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C. Neumann Constraint and Involutive Solutions of Tu Hierarchy *)

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Abstract The commutator representations of Tu hierarchy of equations are presented first and then through nonlinearization of Tu's eigenvalue problem, the involutive solutions of Tu hierarchy are given in this paper.

Key Words and Phrases Tu Hierarchy; Commutator Representation; Involutive System; Involutive Solution

Consider Tu's spectral problem (see [1]):

$$Ly \equiv (-\partial^2 + u + \lambda^{-1}v)y = \lambda y, \qquad \partial = \partial/\partial x. \tag{1}$$

The pair of Lenard's operators are

$$K = \begin{pmatrix} 2\partial & 0 \\ 0 & v\partial + \partial v \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 2\partial \\ 2\partial & \frac{1}{2}\partial^3 - (u\partial + \partial u) \end{pmatrix}. \tag{2}$$

The Lenard's recursive gradient sequence G_j of (1) are defined as follows:

$$G_{-2} = (1, 0)^T,\ G_{-1} = (\tfrac{1}{2}u, 1)^T; \qquad G_{-2}, G_{-1} \in \mathrm{Ker}J,$$
$$KG_{j-1} = JG_j, \qquad j = 0, 1, 2, \cdots \tag{3}$$

$X_m \triangleq JG_m$ ($m=0, 1, 2, \cdots$) are called the Tu vector fields, which produce the Tu hierarchy of soliton equations

$$(u, v)^T_t = X_m(u, v), \qquad m = 0, 1, 2, \cdots \tag{4}$$

with the representative equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = X_1(u, v) \equiv \begin{pmatrix} v_x - \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \\ u_xv + \frac{1}{2}uv_x \end{pmatrix}. \tag{5}$$

Obviously, (5) is reduced to be the well-known KdV equation

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$$u_t = -\frac{1}{4}u_{xxx} + \frac{3}{2}u_x \qquad \text{as } v = 0. \tag{6}$$

§ 1. The Commutator Representations of Tu Hierarchy of Equations

Lemma 1.1 The Gateaux derivative mapping of the spectral operator L defined by (1) is

$$L_{*w}(\xi) \triangleq \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(w + \varepsilon\xi) = \xi_1 + \lambda^{-1}\xi_2 \tag{7}$$

and L_{*w} (simply written as L_* below) is an injective homomorphism, where $w = (u, v)^T$, $\xi = (\xi_1, \xi_2)^T$.

Theorem 1.1 Let $G(x)$ be an arbitrary smooth function and $V = V(G) = -\frac{1}{2}G_x + G\partial$. Then

$$[V, L] \triangleq VL - LV = L_*(K\widetilde{G}) - L_*(J\widetilde{G})L, \tag{8}$$

where $\widetilde{G} = (-\frac{1}{4}G_{xx} + \frac{1}{2}(\partial^{-1}uG_x + uG), G)^T$; K, J and L are determined by (2) and (1), respectively.

Proof

$$[V, L] = -\frac{1}{2}G_{xxx} + (u_x + \lambda^{-1}v_x)G + 2(u + \lambda^{-1}v)G_x - 2G_xL. \tag{9}$$

Substituting (2) and the expression of $\widetilde{G}$ into the right-hand side of (8), by using (7) and carefully calculating $L_*(K\widetilde{G}) - L_*(J\widetilde{G})L$, we can get that $L_*(K\widetilde{G}) - L_*(J\widetilde{G})L$ is equal to the right-hand side of (9) without difficulty.

Theorem 1.2 Let $G_j = (G_j^{(1)}, G_j^{(2)})^T$ be the Lenard's recursive sequence of (1), and $V_j = V(G_j^{(2)}) = -\frac{1}{2}G_{jx}^{(2)} + G_j^{(2)}\partial$, $j = -1, 0, 1, \cdots$. Then

$$[W_m, L] = L_*(X_m), \tag{10}$$

where the operator $W_m = \sum_{j=0}^{m} V_{j-1}L^{m-j}$.

Proof By using Theorem 1.1, directly calculate.

Corollary 1.1 The Tu hierarchy of equations $(u, v)_t^T = X_m(u, v)$, $m = 0, 1, 2, \cdots$, possesses the following commutator representations:

$$L_t = [W_m, L], \qquad m = 0, 1, 2, \cdots \tag{11}$$

Proof $L_t = L_*(u_t, v_t)$, $L_t - [W_m, L] = L_*((u_t, v_t)^T - X_m)$, and L_* is injective. Hence $L_t = [W_m, L] \Leftrightarrow (u_t, v_t)^T = X_m$.

§ 2. The C. Neumann Constraint and Involutive System[2]

Let $\lambda_1, \cdots, \lambda_N$ be N different eigenvalues of the spectral problem (1). Then it is easy to calculate the functional gradients $\nabla_{(u,v)}\lambda_j$ of eigenvalue λ_j with respect to u, v:

$$\nabla_{(u,v)}\lambda_j = \begin{pmatrix} \delta\lambda_j/\delta u \\ \delta\lambda_j/\delta v \end{pmatrix} = \begin{pmatrix} \varphi_j^2(x) \\ \lambda_j^{-1}\varphi_j^2(x) \end{pmatrix}, \qquad j = 1, 2, \cdots, N, \tag{12}$$

which satisfy the linear equations

$$K\nabla_{(u,v)}\lambda_j = \lambda_j \cdot J\nabla_{(u,v)}\lambda_j, \tag{13}$$

where K, J are defined by (2).

The C. Neumann constraint $G_{-1} = \sum_{j=1}^{N}\nabla_{(u,v)}\lambda_j$ yields

$$\langle\Lambda^{-1}\varphi, \varphi\rangle = 1, \; u = 2\langle\varphi, \varphi\rangle, \; v = -\frac{\langle\varphi, \varphi\rangle + \langle\Lambda^{-1}\psi, \psi\rangle}{\langle\Lambda^{-2}\varphi, \varphi\rangle} \qquad (\psi = \varphi_x). \tag{14}$$

In [2], it has been shown that under the constraint condition (14), the Tu's sepctral problem (1) is nonlinearized as a completely integrable Hamiltonian system in Liouville's sense $(R^{2N}, d\psi \wedge d\varphi, H^* = H - \mu F|_{TQ^{N-1}})$, whose involutive system is $F_m^* = F_m - \mu_m F$, where $TQ^{N-1} = \{(\psi, \varphi) \in R^{2N} \mid F = \frac{1}{2}(\langle\Lambda^{-1}\varphi, \varphi\rangle - 1) = 0, G = \langle\Lambda^{-1}\psi, \varphi\rangle = 0\}$,

$$\mu = \frac{(H, G)}{(F, G)}\Big|_{TQ^{N-1}}, \; \mu_m = \frac{(F_m, G)}{(F, G)}\Big|_{TQ^{N-1}},$$

$$H = F_0 = \frac{1}{2}\langle\psi, \psi\rangle + \frac{1}{2}\langle\Lambda\varphi, \varphi\rangle - \frac{1}{2}\langle\varphi, \varphi\rangle^2, \tag{15}$$

$$F_m = \frac{1}{2}\langle\Lambda^m\psi, \psi\rangle + \frac{1}{2}\langle\Lambda^{m+1}\varphi, \varphi\rangle - \frac{1}{2}\langle\varphi, \varphi\rangle\langle\Lambda^m\varphi, \varphi\rangle + \frac{1}{2}\sum_{i+j=m-1}\begin{vmatrix} \langle\Lambda^i\varphi, \varphi\rangle & \langle\Lambda^i\psi, \varphi\rangle \\ \langle\Lambda^j\psi, \varphi\rangle & \langle\Lambda^j\psi, \psi\rangle \end{vmatrix}. \tag{16}$$

In the above formulae, $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, $\varphi = (\varphi_1, \cdots, \varphi_N)^T$, $\psi = (\varphi_{1,x}, \cdots, \varphi_{N,x})^T$, $\varphi_j(x)$ is the eigenfunction corresponding to λ_j, $\langle\cdot, \cdot\rangle$ stands for the standard inner-product in R^N, and the Poisson bracket (E, F) of two functions E, F in the symplectic space $(R^2N, d\psi \wedge d\varphi)$ is determined by

$$(E, F) = \sum_{j=1}^{N}\left(\frac{\partial E}{\partial\varphi_j}\frac{\partial F}{\partial\psi_j} - \frac{\partial E}{\partial\psi_j}\frac{\partial F}{\partial\varphi_j}\right) = \left\langle\frac{\partial E}{\partial\varphi}, \frac{\partial F}{\partial\psi}\right\rangle - \left\langle\frac{\partial E}{\partial\psi}, \frac{\partial F}{\partial\varphi}\right\rangle. \tag{17}$$

§ 3. Involutive Solutions

Consider the canonical equations of F_m^*-flows

$$(F_m^*): \quad \frac{\partial}{\partial t_m}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \partial F_m^*/\partial\psi \\ -\partial F_m^*/\partial\varphi \end{pmatrix} = I\nabla F_m^*, \; \nabla F_m^* = \begin{pmatrix} \partial F_m^*/\partial\varphi \\ \partial F_m^*/\partial\psi \end{pmatrix}, \; I = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \tag{18}$$

where I_N is the $N \times N$ unit matrix. We denote by $g_m^{t_m}$ the solution operator of its initial value problem. Then its solution can be expressed as

$$\begin{pmatrix} \varphi(t_m) \\ \psi(t_m) \end{pmatrix} = g_m^{t_m}\begin{pmatrix} \varphi(0) \\ \psi(0) \end{pmatrix}. \tag{19}$$

Since any two functions F_k^*, F_l^* are in volution, $(F_k^*, F_l^*) = 0$, we have (see [3])

Proposition 3.1 1) Any two canonical systems (F_k^*) and (F_l^*) are compatible; 2) The

Hamiltonian phase-flows $g^{t_0}_{\mathbf{t}}$ and $g^t_{\mathbf{t}}$ commute.

Denote by $x=t_0$, $t=t_m$ the flow variables of $(H^*)=(F_0^*)$ and (F_m^*), respectively. Define

$$\begin{pmatrix} \varphi(x,t_m) \\ \psi(x,t_m) \end{pmatrix} = g_0^x g_m^{t_m} \begin{pmatrix} \varphi(0,0) \\ \psi(0,0) \end{pmatrix}, \tag{20}$$

which is called the involutive solution of the consistent systems (H^*) and (F_m^*) (see [4]). The commutativity of g_0^x, $g_m^{t_m}$ implies that (20) is a smooth function of (x, t_m).

Theorem 3. 1 Let $(\varphi(x,t_m), \psi(x,t_m))^T$ be an involutive solution of the compatible group (H^*), (F_m^*), and let $\langle \Lambda^{-1}\varphi, \varphi\rangle=1$, $u=2\langle\varphi,\varphi\rangle$, $v=-\dfrac{\langle\varphi,\varphi\rangle+\langle\Lambda^{-1}\psi,\psi\rangle}{\langle\Lambda^{-2}\varphi,\varphi\rangle}$. Then

1) the canonical flow equations (H^*), (F_m^*) can be reduced to the spatial part

$$L(u,v)\varphi \equiv (-\varphi_{xx}+u\varphi+v\Lambda^{-1}\varphi)=\Lambda\varphi \tag{21}$$

and the time part

$$\varphi_{t_m}=(W_m+c_1W_{m-1}+\cdots+c_mW_0)\varphi, \tag{22}$$

respectively, of the Lax representation for the Tu hierarchy of evolution equations (23) below with u, v as their potentials, where c_j are independent of x, and W_k $(k=0,1,2,\cdots,m)$ are defined in Theorem 1. 2.

2) $u(x,t_m)=2\langle\varphi,\varphi\rangle$ and $v(x,t_m)=-\dfrac{\langle\varphi,\varphi\rangle+\langle\Lambda^{-1}\psi,\psi\rangle}{\langle\Lambda^{-2}\varphi,\varphi\rangle}$ satisfy the higher-order Tu equation

$$(u,v)^T_{t_m}=X_m+c_1X_{m-1}+\cdots+c_mX_0, \tag{23}$$

where $X_k=JG_k$ $(k=0,1,2,\cdots,m)$ are the Tu vector fields.

Proof By the expression (16) of F_m, we have

$$\frac{\partial F_m}{\partial\varphi}=\Lambda^{m+1}\varphi-\langle\varphi,\varphi\rangle\Lambda^m\varphi-\langle\Lambda^m\varphi,\varphi\rangle\varphi+\sum_{i+j=m-1}(\langle\Lambda^i\psi,\psi\rangle\Lambda^j\varphi-\langle\Lambda^i\psi,\varphi\rangle\Lambda^j\psi),$$

$$\frac{\partial F_m}{\partial\psi}=\Lambda^m\psi+\sum_{i+j=m-1}(\langle\Lambda^i\varphi,\varphi\rangle\Lambda^j\psi-\langle\Lambda^i\psi,\varphi\rangle\Lambda^j\varphi),$$

$$(F_m,G)|_{TQ^{N-1}}=\left(\frac{\partial F_m}{\partial\varphi},\frac{\partial G}{\partial\psi}\right)-\left(\frac{\partial F_m}{\partial\psi},\frac{\partial G}{\partial\varphi}\right)=-(\langle\varphi,\varphi\rangle+\langle\Lambda^{-1}\psi,\psi\rangle)\langle\Lambda^{m-1}\varphi,\varphi\rangle.$$

Hence

$$\mu_m=\frac{(F_m,G)}{(F,G)}\Big|_{TQ^{N-1}}=-\frac{\langle\varphi,\varphi\rangle+\langle\Lambda^{-1}\psi,\psi\rangle}{\langle\Lambda^{-2}\varphi,\varphi\rangle}\langle\Lambda^{m-1}\varphi,\varphi\rangle=v\langle\Lambda^{m-1}\varphi,\varphi\rangle, \tag{24}$$

$$\mu=\frac{(F_0,G)}{(F,G)}\Big|_{TQ^{N-1}}=-\frac{\langle\varphi,\varphi\rangle+\langle\Lambda^{-1}\psi,\psi\rangle}{\langle\Lambda^{-2}\varphi,\varphi\rangle}=v. \tag{25}$$

Under (14), $\dfrac{\partial H^*}{\partial\varphi}=\dfrac{\partial H}{\partial\varphi}-\mu\dfrac{\partial F}{\partial\varphi}=\Lambda\varphi-u\varphi-v\Lambda^{-1}\varphi$, $\dfrac{\partial H^*}{\partial\psi}=\psi=\varphi_x$. Thus $(H^*)=(F_0^*)$ is reduced to (21).

On the other hand, the pair of Lenard operators K, J possess the following properties:

$$J^{-1}K:\quad \begin{pmatrix}\langle\Lambda^j\varphi, & \varphi\rangle\\ \langle\Lambda^{j-1}\varphi, & \varphi\rangle\end{pmatrix}\mapsto\begin{pmatrix}\langle\Lambda^{j+1}\varphi, & \varphi\rangle\\ \langle\Lambda^j\varphi, & \varphi\rangle\end{pmatrix}, \tag{26}$$

$$J^{-1}K:\quad G_{j+1}\mapsto G_j+\text{const.}\cdot G_{-1}+\text{const.}\cdot G_{-2}, \tag{27}$$

where $G_j=(G_j^{(1)}, G_j^{(2)})^T$ are the Lenard recursive gradient sequence. Let the operator $(J^{-1}K)^k$ $(k\geqslant 0, k\in Z)$ act upon $G_{-1}=(\langle\varphi, \varphi\rangle, \langle\Lambda^{-1}\varphi, \varphi\rangle)^T$ and use (26), (27). It is easy to find that there are some constants $c_2, \cdots, c_{k+2}$, such that

$$A_k=\begin{pmatrix}A_k^{(1)}\\A_k^{(2)}\end{pmatrix}\triangleq\begin{pmatrix}\langle\Lambda^{k+1}\varphi, & \varphi\rangle\\ \langle\Lambda^k\varphi, & \varphi\rangle\end{pmatrix}=c_k+c_2c_{k-2}+\cdots+c_kG_0+c_{k+1}G_{-1}+c_{k+2}G_{-2}. \quad (28)$$

Specially, $A_k^{(2)}=\langle\Lambda^k\varphi, \varphi\rangle=\sum_{s=0}^{k+2}c_sG_{k-s}^{(2)}\ (c_0=1, c_1=0)$.

On the tangent bundle of ellipsoid TQ^{N-1}, through a series of careful calculations we obtain

$$\begin{aligned}\varphi_{t_m}&=\frac{\partial F_m^*}{\partial\psi}=\frac{\partial F_m}{\partial\psi}-\mu_m\frac{\partial F}{\partial\psi}=\frac{\partial F_m}{\partial\psi}\\&=\Lambda^m\varphi+\sum_{i+j=m-1}(\langle\Lambda^i\varphi, \varphi\rangle\Lambda^j\varphi-\langle\Lambda^i\varphi, \varphi\rangle\Lambda^j\varphi)\\&=\sum_{j=0}^{m}(-\frac{1}{2}A_{j-1,x}^{(2)}+A_{j-1}^{(2)}\partial)\Lambda^{m-j}\varphi\\&=\sum_{j=0}^{m}\sum_{s=0}^{j+1}(-\frac{1}{2}c_sG_{j-1-s,x}^{(2)}+c_sG_{j-1-s}^{(2)}\partial)L^{m-j}\varphi\\&=\sum_{s=0}^{m}c_s\sum_{k=0}^{m-s}(-\frac{1}{2}G_{k-1,x}^{(2)}+G_{k-1}^{(2)}\partial)L^{m-s-k}\varphi\\&=\sum_{s=0}^{m}c_sW_{m-s}\varphi.\end{aligned}$$

This is (22).

Observe that

$$u_{t_m}=4\langle\varphi, \varphi_{t_m}\rangle, \quad (29)$$

$$\begin{aligned}v_{t_m}=&-\frac{2}{\langle\Lambda^{-2}\varphi, \varphi\rangle^2}[(\langle\varphi, \varphi_{t_m}\rangle+\langle\Lambda^{-1}\psi, \psi_{t_m}\rangle)\langle\Lambda^{-2}\varphi, \varphi\rangle\\&-(\langle\varphi, \varphi\rangle+\langle\Lambda^{-1}\psi, \psi\rangle)\langle\Lambda^{-2}\varphi, \varphi_{t_m}\rangle].\end{aligned} \quad (30)$$

Substituting the two equalities

$$\begin{aligned}\varphi_{t_m}&=\Lambda^m\psi+\sum_{i+j=m-1}(\langle\Lambda^i\varphi, \varphi\rangle\Lambda^j\psi-\langle\Lambda^i\psi, \varphi\rangle\Lambda^j\varphi),\\ \psi_{t_m}&=-\frac{\partial F_m^*}{\partial\varphi}=-\frac{\partial F_m}{\partial\varphi}+\mu_m\frac{\partial F}{\partial\varphi}\\&=-\Lambda^{m+1}\varphi+\langle\varphi, \varphi\rangle\Lambda^m\varphi+\langle\Lambda^m\varphi, \varphi\rangle\varphi-\frac{\langle\varphi, \varphi\rangle+\langle\Lambda^{-1}\psi, \psi\rangle}{\langle\Lambda^{-2}\varphi, \varphi\rangle}\langle\Lambda^{m-1}\varphi, \varphi\rangle\Lambda^{-1}\varphi\\&\quad-\sum_{i+j=m-1}(\langle\Lambda^i\psi, \psi\rangle\Lambda^j\varphi-\langle\Lambda^i\psi, \varphi\rangle\Lambda^j\psi)\end{aligned}$$

into (29) and (30), and noticing $\langle\Lambda^{-1}\varphi, \varphi\rangle=1$ and $\langle\Lambda^{-1}\psi, \varphi\rangle=0$, we get

$$u_{t_m}=4\langle\varphi, \Lambda^m\psi\rangle=2\partial\langle\varphi, \Lambda^m\varphi\rangle=2\partial A_m^{(2)}, \quad (31)$$

$$v_{t_m}=\frac{4(\langle\varphi, \varphi\rangle+\langle\Lambda^{-1}\psi, \psi\rangle)(\langle\Lambda^{m-1}\varphi, \varphi\rangle\langle\Lambda^{-2}\varphi, \psi\rangle-\langle\Lambda^{-2}\varphi, \varphi\rangle\langle\Lambda^{m-1}\varphi, \psi\rangle)}{\langle\Lambda^{-2}\varphi, \varphi\rangle^2}. \quad (32)$$

In addition,

$$2\partial A_m^{(1)} = 4\langle \Lambda^{m-1}\varphi, \psi\rangle,$$

$$\psi_x = \varphi_{xx} = u\varphi + v\Lambda^{-1}\varphi - \Lambda\varphi,$$

$$v_x = \frac{4\langle \Lambda^{-2}\varphi, \psi\rangle}{\langle \Lambda^{-2}\varphi, \varphi\rangle^2}\cdot(\langle\varphi, \varphi\rangle + \langle \Lambda^{-1}\psi, \psi\rangle).$$

Hence

$$\frac{1}{2}\partial^3 A_m^{(2)} - (u\partial + \partial u)A_m^{(2)}$$

$$= 2\langle\psi_x, \Lambda^m\psi\rangle - 2u\langle\psi, \Lambda^m\varphi\rangle + v_x\langle\varphi, \Lambda^{m-1}\varphi\rangle + 2v\langle\psi, \Lambda^{m-1}\varphi\rangle - 2\langle\psi, \Lambda^{m-1}\varphi\rangle$$

$$= \frac{4(\langle\varphi, \varphi\rangle + \langle\Lambda^{-1}\psi, \psi\rangle)(\langle\Lambda^{m-1}\varphi, \varphi\rangle\langle\Lambda^{-2}\varphi, \psi\rangle - \langle\Lambda^{-2}\varphi, \varphi\rangle\langle\Lambda^{m-1}\varphi, \psi\rangle)}{\langle\Lambda^{-2}\varphi, \varphi\rangle^2}$$

$$- 4\langle\Lambda^{m+1}\varphi, \psi\rangle$$

$$= v_{t_m} - 2\partial A_m^{(1)}.$$

i.e.,

$$v_{t_m} = 2\partial A_m^{(1)} + [\frac{1}{2}\partial^3 - (u\partial + \partial u)]A_m^{(2)}. \tag{33}$$

Combining (31) with (33), we have

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = \begin{pmatrix} 0 & 2\partial \\ 2\partial & \frac{1}{2}\partial^3 - (u\partial + \partial u) \end{pmatrix}\begin{pmatrix} A_m^{(1)} \\ A_m^{(2)} \end{pmatrix}$$

$$= JA_m = J(\sum_{i=0}^{m+1} c_i G_{m-i})$$

$$= X_m + c_1 X_{m-1} + \cdots + c_m X_0.$$

The proof is complete.

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