9.1 $\boxed{\text{Definition of Function}}$

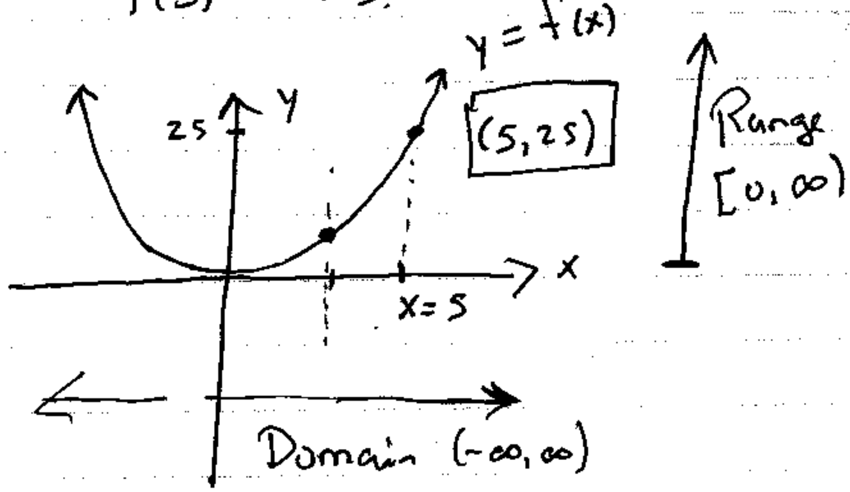
From Calculus or Courses Before Calculus

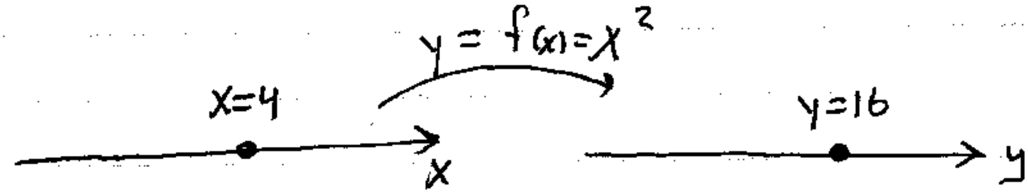
$x = 5 \rightarrow \boxed{y = x^2} \rightarrow 25$

" Consider the function $f(x) = x^2$. "

- Interpreted to mean some rule, put a value of x in, out pops the value x^2

$$f(5) = 25.$$

- 

- 

$\mathbb{R}$ $\quad (x, x^2) \quad$ $\mathbb{R}$

A

Careful Definition of Function

Let A and B be nonempty set. Suppose f is a relation from A to B, that has the following property:

For every $x \in A$, there exist a unique $y \in B$ such that $x f y$.

Then f is called a function from A to B.

EX $A = \{1, 2, 3\}$ $B = \{10, 20, 30\}$

(a) $R_1 = \{(1,10), (3,20)\}$

$x = 2 \in A$ but there is no $y \in B$ such that $2 R y$.

R_1 is not a function from A to B.

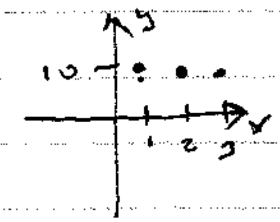
(b) $R_2 = \{(1,10), (2,20), (2,30), (3,30)\}$

$x = 2$ is related to two values $20, 30 \in B$.

Not a function from A to B.

(c) $R_3 = \{(1,10), (2,20), (3,30)\}$

R_3 is a function.



(d) $R_4 = \{(1,10), (2,10), (3,10)\}$

R_4 <u>is</u> a function from A to B.

<u>EX</u> Find all functions from $A = \{1,2\}$ to $B = \{10,20\}$

$f_1 = \{(1,10), (2,10)\}$

$f_2 = \{(1,10), (2,20)\}$

$f_3 = \{(1,20), (2,10)\}$

$f_4 = \{(1,20), (2,20)\}$

The set of all functions from A to B is

denoted by B^A.

$$B^A = \{f_1, f_2, f_3, f_4\}$$

Ex $A = \mathbb{R}, \; B = \mathbb{R}$

$$R = \{ (x, x^2) : x \in \mathbb{R} \}$$

OR $\quad = \{ (x, y) : x \in \mathbb{R}, \; y = x^2 \}$

$(2,4) \in R \qquad (2,3) \notin R$

For every $x \in A = \mathbb{R}$, the only value $y \in B = \mathbb{R}$ that x is related to is $y = x^2$.

So this is a function.

Terminology

Suppose f is a function from A to B.

- $f : A \to B$
- $\operatorname{dom} f = \{ x \in A : x \text{ is related to some } y \in B \} = A$
- $\operatorname{ran} f = \{ y \in B : \text{there is an } x \in A \text{ related to } y \}$

Ex $A = \{1, 2, 3\} \quad B = \{10, 20, 30\}$

$f = \{ (1,10), (2,10), (3,10) \}$ $\operatorname{dom} f = \{1,2,3\}$, $\operatorname{ran} f = \{10\}$

- Suppose $x \in A$. We know there is a unique (one and only one) $y \in B$ such that x is related to y. Denote this y value $f(x)$.

$$\boxed{\begin{array}{l} f(x) \text{ is } \underline{\text{NOT}} \text{ a function.} \\ f(x) \text{ is the element related to } x. \end{array}}$$

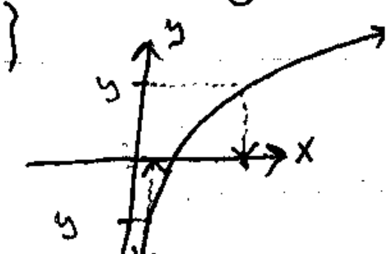
- $f(x)$ is called the <u>image</u> of x.

<u>EX</u> "Consider the function $f(x) = \ln(x)$."

$$f : A \to B$$

- Domain $A = \mathbb{R}^+ = [0, \infty)$; $B = \mathbb{R}$
- Range $\{ y \in \mathbb{R} :$ there is an $x \in \mathbb{R}$ such that $\ln(x) = y \}$

$\mathbb{R}$



Note: "Consider the function $f(x) = \ln(x)$"

Instead, write

"Let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \ln(x)$."

9.3 $\boxed{\text{One-to-One Functions}}$

EX $A = \{1, 2\}$ $B = \{10, 20\}$

$f_1 = \{(1, 20), (2, 10)\}$ $f_2 = \{(1, 20), (2, 20)\}$

There are two different $x \in A$ related to the same $y \in B$.

Definition A function $f: A \to B$ is said to be ~~one~~ one-to-one or injective if the following is true:

Let ~~x,y∈~~ $a, b \in A$. If $a \neq b$, then $f(a) \neq f(b)$.

<u>Note</u> The usual way to show or prove a function f is one-to-one is to the contrapositive is true:

$$\text{Let } a, b \in A. \text{ If } f(a) = f(b), \text{ then } a = b.$$

<u>EX</u> Let $f: \mathbb{R} \to \mathbb{R}$ defined $f(x) = 7x + 13$.

Prove that f is one-to-one.

Proof. Let $a, b \in A$ and suppose $f(a) = f(b)$.

We will show that $a = b$. Since $f(a) = f(b)$,

$$7a + 13 = 7b + 13$$

$$7a = 7b$$

Therefore $a = b$. So f is one-to-one. ∎

Note To show a function f is not one-to-one, find a counterexample, that is, find $a, b \in A$ such that

$$a \neq b \text{ , but } f(a) = f(b).$$

EX Let $f: \mathbb{R} \to \mathbb{R}$ defined $f(x) = x^2 - 3x - 2$.

Show that f is not one-to-one (injective).

Aside: Can we find $a \neq b$, where $f(a) = f(b)$?

$$a^2 - 3a - 2 = b^2 - 3b - 2$$

$$a^2 - 3a = b^2 - 3b$$

$$a^2 - b^2 - 3a + 3b = 0$$

$$(a+b)(a-b) - 3(a-b) = 0$$

$$(a-b)(a+b-3) = 0$$

$$a - b = 0 \qquad a + b - 3 = 0$$

$$a + b = 3$$

$$a = 1, \ b = 2$$

Proof. Consider $a=1$ and $b=2$. Then

$a \neq b$, but

$$f(1) = 1^2 - 3\cdot 1 - 2 = -4$$

$$f(2) = 2^2 - 3\cdot 2 - 2 = 4 - 6 - 2 = -4$$

so $f(1) = f(2)$. Therefore, f is <u>not</u>

one-to-one. $\blacksquare$