tau + \partial \\
(\varepsilon q - q + q) & -q - r^2 + r \tau + \partial
\end{array} \right). \] (10)

Proof Obvious.

Definition 1 Let \( L : u \rightarrow L(u, \varepsilon) \) be the mapping from a potential function into a differential operator. The Gateaux derivative of the mapping \( L \) in the direction \( \xi \) is defined by
\[ L(\xi) = \frac{d}{d\eta}_{\xi \rightarrow 0} L(u + \eta \xi). \] (11)

Lemma 1 For the spectral problem (10), the Gateaux derivative of \( L \) is
\[ L(\xi) = \left( \begin{array}{cc}
-q & \xi \\
(\varepsilon q - q + q) & -q - r^2 + \tau \xi + \partial
\end{array} \right), \]
\[ u = \left( \begin{array}{c}
\xi \\
\tau
\end{array} \right), \quad \xi = \left( \begin{array}{c}
\xi_1 \\
\xi_2
\end{array} \right), \quad \varepsilon = \pm 1, \] (12)
and \( L(\xi) \) (simply written as \( L(\xi) \) below) is an injective homomorphism.

Proof Directly calculate.

Consider the commutator \([V, L]\) of the two operators
\[ V = V_1 + V_2 \partial, \quad L = L(u, \varepsilon) = L_1 + L_2 \partial + L_3 \partial^2, \]
where
where $Z_1 = \varepsilon (-q + qr)H - ((q - q + q - r^2)E + \varepsilon (-q_m + q_e + qr)D) - E_m$, 
$Z_2 = \varepsilon E - \varepsilon (-q_e + qr)F + \varepsilon F_e + (q_m - q_e - 2\varepsilon \lambda)D - H_m$.

In the following we shall separately discuss (15) for $\varepsilon = 1$ and $\varepsilon = -1$.

1. $\varepsilon = 1$.

We hope

$$[V, L] = L \cdot (KG) - L \cdot (JG)L,$$  (16)

i.e.,

$$[V, L] = \begin{pmatrix} - (KG)^{(1)} & (KG)^{(2)} \\ - (JG)^{(1)} + r(KG)^{(1)} + q(KG)^{(3)} & (KG)^{(2)} - 2\tau (KG)^{(2)} - (KG)^{(1)} \\ - (JG)^{(1)} & (JG)^{(2)} \\ 0 & 0 \\ \end{pmatrix} L_1$$

$$+ \begin{pmatrix} 0 & 0 \\ (JG)^{(1)} & 0 \\ - (JG)^{(1)} & (JG)^{(2)} \\ 0 & 0 \\ \end{pmatrix} L_0 + \begin{pmatrix} 0 & 0 \\ (JG)^{(1)} & 0 \\ - (JG)^{(1)} & (JG)^{(2)} \\ 0 & 0 \\ \end{pmatrix} L_2$$

$$+ \begin{pmatrix} 0 & 0 \\ (JG)^{(1)} & 0 \\ - (JG)^{(1)} & (JG)^{(2)} \\ 0 & 0 \\ \end{pmatrix} L_3$$

$$+ \begin{pmatrix} 0 & 0 \\ (JG)^{(1)} & 0 \\ - (JG)^{(1)} & (JG)^{(2)} \\ 0 & 0 \\ \end{pmatrix} \mathcal{F}.$$  (17)

where $K$, $J$ and $L = L(u, 1)$ are defined by (6) and (10) respectively, $G(z) = (G^{(1)}(z), G^{(2)}(z))^T$, $G^{(1)}(z)$ and $G^{(2)}(z)$ are two arbitrary smooth functions on $\Omega$, and $(\cdot)^{(0)} (i=1, 2)$
stands for the i-th component of (·).

In order to get (16), in (15) we should choose
\[
A = A(G) = -\frac{1}{2} G^{-1}(qG^{(1)} + rG^{(2)}) - \frac{1}{2} \frac{G^{(2)}}{r} + \frac{1}{8} \left( \frac{2}{r} G^{(1)} + G^{(2)} \right),
\]
\[
D = D(G) = -\frac{1}{2} G^{-1}(qG^{(1)} + rG^{(2)}) - \frac{1}{2} \frac{G^{(2)}}{r} + \frac{1}{4} \left( \frac{2}{r} G^{(1)} + G^{(2)} \right),
\]
\[
E = E(G) = -\frac{1}{4} \frac{q}{r} (qG^{(1)} + rG^{(2)}) + \frac{1}{8} \left( \frac{2}{r} G^{(1)} + G^{(2)} \right),
\]
\[
F = F(G) = -\frac{1}{4} \frac{q}{r} (qG^{(1)} + rG^{(2)}) - \frac{1}{8} \left( \frac{2}{r} G^{(1)} + G^{(2)} \right),
\]
\[
H = H(G) = \frac{1}{4} (qG^{(1)} + rG^{(2)}) - \frac{1}{8} \left( \frac{2}{r} G^{(1)} + G^{(2)} \right),
\]
(18)

Hence, we have

Theorem 1 Let \(G^{(1)}(x)\) and \(G^{(2)}(x)\) be two given smooth functions on \(Ω\), and \(G \in C^1(Ω, G_2)\). Then the operator equation determined by the pair of Lenard's operators \(K, J\) and the spectral operator \(L = L(u, 1)\),
\[
[V, L] = L \cdot (KG) - L \cdot (JG)L
\]
possesses the operator solution
\[
V = V(G) = \begin{pmatrix} 0 & F(G) \\ E(G) & H(G) \end{pmatrix} + \begin{pmatrix} A(G) & 0 \\ 0 & D(G) \end{pmatrix} \partial_x,
\]
(20)

where \(A(G), D(G), E(G), F(G), H(G)\) are defined by (18).

Proof Substituting the expressions (18) of \(A(G), D(G), E(G), F(G), H(G)\) into the right-hand side of (15) and noticing \(\varepsilon = 1\), through a lengthy calculations we find that the calculated result is equal to the right-hand side of (17). So, Theorem 1 holds.

Now, for \(\varepsilon = 1\), we recursively define the Lenard's gradient sequence \(G_j\) of (1) as follows:
\[
G_{-1} = (0, 0)^T, \quad G_0 = (2, 2r)^T,
\]
\[
JG_{j+1} = KG_j, \quad j = -1, 0, 1, \ldots
\]
(21)

\(X_m = JG_m (m = 0, 1, 2, \ldots)\) are called the vector fields of the spectral problem (1) with \(\varepsilon = 1\), the first few results of calculations being
\[
X_0 = (q, \tau r)^T, \quad G_0 = (2, 2r)^T,
\]
\[
X_1 = \left( \left( \frac{1}{4} \frac{q}{r} q - \frac{1}{2} q^2 - q \right), -q r + \left( \frac{1}{4} \frac{2}{r} q - \frac{1}{2} r^2 - q \right) \right)^T,
\]
\[
X_m = \left( -\tau r, \frac{1}{2} \tau r - \tau^2 - 2q \right)^T.
\]
The hierarchy of evolution equations associated with (1) for \(\varepsilon = 1\) are produced by the vector field \(X_m\), i.e.,
\[
u_m = (q, \tau)^T = X_m(q, \tau), \quad m = 0, 1, 2, \ldots
\]
(22)
with the representative equation
\[
(q, \tau)^T = X_1(q, \tau)
\]
which can be reduced to the well-known MKdV equation

\[ r = \frac{1}{4} r_{xx} - \frac{3}{2} r_x r. \tag{24} \]

as \( q = 0 \).

1. \( \varepsilon = -1 \).

We hope

\[ [V, L] = L_\varepsilon (K_\varepsilon G) - L_\varepsilon (J_\varepsilon G), \tag{25} \]

where \( K, J \) and \( L = L(\alpha, -1) \) are defined by (7) and (10) respectively, \( G(x) = (G^{(1)}(x), \ G^{(2)}(x))^T, \ G^{(1)}(x) \) and \( G^{(2)}(x) \) are two arbitrary smooth functions on \( \Omega \).

In order to solve \( V \) from (25) by using the approach used in case 1, in (15) we should make choice of

\[
A = \hat{\mathcal{A}}(G) = -\frac{1}{2} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)}) - \frac{1}{4} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)})_x = \frac{1}{4} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)})_x,
\]

\[
D = \hat{\mathcal{D}}(G) = -\frac{1}{2} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)}) + \frac{1}{4} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)})_x,
\]

\[
E = \hat{\mathcal{E}}(G) = \frac{1}{4} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)}) - \frac{1}{8} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)})_x + \frac{1}{4} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)})_x,
\]

\[
F = \hat{\mathcal{F}}(G) = -\frac{1}{4} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)}) - \frac{1}{8} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)})_x + \frac{1}{4} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)})_x,
\]

\[
H = \hat{\mathcal{H}}(G) = \frac{1}{4} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)}) - \frac{1}{8} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)})_x + \frac{1}{4} \hat{\theta}^{-1}(qG^{(1)} + rG^{(2)})_x.
\]

So, we get

**Theorem 2** Let \( G^{(1)}(x) \) and \( G^{(2)}(x) \) be two given smooth functions on \( \Omega \), and \( G_0, G \), \( G^{(1)}, G^{(2)} \). Then the operator equation determined by the pair of Lenard's operators \( K, J \) and the spectral operator \( L = L(\alpha, -1) \),

\[ [V, L] = L_\varepsilon (K_\varepsilon G) - L_\varepsilon (J_\varepsilon G) \tag{27} \]

has the operator solution

\[ V = \hat{\mathcal{V}}(G) = \left( \begin{array}{cc} 0 & \hat{F}(G) \\ \hat{E}(G) & H(G) \end{array} \right) \left( \begin{array}{c} \hat{\mathcal{A}}(G) \\ \hat{\mathcal{D}}(G) \end{array} \right), \tag{28} \]

where \( \hat{\mathcal{A}}(G), \hat{\mathcal{D}}(G), \hat{\mathcal{E}}(G), \hat{\mathcal{F}}(G), H(G) \) are defined by (26).

**Proof** Substituting (26) into (15) and noticing \( \varepsilon = -1 \), by directly calculating (15) and decomposing (25) into the form like (17), we see that the operator equation (27) has the operator solution (28).

For \( \varepsilon = -1 \), the Lenard's recursive gradient sequence \( \hat{G}_j \) of (1) are defined by

\[
\begin{align*}
\hat{G}_{-1} &= (0, 0)^T, & \hat{G}_0 &= (0, 2x)^T, \\
J\hat{G}_{j+1} &= K\hat{G}_j, & j &= -1, 0, 1, \ldots
\end{align*}
\]

\( J\hat{G}_m = J\hat{G}_m (m = 0, 1, 2, \ldots) \) are called the vector fields of the spectral problem (1) with \( \varepsilon = -1 \), the first few results being...
The hierarchy of evolution equations associated with (1) for $\varepsilon = -1$ are given by the vector fields $\vec{X}_m$, i.e.,

$$u_m = (q, r)_t = \vec{X}_m(q, r), \quad m = 0, 1, 2, \ldots$$  \hspace{1cm} (30)

with the representative equation

$$(q, r)_t = \vec{X}_1(q, r)$$

which can also be reduced to the remarkable MiKdV equation

$$r_t = \frac{1}{4} r_{mm} - \frac{3}{2} r r_m$$

as $q = 0$.

Combining I $(\varepsilon = 1)$ with II $(\varepsilon = -1)$, we have two theorems below, which describe the close connection between the commutator representations for the hierarchies of evolution equations (22), (30) and the operator solutions of the operator equation (19), (27).

**Theorem 3**  Let $G_1 = (G_1^{(1)}, G_1^{(2)})^T$ and $\hat{G}_1 = (\hat{G}_1^{(1)}, \hat{G}_1^{(2)})^T$ be the Lenard's recursive gradient sequences of (1) for $\varepsilon = 1$ and $\varepsilon = -1$, respectively. Let $V = V(G_1)$ and $\hat{V} = \hat{V}(\hat{G}_1)$ be separately determined by (20) with $G = G_1$ and (28) with $G = \hat{G}_1$. Then

$$[W_m, L] = L_m(X_m), \quad m = 0, 1, 2, \ldots$$  \hspace{1cm} (32)

$$[W_m, L] = L_m(\hat{X}_m), \quad m = 0, 1, 2, \ldots$$  \hspace{1cm} (33)

where $W_m = \sum_{j=0}^{m} V_{j-1} L^{m-j}$, $\hat{W}_m = \sum_{j=0}^{m} \hat{V}_{j-1} \hat{L}^{m-j}$, $L = L(u, 1)$ in (32), $L = L(u, -1)$ in (33).

**Proof**  From Theorem 1 and (21), we have

$$[W_m, L] = \sum_{j=0}^{m} [V_{j-1}, L] L^{m-j}$$

$$= \sum_{j=0}^{m} (L, (KG_{j-1}) - L, (JG_{j-1}) L^{m-j})$$

$$= \sum_{j=0}^{m} (L, (JG_{j}) L^{m-j} - L, (JG_{j-1}) L^{m-j+1})$$

$$= L_m(X_m)$$

Similarly, from Theorem 2 and (29) we can obtain (33).

**Theorem 4**  The two hierarchies of evolution equations (22) and (30) possess the commutator representations

$$L_m = [W_m, L], \quad L = L(u, 1), \quad m = 0, 1, 2, \ldots$$  \hspace{1cm} (34)
and
\[ L_s = [W_s, L], \quad L = L(u, -1), \quad m = 0, 1, 2, \ldots \]  \hspace{1cm} (35)
respectively.

Proof
\[ L_s = \begin{pmatrix} -\epsilon q_i & \eta_i \\ \eta_i & \epsilon q_i - 2\eta_i \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\epsilon q_i \end{pmatrix} \mathbf{a} = L_s(u). \]
For \( \varepsilon = 1 \),
\[ L_s - [W_s, L] = L_s(u) - L_s(X_s) = L_s(u - X_s). \]
For \( \varepsilon = -1 \),
\[ L_s - [W_s, L] = L_s(u) - L_s(X_s) = L_s(u - X_s). \]
In addition, noting that \( L_s \) is injective, we obtain \( u - X_s = 0, \ u - X_s = 0 \) if and only if \( L_s = [W_s, L], \ L_s = [W_s, L] \), respectively. Those are the desired results.

Corollary 1 (22) and (30) are the natural compatible conditions of \( L(u, 1)\psi = \lambda \phi, \ \phi = W_s \psi \) and \( L(u, -1)\psi = \lambda \phi, \ \phi = W_s \psi \), respectively.

From (32) and (33), we get the results immediately.

Corollary 2 The potential vector \( u = (q, r)^T \) is a finite gap, namely, it satisfies some stationary nonlinear evolution equation
\[ \sum_{i=0}^{N} a_i x_{N-i} = 0 \quad \text{or} \quad \sum_{i=0}^{N} \beta_i x_{N-i} = 0 \ \quad (N \geq 0), \]  \hspace{1cm} (36)
if and only if
\[ [\sum_{i=0}^{N} a_i W_{N-i}, \quad L] = 0, \quad L = L(u, 1) \]
\[ [\sum_{i=0}^{N} \beta_i W_{N-i}, \quad L] = 0, \quad L = L(u, 1) \ \quad (N \geq 0), \]  \hspace{1cm} (37)
where \( a_i, \ \beta_i (0 \leq i \leq N) \) are some constants.

As a special case of Theorem 4, we obtain the commutator representations for the MkdV hierarchy if letting \( q = 0 \).

Corollary 3 The MkdV hierarchy of equations
\[ r_s = J L_s r, \quad m = 0, 1, 2, \ldots \]  \hspace{1cm} (38)
have the commutator representations
\[ L_s = [W_s, L], \quad m = 0, 1, 2, \ldots \]  \hspace{1cm} (39)
with
\[ L = \begin{pmatrix} 0 & \mathbf{r} + \mathbf{a} \\ 0 & -\mathbf{r}^2 + \mathbf{r} + \mathbf{a} \end{pmatrix}, \]  \hspace{1cm} (40)
\[ W_s = \sum_{i=0}^{N} \begin{pmatrix} 0 & -\frac{1}{4} G_{s-1, i} - \frac{1}{8} \mathbf{G}_{s-1, i} \\ 0 & \frac{1}{4} \mathbf{G}_{s-1, i} - \frac{1}{8} \mathbf{G}_{s-1, i} \end{pmatrix} \]
where $\mathbf{J} = \mathbf{J}_0, \mathbf{J}_1, \mathbf{J}_2, \ldots$ is recursively determined by the following relations: $G_0 = \mathbf{J}_0 = 0, G_{-1} = 1, G_{-1} = 0$. 

Remark On the nonlinearity of the spectral problem (1) and its Lax operator algebra, we have got some results, which are left to a forthcoming paper.

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References