8.4 $\boxed{\text{Properties of Equivalence Classes}}$ (Continued from Thursday)

Suppose R is an equivalence relation on the nonempty set A.

(a) If $a \in A$, then $a \in [a]$

(b) If $[a] = [b]$, then aRb.

(c) If aRb, then $[a] = [b]$.

Proof. Suppose $a, b \in A$ and aRb. We will show $[a] = [b]$. This will be done by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Without loss of generality, we will only show $[a] \subseteq [b]$.

Let x be any any element of $[a]$. Then x is related to a, xRa. We ~~kn~~ know aRb, so xRb by transitivity since R is an equivalence relation. Therefore, since xRb we have $x \in [b]$. So $[a] \subseteq [b]$.

We need to show $x \in [b]$, xRb

Similarly, $[b] \subseteq [a]$. Therefore $[a] = [b]$. ∎

Ex Consider the relation R ~~def~~ on $\mathbb{Z}$ defined xRy if $x+y$ is even. This was shown to be an equivalence relation, and the distinct equivalence classes? are

$$\{\ldots, -3, -1, 1, 3, 5, \ldots\} \quad \{\ldots, -4, -2, 0, 2, 4, \ldots\}$$

(a) $[5] = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$ $\quad 5 \in [5]$

(b) $[3] = [1]$ $\quad 3+1=4$ is even, $3R1$.

(c) $2R4$ $\quad [2] = [4]$

$2+4$ is even

8.5 $\boxed{\text{Congruence Modulo } n}$

Recall that for integers a, b, and $n \geq 2$, we say <u>a is congruent to b modulo n</u> if $n \mid (a-b)$. In this case, we write

$$a \equiv b \pmod{n}$$

~~Thm~~

<u>Result</u> Let R be the relation on $\mathbb{Z}$ defined xRy if $x \equiv y \pmod{7}$. Prove that R is an equivalence relation.

Proof. We will show R is reflexive, symmetric, and transitive.

First, let x be any element of $\mathbb{Z}$. Since $0 = 7 \cdot 0$, $7 \mid 0$. Then $7 \mid (x-x)$, so $x \equiv x \pmod{7}$. Then xRx, therefore R is reflexive.

xRx
$7 \mid (x-x)$
$7 \mid 0$
$0 = k \cdot 7$

Next, suppose $x, y \in \mathbb{Z}$ with xRy. Since xRy, $x \equiv y \pmod 7$, so $7 \mid (x-y)$. Then $7 \mid -(x-y)$, so $7 \mid (y-x)$. Therefore $y \equiv x \pmod 7$, so yRx. Therefore R is symmetric.

yRx
$y \equiv x \pmod 7$
$7 \mid (y-x)$
$7 \mid (x-y)$
$x \equiv y$ mod

Finally, suppose $x, y, z \in \mathbb{Z}$ with xRy and yRz. Then $x \equiv y \pmod 7$ and $y \equiv z \pmod 7$, so $7 \mid (x-y)$ and $7 \mid (y-z)$. Then

$$x - y = 7k, \quad y - z = 7\ell$$

for some $k, \ell \in \mathbb{Z}$. So by adding,

$$(x-y)+(y-z) = 7k + 7\ell$$

$$x - z = 7(k+\ell).$$

$x \equiv y \pmod 7$
$7 \mid (x-y)$
$y \equiv z$ (mod 7
$7 \mid (y-z$
$x-y=7k$
$y-z=7$
$x-z$

Since $k+\ell \in \mathbb{Z}$, this shows $7 \mid (x-z)$, so $x \equiv z \pmod 7$. Therefore xRz, so R is transitive.

Therefore, R is an equivalence relation. ∎

<u>Theorem</u> ~~F~~ Suppose n is any integer $n \geq 2$.

Let R be the relation on $\mathbb{Z}$ defined by

xRy if $x \equiv y \pmod{n}$. Then R is an equivalence relation.

(Proof - See book)

<u>Note</u> Most people say "congruence modulo n" is an equivalence relation." As an equivalence relation, it will partition $\mathbb{Z}$ into distinct equivalence classes.

<u>EX</u> Congruence modulo 3 on $\mathbb{Z}$. Determine the equivalence classes.

$$[0] = \{x \in \mathbb{Z} : x \equiv 0 \pmod 3\} = \{\ldots, -6, -3, 0, 3, 6, \ldots$$

$$[1] = \{x \in \mathbb{Z} : \underset{3 \mid (x-1)}{x \equiv 1 \pmod 3}\} = \{\ldots, -5, -2, 1, 4, 7, \ldots$$

$$[2] = \{x \in \mathbb{Z} : \underset{3 \mid x-2}{x \equiv 2 \pmod 3}\} = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$$

$\mathbb{Z} = \{\ldots, -3, 0, 3, 6, \ldots\} \cup \{\ldots, -2, 1, 4, 7, \ldots\} \cup \{\ldots, -1, 2, 5, 8, \ldots$

<u>Note</u> We can use mod n to generate other ~~equivalence~~ equivalence relations.

<u>EX</u> Consider the relation R on $\mathbb{Z}$ defined by aRb if $3a+5b \equiv 0 \pmod 8$. Prove this is an equivalence relation.

Proof. We will show R is reflexive, symmetric, and transitive.

aRa
$3a+5a \equiv 0$ mod 8
$8a \equiv 0$ mod 8.

First, suppose $a \in \mathbb{Z}$. Then $8 \mid 8a$, so $8 \mid (8a-0)$, then $8a \equiv 0 \pmod 8$, ~~so~~ Therefore $3a+5a \equiv 0 \pmod 8$, so aRa.

$3a+5b \equiv 0$ mod 8

Next, suppose $a, b \in \mathbb{Z}$ with aRb.

$3b+5a \equiv 0$ mod 8

Then ~~a≡b~~ $3a+5b \equiv 0 \pmod{8}$, so ~~3a+5~~ $8 \mid (3a+5b-0)$, meaning $3a+5b = 8k$ for some $k \in \mathbb{Z}$. Now

$$(3b+5a)+(3a+5b) = 8a+8b = 8(a+b).$$

So

$$(3b+5a) + 8k = 8(a+b)$$

$$3b+5a = 8(a+b-k).$$

Since $a+b-k \in \mathbb{Z}$, this shows $8 \mid (3b+5a)$, so $3b+5a \equiv 0 \pmod{8}$. Therefore bRa.

Finally, suppose $a,b,c \in \mathbb{Z}$ with aRb and bRc. Then $3a+5b \equiv 0 \pmod{8}$ and $3b+5c \equiv 0 \pmod{8}$. So $8 \mid (3a+5b-0)$ and $8 \mid (3b+5c-0)$. Then

$$3a+5b = 8k, \quad 3b+5c = 8l$$

for some $k,l \in \mathbb{Z}$.

By adding,

$$(3a+5b) + (3b+5c) = 8k+8\ell$$

So

$$3a + 5c = 8k+8\ell-8b$$

$$3a+5c = 8(k+\ell-b).$$

Then $8 \mid (3a+5c-0)$, so $3a+5c \equiv 0 \pmod{8}$.

Therefore aRc, so R is transitive.

Therefore, R is an equivalence relation. ∎

Ex Suppose R is the relation on $\mathbb{Z}$ defined xRy if $2x+5y \equiv 0 \pmod 8$. Show this is not an equivalence relation.

Aside: xRx : $2x+5x \equiv 0 \pmod 8$
$7x \equiv 0 \pmod 8$

Disproof. Consider $x=1$. Since $2x+5x = 2\cdot1+5\cdot1=7$ and $7 \not\equiv 0 \pmod 8$, $1 \not R 1$. Therefore, R is not reflexive, and so cannot be an equivalence relation. $\blacksquare$