$\boxed{\text{From HW \#6}}$

Prove: If $r \in \mathbb{R}$ with $0<r<1$, then $\frac{1}{r(1-r)} \geq 4$.

Proof. Suppose $r \in \mathbb{R}$ with $0<r<1$, and

$$\cancel{\frac{1}{r(1-r)} \geq 4.}$$

Never assume what you are trying to show is true.

Aside:
$$\frac{1}{r(1-r)} \geq 4$$

$$1 \geq 4r(1-r)$$

$$1 \geq 4r - 4r^2$$

$$4r^2 - 4r + 1 \geq 0$$

$$(2r-1)^2 \geq 0$$

Now work/write backwards..

Proof. Suppose $r \in \mathbb{R}$ with $0<r<1$. Then

$$(2r-1)^2 \geq 0.$$

So

$$4r^2 - 4r + 1 \geq 0$$

then

$$1 \geq 4r - 4r^2$$

so

$$1 \geq 4r(1-r).$$

Since $0<r<1$, we know $r>0$ and $1-r>0$. So

$$\frac{1}{r(1-r)} \geq 4 \qquad \blacksquare$$

~~Prove that there are no positiv~~

~~For~~

Prove: For all positive real numbers x and y, $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

Proof. We will show this by using a proof by contradiction. Assume there exists positive real numbers x and y such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$. Then

$$x + y = (\sqrt{x} + \sqrt{y})^2$$

so

$$x + y = x + 2\sqrt{x}\sqrt{y} + y.$$

Then

$$0 = 2\sqrt{x}\sqrt{y}$$

so

$$0 = \sqrt{x}\sqrt{y}$$

then

$$0 = xy.$$

So tells us. $x = 0$ or $y = 0$. But this contradicts the assumption that x, y are both positive. So the original statement must be true. ∎

7.3 Testing Statements

Given a statement that we don't know is true or false, first "determine" if it is true or false, then prove or disprove.

<u>EX</u> Prove or disprove: There exists a real number x such that $x^6 + 2x^2 + 1 = 0$.

Strategy #1: Observation

$$x^6 \geq 0, \quad 2x^2 \geq 0, \quad 1 > 0$$

so this is probably false.

Disproof: This statement is false. Let $x \in \mathbb{R}$. Then $x^6 \geq 0$, $2x^2 \geq 0$, $1 > 0$ so

$$x^6 + 2x^2 + 1 > 0.$$

Therefore $x^6 + 2x^2 + 1 \neq 0$. ∎

<u>EX</u> Prove or disprove: There exists a real number x such that $x^3 < x < x^2$.

Strategy # 2: Guess

$x = 1 \qquad 1^3 \not< 1 \not< 1^2$

$x = 2 \qquad 2^3 \not< 2 \quad 2^2$

$x = 3 \qquad 3^3 \not< 3 \quad 3^2$

$x = \frac{1}{2} \qquad \left(\frac{1}{2}\right)^3 \quad \frac{1}{2} \not< \left(\frac{1}{2}\right)^2$

$x = -2 \qquad (-2)^3 \quad (-2) \quad (-2)^2$

$\qquad\qquad\quad -8 < -2 < 4$

Proof. Consider $x = -2$. Then $(-2)^3 = -8$, $(-2)^2 = 4$
so $(-2)^3 < (-2) < (-2)^2$. ∎

EX Prove or Disprove: For every $n \in \mathbb{N}$, n^2+5n is odd.

Strategy #3: Look for a counterexample

$n=2$ $\qquad 2^2+5\cdot 2 = 14$ not odd

Disproof. This statement is false. Consider $n=2$, which is a natural number. Note $n^2+5n=14$, which is not odd. So $n=2$ is a counterexample. ∎

Strategy #4: Try a direct proof.

"Proof". Let $n \in \mathbb{N}$. Either n is even or odd.

Case ① Suppose n is even. Then $n=2k$ for some $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^2+5n &= (2k)^2+5(2k) \\ &= 4k^2+10k \\ &= 2(2k^2+5k) \end{aligned}$$

This is <u>even</u>. So this statement is probably FALSE; go back and find a counterexample.

EX Prove or disprove: If $n \in \mathbb{N}$, then $5 \mid (2 \cdot 4^n + 3 \cdot 9^n)$.

$n=1 \qquad 2 \cdot 4^1 + 3 \cdot 9^1 = 35 \qquad 5 \mid 35$

$n=2 \qquad 2 \cdot 4^2 + 3 \cdot 9^2 = 275 \qquad 5 \mid 275$

$n=3 \qquad 2 \cdot 4^3 + 3 \cdot 9^3 = 2{,}315 \qquad 5 \mid 2{,}315$

This appears to be true.

Proof. We will show this is true by induction.

Consider $n=1$. Then $2 \cdot 4^1 + 3 \cdot 9^1 = 35$, so

$5 \mid (2 \cdot 4^n + 3 \cdot 9^n)$ when $n=1$.

Now suppose $n=k$ is any natural number such that

$5 \mid (2 \cdot 4^n + 3 \cdot 9^n)$. Then

$$5 \mid (2 \cdot 4^k + 3 \cdot 9^k).$$

We need to show $5 \mid (2 \cdot 4^{k+1} + 3 \cdot 9^{k+1})$. Since

$5 \mid (2 \cdot 4^k + 3 \cdot 9^k)$, $2 \cdot 4^k + 3 \cdot 9^k = 5\ell$ for

some $\ell \in \mathbb{Z}$. Then

$$\begin{aligned} 2 \cdot 4^{k+1} + 3 \cdot 9^{k+1} &= 4(2 \cdot 4^k) + 3 \cdot 9^{k+1} \\ &= 4(5\ell - 3 \cdot 9^k) + 3 \cdot 9^{k+1} \end{aligned}$$

$$
\begin{aligned}
2\cdot 4^{k+1} + 3\cdot 9^{k+1} &= 20\ell - 12\cdot 9^k + 3\cdot 9^{k+1} \\
&= 20\ell + 9^k(-12 + 3\cdot 9) \\
&= 20\ell + 9^k(15) \\
&= 5(4\ell + 3\cdot 9^k).
\end{aligned}
$$

Since $4\ell + 3\cdot 9^k \in \mathbb{Z}$, this shows

$$5 \mid (2\cdot 4^{k+1} + 3\cdot 9^{k+1}).$$

Therefore, $5 \mid (2\cdot 4^n + 3\cdot 9^n)$ for all $n \in \mathbb{N}$. ∎