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谱问题、发展方程族及 Lax 表示的等价性*

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A 摘要 讨论了四对谱问题、相应的保谱方程族以及它们 Lax 表示的等价性, 并利用这种等价关系可导出一些方程的 Lax 表示.

关键词 谱问题, 发展方程族, Lax 表示, 等价性, 非线性
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从理论或应用的观点来看, 寻求等价的非线性发展方程族或给出这些非线性发展方程族的等价类是十分有趣的事情. 本文讨论了下述四对谱问题^[1-6]

$$\psi_x = U_1 \psi, \quad U_1 = \begin{pmatrix} \beta_1 \lambda + u_1 & q \\ r & \beta_2 \lambda + u_2 \end{pmatrix}, \quad \beta_1 \neq \beta_2, \quad (1)_1$$

$$\varphi_x = \tilde{U}_1 \varphi, \quad \tilde{U}_1 = \begin{pmatrix} -\bar{\lambda} + s & q \\ r & \bar{\lambda} - s \end{pmatrix}, \quad (1)_2$$

$$\psi_x = U_2 \psi, \quad U_2 = \begin{pmatrix} -q & \lambda q \\ -\lambda & \lambda^2 + r \end{pmatrix}, \quad (2)_1$$

$$\varphi_x = \tilde{U}_2 \varphi, \quad \tilde{U}_2 = \begin{pmatrix} -i\bar{\lambda} + \bar{r} & \bar{q} + \bar{r} \\ r & i\bar{\lambda} - \bar{r} \end{pmatrix}, \quad i^2 = -1, \quad (2)_2$$

$$\psi_x = U_3 \psi, \quad U_3 = \begin{pmatrix} a_1 \lambda_1 + q & r \\ a_3 & a_2 \lambda + s \end{pmatrix}, \quad a_1 \neq a_2, a_3 \neq 0, \quad (3)_1$$

$$\varphi_x = \tilde{U}_3 \varphi, \quad \tilde{U}_3 = \begin{pmatrix} -\frac{1}{2}\bar{\lambda} + \frac{1}{2}\bar{q} & -\bar{r} \\ 1 & \frac{1}{2}\bar{\lambda} - \frac{1}{2}\bar{q} \end{pmatrix}, \quad (3)_2$$

$$\psi_x = U_4 \psi, \quad U_4 = \begin{pmatrix} u & \lambda \\ -1 & -u \end{pmatrix}, \quad (4)_1$$

$$\varphi_x = \tilde{U}_4 \varphi, \quad \tilde{U}_4 = \begin{pmatrix} -i\bar{\lambda} & u \\ u & i\bar{\lambda} \end{pmatrix}, \quad i^2 = -1 \quad (4)_2$$

和相应的非线性保谱发展方程族以及 Lax 表示的等价性. 这里 $\lambda, \bar{\lambda}$ 为各自的谱参数, $a_1, a_2, a_3, \beta_1, \beta_2$ 均为常数, 其余均为各自的位势函数.

命题 1 谱问题 (1)₁ 与 (1)₂, (2)₁ 与 (2)₂, (3)₁ 与 (3)₂, (4)₁ 与 (4)₂ 分别等价.

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证明 对这四对谱问题分别做如下变换:

$$\psi = \varphi \cdot \exp\left(\frac{\beta_1 + \beta_2}{2}\lambda x + \sigma^{-1}\frac{u_1 + u_2}{2}\right), \quad \bar{\lambda} = \frac{1}{2}\beta\lambda, \beta = \beta_2 - \beta_1, s = \frac{1}{2}(u_1 - u_2); \quad (5)$$

$$\varphi = \begin{pmatrix} 1 & 0 \\ -1 & \lambda \end{pmatrix} \psi \cdot \exp\left(-\frac{\lambda^2}{2}x - \sigma^{-1}\frac{r - q}{2}\right), \quad \lambda^2 = 2i\bar{\lambda}, q = \bar{q} + \bar{r}, r = \bar{q} - \bar{r}; \quad (6)$$

$$\psi = \begin{pmatrix} \alpha_3^{-1} & 0 \\ 0 & 1 \end{pmatrix} \varphi \cdot \exp\left(\frac{\alpha_1 + \alpha_2}{2}\lambda x + \sigma^{-1}\frac{q + s}{2}\right), \quad \bar{\lambda} = (\alpha_2 - \alpha_1)\lambda, \bar{q} = q - s, \bar{r} = -\alpha_3 r; \quad (7)$$

$$\varphi = \begin{pmatrix} 1 & i\lambda \\ 1 & -i\lambda \end{pmatrix} \psi, \quad \lambda = \bar{\lambda}^2, \quad (8)$$

并注意这些变换的可逆性,即知本命题成立.

由文[7],可从谱问题(1)₂产生 D-AKNS 族发展方程:

$$(q, r, s)_t^T = X_m(q, r, s) \stackrel{\Delta}{=} J(J^{-1}K)^{m+1}G_{-1}, \quad G_{-1} = a(2r, 2q, 4) \in \text{Ker}J, \quad (9)$$

这里 $a = \text{const}, m = 0, 1, 2, \dots; K, J$ 是两个算子,具体形式见文[7].代入 $s = \frac{1}{2}(u_1 - u_2)$,并记

$$X_m(q, r, s) = (X_m^{(1)}(q, r, \frac{1}{2}(u_1 - u_2)), X_m^{(2)}(q, r, \frac{1}{2}(u_1 - u_2)), X_m^{(3)}(q, r, \frac{1}{2}(u_1 - u_2)))^T,$$

则有

$$\begin{aligned} q_t &= X_m^{(1)}(q, r, \frac{1}{2}(u_1 - u_2)), r_t = X_m^{(2)}(q, r, \frac{1}{2}(u_1 - u_2)), \\ (u_1 - u_2)_t &= 2X_m^{(3)}(q, r, \frac{1}{2}(u_1 - u_2)). \end{aligned} \quad (10)$$

设 δ_{2m} 是一个任意的光滑函数,令 $u_{2x} = \delta_{2m,x}$,则 $u_{1t} = 2X_m^{(3)}(q, r, \frac{1}{2}(u_1 - u_2)) + \delta_{2m,x}$,从而(10)变为

$$\begin{aligned} q_t &= X_m^{(1)}(q, r, \frac{1}{2}(u_1 - u_2)), \\ r_t &= X_m^{(2)}(q, r, \frac{1}{2}(u_1 - u_2)), \\ u_{1t} &= 2X_m^{(3)}(q, r, \frac{1}{2}(u_1 - u_2)) + \delta_{2m,x}, u_{2x} = \delta_{2m,x}, \quad m \geq 0, m \in \mathbb{Z}, \end{aligned} \quad (11)$$

这恰是文[1]按 Loop 代数格式导出的非线性发展方程族,即二者是等价的.

(1)₂ 可改写为

$$L\varphi \equiv L(q, r, s)\varphi = \bar{\lambda}\varphi, \quad L = \begin{pmatrix} -\partial + s & q \\ -r & \partial + s \end{pmatrix}, \quad \partial = \partial/\partial x, \quad (12)$$

那么 D-AKNS 族(9)具有 Lax 表示^[7]:

$$L_t = [W_m, L], \quad m = 0, 1, 2, \dots, \quad (13)$$

W_m 的表达式见[7].在方程族(11)中注意 $(\frac{u_1 - u_2}{2})_t = X_m^{(3)}(q, r, \frac{1}{2}(u_1 - u_2))$,且(9)与(11)等价,故(11)具有 Lax 表示:

$$\tilde{L}_t = [\tilde{W}_m, \tilde{L}], \quad m = 0, 1, 2, \dots, \quad (14)$$

这里, $\tilde{L} = \frac{2}{\beta}L(q, r, \frac{1}{2}(u_1 - u_2)), \tilde{W}_m = W_m|_{s=\frac{1}{2}(u_1 - u_2)}$.因而,我们得到

命题 2 二方程族(9)与(11)是等价的,且从(9)的 Lax 表示(13)导出(11)的 Lax 表示. 类似地,还可推得其它三对谱问题(2)₁与(2)₂, (3)₁与(3)₂, (4)₁与(4)₂分别对应的发展方程族及 Lax 表示的等价性,这里,因篇幅所限,就略去证明了.

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Equivalence of Some Spectral Problems, Corresponding Evolution Equations Hierarchies and Lax Representations

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Abstract

We discuss the equivalence of four pairs of spectral problems, their corresponding evolution equations hierarchies and Lax representations. Thereby the Lax representations of some evolution equations are deduced.

Keywords spectral problems, evolution equation hierarchy, Lax representations.