5.3 $\boxed{\text{A Review of Three Proof Techniques}}$

Suppose we want to prove

For all $x \in S$, if $P(x)$ then $Q(x)$.

is true.

① Direct Proof

Proof. Suppose $x \in S$ and $P(x)$ is true.

[Then show $Q(x)$ must also be true.]

② Proof by Contrapositive

Proof. We will show the contrapositive is true. Let $x \in S$ and suppose $Q(x)$ is not true.

[Then show $P(x)$ must also be not true.]

③ Proof by Contradiction

Proof. Assume that the statement is false. Then there exists $x \in S$ such that $P(x)$ is true but $Q(x)$ is false.

[Then show we get a contradiction.]

<u>Result</u> Let x be a nonzero real number. If $x+\frac{1}{x}<2$, then $x<0$.

Direct Proof

Proof. Suppose x is a nonzero real number and $x+\frac{1}{x}<2$. Since $x\neq 0$, we know that $x^2>0$. Then

$$x^2\left(x+\frac{1}{x}\right)<2\cdot x^2$$

Aside:

$$x+\frac{1}{x}<2$$

$$\frac{x^2+1}{x}<2$$

$$x^2\cdot\frac{x^2+1}{x}<2\cdot x^2$$

$$x(x^2+1)<2x^2$$

So

$$x^3+x<2x^2$$

Then

$$x^3+x-2x^2<0,$$

$$x(x^2-2x+1)<0,$$

$$x(x-1)^2<0.$$

Since $x(x-1)^2<0$, we can't have $(x-1)^2=0$.

So $(x-1)^2>0$, then

$$\frac{x(x-1)^2}{(x-1)^2}<\frac{0}{(x-1)^2}$$

Therefore $x<0$. ■

<u>Result</u> Let x be a nonzero real number. If $x + \frac{1}{x} < 2$, then $x < 0$.

Proof by Contrapositive

Proof. Suppose x is a nonzero real number and $x \geq 0$. So $x > 0$.

Aside:

$x + \frac{1}{x} \not< 2$

$x + \frac{1}{x} \geq 2$

$x(x + \frac{1}{x}) \geq 2x$

$x^2 + 1 \geq 2x$

$x^2 - 2x + 1 \geq 0$

Note that
$$(x-1)^2 \geq 0.$$
Then
$$x^2 - 2x + 1 \geq 0$$
So
$$x^2 + 1 \geq 2x.$$
Recall $x > 0$, so
$$\frac{x^2+1}{x} \geq \frac{2x}{x}.$$
Then
$$x + \frac{1}{x} \geq 2.$$
■

Result Let x be a nonzero real number. If $x+\frac{1}{x}<2$, then $x<0$.

Proof by contradiction

Proof. Assume this is a false statement. Then there exists a nonzero real number x such that $x+\frac{1}{x}<2$ but $x\geq 0$.

Since $x\neq 0$ and $x\geq 0$, then $x>0$. Then

$$x+\frac{1}{x}<2$$

So

$$x\left(x+\frac{1}{x}\right)<2(x).$$

Then

$$x^2+1<2x$$

So

$$x^2-2x+1<0$$

and

$$(x-1)^2<0.$$

But $(x-1)^2\geq 0$, so this is a contradiction.

So the assumption that the statement is false, is false. Therefore the statement is true. ∎

5.4 $\boxed{\text{Existence and Uniqueness Proofs}}$

Suppose we want to prove

There exists $x \in S$ such that $P(x)$

is true. In the simplest case, all we have to do is give one $x \in S$ that makes $P(x)$ true.

Result There exists a real number whose cube is equal to its square.

Aside:
$x^3 = x^2$
$x^3 - x^2 = 0$
$x^2(x-1) = 0$
$x = 1, 0$

Proof. Consider the real number 1. Note that $1^3 = 1$ and $1^2 = 1$, so its cube is equal its square. ∎

Result There exists an integer n such that $2 \mid n$ and $4 \nmid n$.

Proof. Consider $n = 6$. Note $2 \mid 6$ and $4 \nmid 6$. ∎

In many cases, we may not easily find something that exists.

Note: $2^{\frac{1}{2}}$ $\quad a^b$ $\quad a, b$ are rational numbers

$a = \frac{2}{1} \quad b = \frac{1}{2}$

$2^{\frac{1}{2}} = \sqrt{2}$ is irrational.

$(\text{rational})^{\text{rational}}$ —

Result There exist rational numbers a and b such that a^b is ~~ra~~ irrational.

Proof. Consider $a = 2$ and $b = \frac{1}{2}$. Then a, b are rational, ~~ba~~ and $a^b = 2^{\frac{1}{2}} = \sqrt{2}$ is irrational. $\blacksquare$

Result There exists irrational numbers a and b such that a^b is rational.

Proof. First, let's start with $\sqrt{2}^{\sqrt{2}}$. This is either rational or irrational.

Case ① Suppose $\sqrt{2}^{\sqrt{2}}$ is rational. Then let $a = \sqrt{2}$, $b = \sqrt{2}$. Then a, b are irrational, a^b is rational.

Case (2) Suppose $\sqrt{2}^{\sqrt{2}}$ is irrational. Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then

a and b are irrational, and

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}}$$
$$= \sqrt{2}^{2}$$
$$= 2$$
$$= \frac{2}{1},$$

is rational.

So in either case, there are irrational numbers a and b such that a^b is rational. ∎

Uniqueness Proofs (Unique: Only one exists)

Ex (a) There exists $x \in \mathbb{Z}$ such that $x^3 = x^2$.

Proof. Consider $x = 1$. Note that $1^3 = 1 = 1^2$. ∎

(b) There exists a unique $x \in \mathbb{N}$ such that $x^3 = x^2$.

$\underbrace{\text{There exists}}_{\text{At least one}}$ $\quad$ $\underbrace{\text{unique}}_{\text{Only One}}$

Proof. First, consider $x = 1$. Note 1 is a natural number and $1^3 = 1 = 1^2$.

is different than $y \neq 1$

Next, suppose that y is another natural number such that $y^3 = y^2$. Then

$$y^3 = y^2$$

$$y^3 - y^2 = 0$$

$$y^2(y-1) = 0$$

So $y = 0$ or $y = 1$. But $y \in \mathbb{N}$ $(y \neq 0)$, and $y \neq 1$. This a contradiction. So no other natural number y exists such that $y^3 = y^2$. ∎