9.3 Onto Functions

Preliminary Example

$A = \{1,2,3\}$ $B = \{10,20,30\}$ $f: A \to B$

$f_1 = \{(1,10),(2,10),(3,20)\}$ $\operatorname{ran} f_1 = \{10,20\}$

$30 \cancel{10} \in B$, but 10 is not the image of any $a \in A$.

$f_2 = \{(1,10),(2,30),(3,20)\}$ $\operatorname{ran} f_2 = \{10,20,30\} = B$

every $b \in B$ is the image of some $a \in A$.

Definition A function $f: A \to B$ is said to be *onto* or *surjective* if the following is true:

For every $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

Note To show $f: A \to B$ is onto, we have to show that for every $b \in B$, there is some $a \in A$ such that $f(a) = b$.

EX Let $f: \overset{\text{Domain}}{\mathbb{R}} \to \underset{B}{\mathbb{R}}$ defined $f(x) = 7x + 13$.
Prove f is onto.

Aside: $b \in \mathbb{R}$. Find a so that

$$f(a) = b$$

$$7a + 13 = b$$

$$7a = b - 13$$

$$a = \frac{b-13}{7}$$

Is $\frac{b-13}{7}$ in the domain of f? $\frac{b-13}{7} \in \mathbb{R}$.

Proof. Let $b \in \mathbb{R}$. Consider $a = \frac{b-13}{7}$.

Since $b \in \mathbb{R}$, ~~so is~~ a is also a real number,

so $a \in \text{dom}(f)$. Now

$$
\begin{aligned}
f(a) &= 7a + 13 \\
&= 7\left(\frac{b-13}{7}\right) + 13 \\
&= b - 13 + 13 \\
&= b.
\end{aligned}
$$

Since there exists an $a \in \text{dom}(f)$ such that $f(a) = b$,

the function f is onto. ∎

Note A function $f: A \to B$ is not onto if

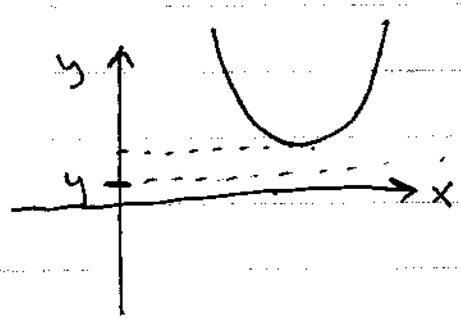
There exists some $b \in B$ such that

for every $a \in A$, $f(a) \neq b$.

EX Let $f: \mathbb{R} \to \mathbb{R}$ defined $f(x) = x^2 - 6x - 2$.

Show f is not onto.

Aside: $f(x) = x^2 - 6x - 2$



$$
\begin{aligned}
f(x) &= (x^2 - 6x) - 2 \\
&= (x^2 - 6x + 9) - 2 - 9 \\
&= (x-3)^2 - 11 \geq -11
\end{aligned}
$$

So it looks like -12 is not in the range of f.

Proof. Consider $b = -12$. We will show that -12 is not the image of any $a \in \text{dom}(f)$.

Let a be any element of $\text{dom}(f) = \mathbb{R}$.

$$\begin{aligned} f(a) &= a^2 - 6a - 2 \\ &= (a^2 - 6a + 9) - 2 - 9 \\ &= (a-3)^2 - 11. \end{aligned}$$

Since $(a-3)^2 \geq 0$,

$$f(a) = (a-3)^2 - 11 \geq 0 - 11 = -11.$$

Since $f(a) \geq -11$, we know $f(a) \neq -12$.

So -12 is ~~a~~ not the image of any a.

Therefore f is not onto. ■

<u>Note</u> $f : \mathbb{R} \to [-11, \infty)$ by $f(x) = x^2 - 6x - 2$

~~is~~ can be shown to be <u>onto</u>.

9.4 $\boxed{\text{Bijective Functions}}$

A function that is injective (one-to-one) and surjective (onto) is called <u>bijective</u>.

<u>EX</u> Show the function $f: \mathbb{R}^+ \to \mathbb{R}$ defined $f(x) = \ln(x)$ is bijective.

Proof. First it will be shown that f is injective (or one-to-one). Suppose $f(a) = f(b)$ for $a, b \in \mathbb{R}^+$. Then

$$\ln(a) = \ln(b)$$

$$e^{\ln(a)} = e^{\ln(b)}$$

$$a = b.$$

Therefore, f is injective.

Aside:
$f(a)=b$
$\ln(a)=b$
$a=e^b$

Next it will be shown that f is surjective (onto).

Let $b \in \mathbb{R}$. Consider $a = e^b$. Since $b \in \mathbb{R}$, $e^b > 0$,

so $a = e^b \in \mathbb{R}^+$. Then

$$f(a) = \ln(a) = \ln(e^b) = b.$$

So b is the image of some $a \in \mathbb{R}^+$. Therefore

f is surjective.

Therefore, f is bijective. $\blacksquare$

$\underline{EX}$ Show that $f: \mathbb{R} \to [-1,1]$ defined by

$f(x) = \sin(x)$ is <u>not</u> bijective.

~~Ads'~~

Aside: Show f is not injective or not surjective
one-to-one onto

$$f(0) = \sin(0) = 0$$
$$f(\pi) = \sin(\pi) = 0$$

Proof. Consider $x=0$ and $y=\pi$. Note $x \neq y$ but $f(0)=0$ and $f(\pi)=0$, so $f(0)=f(\pi)$. This shows f is not injective, and so f is not bijective. ∎

<u>EX</u> Suppose $A = [0,1]$. Is there a bijective function $f: A \to A$?

Simplest case - the identity function i_A defined $i_A(x) = x$.

One-to-One Correspondence

If $f: A \to B$ is bijective, then there is an exact matching of each element of A with each element of B.

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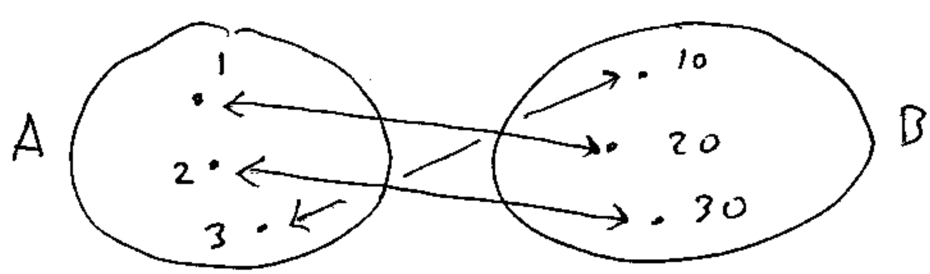
<u>EX</u> (a) $A = \{1, 2, 3\}$ $B = \{10, 20, 30\}$

$$f = \{(1,20), (2,30), (3,10)\}$$

$f: A \to B$ is bijective.



<u>EX</u> $A = \{1, 2, 3, 4, \dots\}$

$B = \{2, 4, 6, 8, \dots\}$

$f = \{(1,2), (2,4), (3,6), (4,8), \dots\}$

$f: A \to B$ is one-to-one and onto, so it is bijective. There is a matching of the elements of A & B.

Sets A, B are said to have the same cardinality.