Each of the following proves or disproves a statement. What is the statement?

1. **Proof.** Assume that n is an odd integer. Then $n = 2k + 1$ for some integer k. Then

$$3n - 5 = 3(2k + 1) - 5 = 6k + 3 - 5 = 6k - 2 = 2(3k - 1).$$

Since $3k - 1$ is an integer, $3n - 5$ is even. ■

If n is an odd integer, then $3n-5$ is even.

2. **Proof.** We will show that this is true by showing that the contrapositive statement is true. Assume that x is even. Then $x = 2a$ for some integer a. Then

$$3x^2 - 4x - 5 = 2(2a)^2 - 4(2a) - 5 = 2(6a^2 - 4a - 3) + 1.$$

Since $6a^2 - 4a - 3$ is an integer, this shows $3x^2 - 4x - 5$ is odd. ■

If $3x^2-4x-5$ is even, then x is odd.

3. **Proof.** Consider $n = 4$. Notice that 4 is an even integer, and $n^2 + n + 3 = 23$ is prime. ■

There exists an even integer n such that n^2+n+3 is prime.

4. **Disproof.** This result is not true. Consider $a = \sqrt{2}$ and $b = 2$. Then a is an irrational number and b is a rational number, but $a^b = (\sqrt{2})^2 = 2$ is rational. Therefore $a = \sqrt{2}$ and $b = 2$ is a counterexample. ■

If a is an irrational number and b is a rational number then a^b is irrational.

5. **Proof.** Suppose a is a nonzero rational number and b is an irrational number, but assume to the contrary that ab is rational. Then there exists integers p and q such that $a = p/q$ and integers r and s such that $ab = r/s$. Since $a \neq 0$,

$$b = \frac{r/s}{a} = \frac{r/s}{p/q} = \frac{r}{s}\frac{q}{p} = \frac{rq}{ps}.$$

Since p, q, r, s are integers, so are rq and ps. This shows b is a rational number, which is a contradiction since b is irrational. Therefore the assumption that ab is rational is false. ■

If a is a nonzero rational number and b is an irrational number, then ab is irrational.

7.3 Proving and Disproving Statements

<u>EX</u> There exists irrational numbers a and b such that $a+b$ is rational.

Aside: $\sqrt{2} + \sqrt{3} \stackrel{?}{=}$ rational

$\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ irrational

$(\sqrt{2})^{\sqrt{2}} + (\quad) = ?$

$(\sqrt{2}) + (\underset{-1\cdot\sqrt{2}}{-\sqrt{2}}) = 0 = \frac{0}{1}$

Proof. This statement is true. Consider $a = \sqrt{2}$ and $b = -\sqrt{2}$, which are both irrational. Then $a+b = 0$ which is rational. ∎

EX For all irrational numbers a and b, $a+b$ is rational.

If we were to show this is true:

Let a, b be any irrational numbers. Show $a+b$ is rational.

If we were to show this is false:

Find irrational a & b where $a+b$ is irrational.

Disproof. This statement is false. Consider

$a = \sqrt{2}$ and $b = \sqrt{2}$ which are irrational,

but $a+b = 2\sqrt{2}$ is also irrational. ■

<u>EX</u> Let A be a set. If $A \cup B \neq \emptyset$ for every set B, then $A \neq \emptyset$.

Aside: $A = \{1, 2\}$ $\quad \{1,2\} \cup B \neq \{\ \}$ $\quad A \neq \emptyset$.

$A = \{\ ?\ \}$ $\quad \{\ ?\ \} \cup B \neq \{\ \}$ $\quad A \overset{?}{\neq} \emptyset$

$\{\ ?\ \} \cup \{\ \} \neq \{\ \}$ $\quad A \neq \emptyset$

Proof. This statement is true. Let A be ~~an~~ a set. Suppose $A \cup B \neq \emptyset$ for every set B. Then this is also true when $B = \emptyset$, so

$$A \cup \emptyset \neq \emptyset.$$

But $A \cup \emptyset = A$, so this shows $A \neq \emptyset$. ∎

EX If A and B are sets with $A \cap B = \emptyset$,

then $A = \emptyset$ or $B = \emptyset$.

Aside: $A = \{1,2\}$ $B = \{3,4\}$ $A \cap B = \emptyset$

Disprove: This statement is false. Consider

$A = \{1,2\}$ and $B = \{3,4\}$. Then $A \cap B = \emptyset$,

but $A \neq \emptyset$ and $B \neq \emptyset$. ∎

EX For every positive real number b, there exists an irrational number a such that $0 < a < b$.

Aside:

$$b = 10 \qquad a = \sqrt{2} < 10$$

$$b = 100 \qquad a = \sqrt{2} < 100$$

$$b = 1 \qquad a = \frac{1}{100} \cdot \sqrt{2} < 1$$

$$b = \frac{1}{100} \qquad a = \frac{1}{1000} \cdot \sqrt{2} < \frac{1}{100}$$

$$\frac{1}{n} \sqrt{2} < b$$

$$\sqrt{2} < b \cdot n$$

$$\frac{\sqrt{2}}{b} < n$$

Proof. This statement is true. Let b be any positive real number. Pick an integer $n > \frac{\sqrt{2}}{b}$. Then

$$b \cdot n > \sqrt{2}$$

So

$$b > \frac{1}{n} \cdot \sqrt{2}.$$

Since $\frac{1}{n}$ is rational and $\sqrt{2}$ is irrational, $\frac{1}{n} \cdot \sqrt{2}$ is also irrational and $0 < \frac{1}{n} \cdot \sqrt{2} < b$. ∎

Ex There exists a positive real number b such that $0 < a < b$ for every irrational number a.

Aside: $b = 10$ $\quad 0 < a < 10$ for every irrational a? NO

$1000\sqrt{2}$ is not less than 10

$b = 1000$ $\quad$ no; $10000000\sqrt{2}$ is not less than 1000

$b < n\sqrt{2}$ $\quad$ Seems like we can always make $n\sqrt{2}$ larger than b...

$\frac{b}{\sqrt{2}} < n$

Disprove. This statement is false. Let b be any positive real number. Pick an integer $n > \frac{b}{\sqrt{2}}$.

Then $n\sqrt{2} > b$, where $n\sqrt{2}$ is irrational.

So it is not the case that all irrational numbers a satisfy $0 < a < b$. ∎

Ch 8 $\boxed{\text{Equivalence Relations}}$

8.1 Relations

$x < y$ $\qquad\qquad$ $A \subseteq B$

$x = y$ $\qquad\qquad$ $a \mid b$

$\cancel{x} \cdot y = x^2$

The idea is that elements or objects are related to other elements or ~~objection~~ objects in some way.

Cartesian Product of Two Sets (1.6)

The Cartisian product of two sets A and B is

$$A \times B = \{ (a,b) : a \in A, \ b \in B \}$$

Ex $A = \{0,1\}$ $\quad$ $B = \{1,2\}$

$$A \times B = \{ (0,1), (0,2), (1,1), (1,2) \}$$

$$A \times A = \{ (0,0), (0,1), (1,0), (1,1) \}$$

A Relation from Set A to Set B

Let A and B be two sets. A subset of $A \times B$ is called a relation from A to B.

EX $A = \{0,1,2\}$ $B = \{3,4\}$

$$A \times B = \{(0,3), (0,4), (1,3), (1,4), (2,3), (2,4)\}$$

A relation is any subset

$$R = \{(0,3), (1,4), (2,3)\}$$

Another relation is

$$R = \{(0,4), (2,4)\}$$

Another relation

$$R = \{\ \}$$

Terminology

Suppose R is a relation from A to B.

- $(a,b) \in R$ we say a is related to b $\quad a\,R\,b$

- $(a,b) \notin R$ we say a is not related to b $\quad a \not R\, b$

- domain of R $\quad \operatorname{dom} R = \{a \in A : (a,b) \in R\}$

- range of R $\quad \operatorname{ran} R = \{b \in B : (a,b) \in R\}$

Ex $\quad A = \{0,1,2\} \quad B = \{3,4\}$

$$R = \{(0,3), (1,4), (2,3)\}$$

$(0,3) \in R$ means $0\,R\,3$
$(0,4) \notin R$ means $0 \not R\, 4$

$\operatorname{dom} R = \{0,1,2\} \quad \operatorname{ran} R = \{3,4\}$