$\boxed{\text{Properties of Relations}}$ (Continued)

~~R is~~

<u>EX</u> (Continued) R is the relation on $\mathbb{Z}$ defined

$$xRy \text{ if } x+y \text{ is even.}$$

(c) Prove or disprove: If xRy and yRz, then xRz.

Aside: $x+y$ is even $\quad 3+\cancel{3}5$

$y+z$ is even $\quad 5+7$.

$x+z$ is even? $\quad 3+7$

$$x+y=2k \qquad \cancel{y}+z=2l$$

$$y=2k-x \rightarrow 2k-x+z=2l$$

$$x+z$$

$$x = 2k-y$$

$$z = 2l-y$$

$$\overline{x+z = 2k+2l-2y}$$

Proof. Suppose xRy and yRz. We will show that xRz. Since xRy and yRz, this tells us $x+y$ and $y+z$ are even. So, there exists integers k and l such that

$$x+y=2k, \qquad y+z=2l.$$

So $x = 2k - y$ and $z = 2l - y$. Then

$$x + z = 2k - y + 2l - y = 2(k + l - y).$$

Since $k + l - y \in \mathbb{Z}$, this shows $x + z$ is even, hence xRz. ∎

Transitive Property

<u>Definitions</u> Suppose R is a relation on the set A.

(a) R is called <u>reflexive</u> if the following is true:

For all $x \in A$, xRx.

(b) R is called <u>symmetric</u> if the following is true:

If xRy, then yRx.

(c) R is called <u>transitive</u> if the following is true:

If xRy and yRz, then xRz.

Ex Let R be the relation on $\mathbb{R}$ defined

$$x R y \quad \text{if} \quad x<y.$$

(a) Is R reflexive? For all $x \in \mathbb{R}$, $x<x$.

No, R is not reflexive. $4 \not< 4$, so 4 is a counterexample.

(b) Is R symmetric? If $x<y$, then $y<x$.

No, R is not symmetric. Consider $x=1$ and $y=10$.

Then $1<10$ but $10 \not< 1$.

(c) Is R transitive? If $\underset{1}{x}<\underset{2}{y}$ and $\underset{2}{y}<\underset{3}{z}$, then $x<z$.

Yes, R is transitive. Suppose $x<y$ and $y<z$.

Then $x<y<z$, so $x<z$.

EX $S = \{a, b, c\}$. Consider the relation R on S

$$R = \{ (a,a), \underline{\underline{(a,b)}}, (c,c), (b,a) \}$$

(a) Reflexive? aRa $\quad b \not R b$ $\quad cRc$ Not reflexive.

(b) Symmetric?

$aRa \quad aRa$

$aRb \quad bRa$

$cRc \quad cRc$

$bRa \quad aRb$

Symmetric

(c) Transitive

$aRa, aRb \quad aRb$

$aRb, bRa \quad aRa$

$bRa, aRa \quad bRa$

$bRa, aRb \quad b \not R b$

Not Transitive

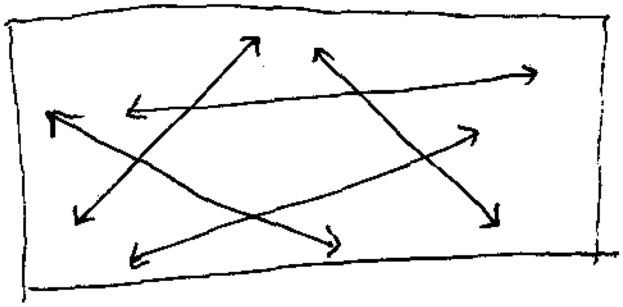
8.3 Equivalence Relations

Any given relation may or may not be reflexive, symmetric or transitive. But when a relation <u>is</u> reflexive, symmetric, and transitive, it is called an <u>equivalence relation</u>.

<u>EX</u> The relation R on $\mathbb{Z}$ defined xRy if $x+y$ is even, <u>is</u> an equivalence relation.

<u>EX</u> The relation R on $\mathbb{R}$ defined xRy if $x<y$, is <u>not</u> an equivalence relation.

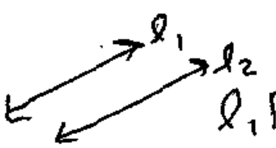
<u>EX</u> Let S be the set of all lines in a plane.



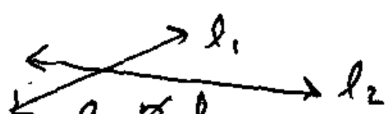
infinitely many

Consider the relation R on S defined

$$\ell_1 R \ell_2 \text{ if } \ell_1 \text{ is parallel to } \ell_2.$$

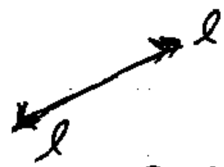


$\ell_1 R \ell_2$



$\ell_1 \not{R} \ell_2$

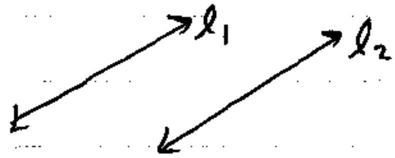
(a) Suppose ℓ is any line.



ℓ is parallel to ℓ, so $\ell R \ell$.

R is ~~symmetric~~ reflexive.

(b) Suppose ℓ_1 and ℓ_2 are lines with $\ell_1 R \ell_2$.



ℓ_1 is parallel to ℓ_2

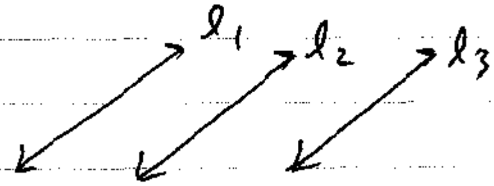
So

ℓ_2 is parallel to ℓ_1

So $\ell_2 R \ell_1$

R is symmetric.

(c) Suppose ℓ_1, ℓ_2, ℓ_3 with $\ell_1 R \ell_2$ and $\ell_2 R \ell_3$.



ℓ_1 is parallel to ℓ_2

ℓ_2 is parallel to ℓ_3

So

ℓ_1 is parallel to ℓ_3

R is transitive.

So R is an equivalence relation.

Equivalence Classes

Preliminary Examples:

(a) Relation R on $\mathbb{Z}$ defined xRy if $x+y$ is even.

~~5~~ Suppose we start with some integer, say 5. Find all elements $x \in \mathbb{Z}$ ~~that~~ that are related to 5.

$$x+5 \text{ is even}$$

This is the set

$$\{x \in \mathbb{Z} : x \text{ is odd}\}.$$

Suppose we start with 4. Find all elements $x \in \mathbb{Z}$ that are related to 4.

$$x+4 \text{ is even}$$

This is the set

$$\{x \in \mathbb{Z} : x \text{ is even}\}$$

Note that

$$\mathbb{Z} = \{x \in \mathbb{Z} : x \text{ is odd}\} \cup \{x \in \mathbb{Z} : x \text{ is even}\}$$

Called a Partition of $\mathbb{Z}$

(b) $S = \{1,2,3,4,5,6\}$

$R = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (6,6),$

$(1,3), (1,6), (6,1), (6,3), (3,1), (3,6),$

$(2,4), (4,2) \}$

This is an equivalence relation (Reflexive, Symmetric, Transitive).

$$\begin{array}{lll}
1 & \{1,6,3\} \checkmark & = [1] \\
2 & \{2,4\} \checkmark & = [2] \\
3 & \{3,6,1\} \checkmark & = [3] \\
4 & \{4,2\} \checkmark & = [4] \\
5 & \{5\} & = [5] \\
6 & \{6,1,3\} \checkmark & = [6]
\end{array}$$

§ $S = \{1,3,6\} \cup \{2,4\} \cup \{5\}$

Partition of S

<u>Definition</u> Suppose R is an <u>equivalence relation</u> on the set A. Let $a \in A$. The set of all elements $x \in A$ that are related to a is called an <u>equivalence class</u>.

$$[a] = \{x \in A : xRa\}.$$

8.4 $\boxed{\text{Properties of Equivalence Classes}}$

Suppose R is an equivalence relation on the (nonempty) set A.

(a) If $a \in A$, then $a \in [a]$.

Proof. Let $a \in A$. Since R is an equivalence relation, it must be reflexive, so aRa. Then $a \in \{x \in A : xRa\} = [a]$. ∎

(b) If $[a]=[b]$, then aRb.

Proof. Suppose $[a]=[b]$. We know $a\in[a]$ (by Part (a)), and since $[a]=[b]$, ~~$b\in$~~ $a\in[b]$. But $[b]=\{x\in A : xRb\}$. Since $a\in[b]$, this means aRb. $\blacksquare$

(c) If aRb, then $[a]=[b]$.

(Continued on Tuesday)