6.1 $\boxed{\text{Mathematical Induction}}$

Suppose we wish to prove

For every $n \in \mathbb{N}$, $P(n)$.

Can prove sometimes with direct proof, contrapositive, or contradiction. <u>Another</u> approach is induction.

Let $T = \{n \in \mathbb{N} : P(n) \text{ is true.}\}$. If we can show $T = \mathbb{N}$, then $P(n)$ is true for all $n \in \mathbb{N}$.

$$\left.\begin{array}{l} \bullet\ T \subseteq \mathbb{N} \\ \bullet\ \mathbb{N} \subseteq T \text{ if } T \text{ has the properties} \\ \qquad 1 \in T, \quad \text{If } k \in T, \text{ then } k+1 \in T. \end{array}\right\} \begin{array}{l}\text{More}\\ \text{on}\\ \text{Thursday}\end{array}$$

So then in this case, $T = \mathbb{N}$.

The Principle of Mathematical Induction

Suppose for each $n \in \mathbb{N}$, $P(n)$ is a statement.

Suppose we can show

(a) $P(n)$ is true when $n=1$ $\quad$ $1 \in T$

(b) If $P(n)$ is true when $n=k$, then $P(n)$ is true when $n=k+1$. $\quad$ $k \in T \Rightarrow k+1 \in T$

Then $P(n)$ is true for every $n \in \mathbb{N}$.

Result For every $n \in \mathbb{N}$, $3 \mid (7^n - 4^n)$.

Proof. This will be shown using induction.

First, consider the case $n = 1$. Then

$$7^1 - 4^1 = 3$$

Base step

so $3 \mid (7^n - 4^n)$ when $n = 1$.

Now suppose k is any positive integer for which $3 \mid (7^n - 4^n)$ when $n = k$. We will show that $3 \mid (7^{k+1} - 4^{k+1})$. Since $3 \mid (7^k - 4^k)$, we have

$$7^k - 4^k = 3m$$

Induction step

for some $m \in \mathbb{Z}$. Now

$$\begin{aligned}
7^{k+1} - 4^{k+1} &= 7(7^k) - 4^{k+1} \\
&= 7(3m + 4^k) - 4^{k+1} \\
&= 21m + 7 \cdot 4^k - 4^{k+1} \\
&= 21m + 4^k(7-4) \\
&= 21m + 3 \cdot 4^k = 3(7m + 4^k).
\end{aligned}$$

So $3 \mid (7^{k+1} - 4^{k+1})$.

Therefore by induction, $3 \mid (7^n - 4^n)$ for all $n \in \mathbb{N}$. ∎

<u>Result</u> Let $a \geq 0$. For every $n \in \mathbb{N}$, $(1+a)^n \geq 1 + n \cdot a$.

Proof. This will be proven by induction.

First, consider $n = 1$. Then $(1+a)^n = (1+a)^1$ and $1 + n \cdot a = 1 + a$, so $(1+a)^n = 1 + n \cdot a$ when $n = 1$. So $(1+a)^n \geq 1 + n \cdot a$ when $n = 1$.

Next, suppose k is a positive integer such that $(1+a)^k \geq 1 + k \cdot a$. We will show $(1+a)^{k+1} \geq 1 + (k+1)a$. Since

$$(1+a)^{k+1} = (1+a)^k \cdot (1+a)$$

and $(1+a)^k \geq 1 + k \cdot a$,

$$(1+a)^{k+1} \geq (1+ka)(1+a) = 1 + ka + a + ka^2$$

So

$$(1+a)^{k+1} \geq \left[1 + (k+1)a + \underline{\underline{ka^2}}\right] \geq \left[1 + (k+1)a\right]$$

Since $ka^2 \geq 0$.

So $(1+a)^{k+1} \geq 1+(k+1)a$.

Therefore by induction, $(1+a)^n \geq 1+na$ for all $n \in \mathbb{N}$. ∎

<u>Note</u> Not all "For all $n \in \mathbb{N}$, $P(n)$" have to be proven by induction.

(a) For all $n \in \mathbb{N}$, n^2-n is even.

Direct proof, by cases

(b) For all $n \in \mathbb{N}$, $1+2+\cdots+n = \frac{n(n+1)}{2}$.

Alternate proof to induction:

Proof. Let $n \in \mathbb{N}$. Let

$$S = 1+2+\cdots+n.$$

Then

$$S = n+(n-1)+\cdots+1$$

So

$$2S = (n+1)+(n+1)+\cdots+(n+1)$$

$$2S = n\cdot(n+1)$$

Then $$S = \frac{n(n+1)}{2}.$$ ∎

6.2 $\boxed{\text{A More General Principle of Math. Induction}}$

Suppose we wish to prove (Let $m \in \mathbb{Z}$.)

For all integers $n \geq m$, $P(n)$

(if $m=1$, then this is $n \in \mathbb{N}$)

We can show this is true if

(a) $P(n)$ is true for $n=m$,

(b) If $P(n)$ is true for $n=k$, then $P(n)$ is true for $n=k+1$

are both true.

<u>Result</u> For $\overset{\text{integers}}{n} \geq 5$, $2^n > n^2$.

Aside: $2^k > k^2 \overset{?}{\Longrightarrow} 2^{k+1} > (k+1)^2 = k^2+2k+1$

$$2^{k+1} = 2 \cdot 2^k$$

$$> 2 \cdot k^2 \overset{?}{\geq} k^2+2k+1$$

$$2k^2 \overset{?}{\geq} k^2 + 2k + 1$$

$$k^2 \geq 2k+1$$

$$k^2 - 2k \geq 1$$

$$k(k-2) \geq 1$$

$\leftarrow$ $k \geq 5$, $k-2 \geq 3$, so $k(k-2) \geq 15 > 1$.

Proof. This will ~~pro~~ be proved by induction.

First, consider $n = 5$. Then $2^5 = 32$ and $5^2 = 25$,

so $2^5 > 5^2$. So $2^n > n^2$ when $n = 5$.

Next, suppose k is an integer $k \ge 5$ such that $2^k > k^2$.

We will show that $2^{k+1} > (k+1)^2$. Since $k \ge 5$,

$k - 2 \ge 3$, so $k(k-2) \ge 15$. Then

$$
\begin{aligned}
k(k-2) &\ge 1, \\
k^2 - 2k &\ge 1 \\
k^2 &\ge 1 + 2k \\
k^2 + k^2 &\ge k^2 + 2k + 1 \\
2k^2 &\ge (k+1)^2.
\end{aligned}
$$

So

$$2^{k+1} = 2 \cdot \underline{\underline{2^k}} > 2\left(\underline{\underline{k^2}}\right) \ge (k+1)^2.$$

Then $2^{k+1} \ge (k+1)^2$.

Therefore $2^n \ge n^2$ for every integer $n \ge 5$. $\blacksquare$

<u>Result</u> Suppose $A_1, \ldots, A_n$ are sets with $n \geq 2$. Then

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_n$$

Note: $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof. This will be proved by induction.

First, consider $n=2$. Then

$$\overline{A_1 \cup \cdots \cup A_n} = \overline{A_1 \cup A_2} = \overline{A}_1 \cap \overline{A}_2 = \overline{A}_1 \cap \cdots \cap \overline{A}_n.$$

So this is true when $n=2$.

Next, suppose $k \geq 2$ is any integer for which

$$\overline{A_1 \cup \cdots \cup A_k} = \overline{A}_1 \cap \cdots \cap \overline{A}_k$$

Now consider $k+1$ sets,

$$\begin{aligned}
\overline{A_1 \cup \cdots \cup A_k \cup A_{k+1}} &= \overline{(A_1 \cup \cdots \cup A_k) \cup (A_{k+1})} \\
&= \overline{(A_1 \cup \cdots \cup A_k)} \cap \overline{A_{k+1}} \\
&= (\overline{A}_1 \cap \cdots \cap \overline{A}_k) \cap \overline{A_{k+1}} \\
&= \overline{A}_1 \cap \cdots \cap \overline{A}_k \cap \overline{A_{k+1}}
\end{aligned}$$

So by induction this is true for any number of sets $n \geq 2$. ∎