5.1 $\boxed{\text{Counterexamples}}$

Counterexamples are used to prove that certain statements are false.

For All

Ex For all $n \in \mathbb{Z}$, n is even.

This is not true. Consider $n = 5$. The integer 5 is not even.

Ex For all $x \in \mathbb{R}$, $|x| = x$.

This is false. Consider $x = -11$. The real number $x = -11$ ~~sat~~ satisfies $|-11| = 11 \neq -11$.

Ex For all $x \in \mathbb{R}$, $\dfrac{x^2+x}{x^2-x} = \dfrac{x+1}{x-1}$.

Aside: $x=1$ $\quad \dfrac{1+1}{1-1} \neq \dfrac{1+1}{1-1}$ because $\frac{2}{0}$ is not defined.

This statement is not true, because when $x=1$ we can't say that $\dfrac{1+1}{1-1} = \dfrac{1+1}{1-1}$ since $\frac{2}{0}$ is not defined.

Aside: $\dfrac{x^2+x}{x^2-x} = \dfrac{x(x+1)}{x(x-1)} = \dfrac{x+1}{x-1}$ By dividing $\frac{x}{x}$, this assumes $x \neq 0$

$x=0$ $\quad \dfrac{0+0}{0-0} \neq \dfrac{0+1}{0-1} = -1$

$$\frac{x^2+x}{x^2-x} = \frac{x+1}{x-1} \quad \text{for } x \in \mathbb{R} - \{0,1\}$$

If Then

For all $x \in S$, $P(x) \Rightarrow Q(x)$.
(Let)

This is not true if we can find $x \in S$ such that $P(x) \Rightarrow Q(x)$ is false. This means if $P(x)$ is true and $Q(x)$ is false.

<u>EX</u> Let $n \in \mathbb{Z}$. If n is odd, then $n^2 - n$ is odd.

This is not true. Consider $n = 3$. ~~So~~ The integer 3 is odd, but $3^2 - 3 = 6$ is not odd.

<u>EX</u> Let $a, b \in \mathbb{R} - \{0\}$. If x, y are positive real numbers, then $\frac{a^2}{2b^2} x^2 + \frac{b^2}{2a^2} y^2 > xy$.

Aside: This ‡ is false (?). So there must exist x, y such that

$$\frac{a^2}{2b^2} x^2 + \frac{b^2}{2a^2} y^2 > xy$$

is not true.

$$\frac{a^2}{2b^2} x^2 + \frac{b^2}{2a^2} y^2 > xy$$

$$\frac{a^4 x^2 + b^4 y^2}{2a^2 b^2} > xy$$

$$a^4 x^2 + b^4 y^2 > 2a^2 b^2 \cdot xy$$

$$a^4 x^2 - 2a^2 b^2 xy + b^4 y^2 > 0$$

$$\cancel{\left(a^2\sqrt{x} - b^2\sqrt{y}\right)^2 > 0}$$

$$\left(a^2 x - b^2 y\right)^2 > 0$$

Could $a^2 x - b^2 y = 0$? $x = b^2$ $y = a^2$

This statement is not true. ~~Let~~ Consider $x = b^2$ and $y = a^2$. Since a, b are not 0 $\left(a, b \in \mathbb{R} - \{0\}\right)$, $x = b^2$ and $y = a^2$ are positive (not 0). Also

$$\frac{a^2}{2b^2}x^2 + \frac{b^2}{2a^2}y^2 = \frac{a^2}{2b^2}(b^2)^2 + \frac{b^2}{2a^2}(a^2)^2$$

$$= \frac{a^2b^4}{2b^2} + \frac{b^2a^4}{2a^2}$$

$$= \frac{a^2b^2}{2} + \frac{b^2a^2}{2}$$

$$= a^2b^2$$

$$= y \cdot x$$

So $\frac{a^2}{2b^2}x^2 + \frac{b^2}{2a^2}y^2 \not> xy$.

5.2 $\boxed{\text{Proof by Contradiction}}$

$$\left.\begin{array}{l} X=2 \quad \text{and} \quad X \neq 2 \\ 1>2 \\ 1<2 \quad \text{and} \quad 2<1. \end{array}\right\} \text{These are contradictory statements}$$

A proof by contradiction first <u>assumes</u> what we are trying to prove is false. Then (somehow) use this to arrive at a contradiction.

<u>Ex</u> There is no largest integer.

Proof. Assume this is not the case, that is, there <u>is</u> a largest integer n. But $n+1$ is also an integer, and

$$n < n+1,$$

so n is <u>not</u> the largest integer. So by contradiction there is no largest integer. ∎

__EX__ There is no smallest positive ~~integer~~ real number.

Proof. Assume, to the contrary, that there __is__ a smallest positive real number r. Consider $\frac{r}{2}$. Since $r > 0$, $r > \frac{r}{2} > 0$. So $\frac{r}{2}$ is a positive real number smaller than r. So r is not the smallest real number. This is r a contradiction, so there is no smallest positive real number. ■

<u>EX</u> ~~If x^2 let $x \in \mathbb{R}$~~. If x^2 is even, then x is even.

Let $x \in \mathbb{Z}$.

Proof. Assume that this is not true. Then there exists some $x \in \mathbb{Z}$ such that x^2 is even but x is odd. Then $x = 2k+1$ for some $k \in \mathbb{Z}$.

Then

$$x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2+2k)+1.$$

Since $2k^2 + 2k \in \mathbb{Z}$, this means x^2 is odd.

This is a contradiction, so the statement that if x^2 is even, then x is even must be true. ∎

Rational + Irrational Numbers

$\frac{1}{2}, \frac{11}{4}, -\frac{2}{3}, 4 \ldots$ rational numbers $\frac{p}{q}$

$\sqrt{2}, \pi,$ irrational? why?

Result $\sqrt{2}$ is irrational.

Proof. Assume to the contrary that $\sqrt{2}$ is rational.

then $\sqrt{2} = \frac{p}{q}$ for some integers p, q. We

can assume without loss of generality that p and q

have no common factors. So

$$\sqrt{2} = \frac{p}{q}$$

tells us

$$\sqrt{2}\, q = p$$

So

$$2q^2 = p^2.$$

Since $q^2 \in \mathbb{Z}$, this says that p^2 is even. So

p is even. Then $p = 2k$ for some $k \in \mathbb{Z}$.

then
$$2q^2 = (2k)^2$$
So
$$2q^2 = 4k^2$$
then $q^2 = 2k^2$. So q^2 is even. then q is even. But if p and q are both even, they have a common factor of 2. This is a contradiction because p, q had no common factors.

So $\sqrt{2}$ is irrational. ■