Comments on Proof Writing

① Avoid $\exists, \forall, \Rightarrow, \ldots$

② Start each sentence with a word.

Let $x = 2p+1$ and $y = 2q+1$ for some $p, q \in \mathbb{Z}$.
Then,
$$(2p+1) - (2q+1) = \cdots$$

③ $P \Leftrightarrow Q$ ~~don't~~ doesn't have a contrapositive.

$P \Rightarrow Q$ has a contrapositive

$Q \Rightarrow P$ " " "

④ Sometimes contrapositives don't help.

$5x-1$ is odd if and only if $2x+4$ is even.

($\Rightarrow$) If $5x-1$ is odd, then $2x+4$ is even.
C.P. If $2x+4$ is odd, then $5x-1$ is even.

Both statements are similar in terms of the amount of work.

(5) Once the proof is started, don't write down the statement you are trying to show, UNLESS you write "We will show that..."

Prove: $5x-1$ is odd if and only if $2x+4$ is even.

First, it will be shown that

Proof. If $5x-1$ is odd, then $2x+4$ is even.

Suppose $5x-1$ is odd. Then...

Continued from Tuesday: $A = \{x \in \mathbb{Z} : 10 \mid x\}$, $B = \{x \in \mathbb{Z} : 5 \mid x\}$, $C = \{x \in \mathbb{Z} : 2 \mid x\}$. Prove $A = B \cap C$.

Proof. First, we will show that $A \subseteq B \cap C$.

Let $x \in A$. Then $10 \mid x$, so $x = 10k$ for some $k \in \mathbb{Z}$. Since $x = 5(2k)$ where $2k \in \mathbb{Z}$, $5 \mid x$, so $x \in B$. Also, $x = 2(5k)$ where $5k \in \mathbb{Z}$, so $2 \mid x$, showing $x \in C$. Since $x \in B$ and $x \in C$, we must have $x \in B \cap C$.

Next, we will show that $B \cap C \subseteq A$.

Let $x \in B \cap C$, so $x \in B$ and $x \in C$. Then $5 \mid x$ and $2 \mid x$, so

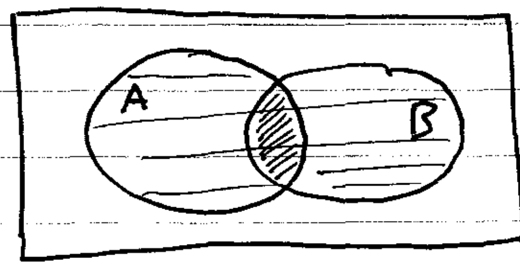
$$x = 5k, \quad x = 2l$$

for some $k, l \in \mathbb{Z}$. So $5k = 2l$, showing that ~~k~~ $l = \frac{5k}{2}$. Since $l \in \mathbb{Z}$, k must be divisible by 2. So $k = 2m$ for some $m \in \mathbb{Z}$.
Then ~~k~~ $x = 5k = 5 \cdot 2m = 10m$. This shows $10 \mid x$, so $x \in A$.

Therefore, $A = B \cap C$. ∎

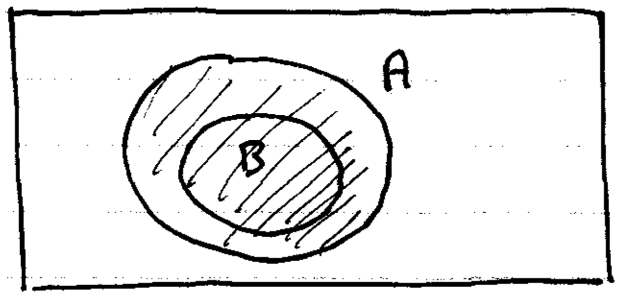
Other Proofs Involving ~~Sets~~ General Sets

Result Let A, B be sets. Then $(A \cap B) \subseteq (A \cup B)$.



Proof. ~~Let~~ Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$. In particular, $x \in A$, so $x \in A \cup B$. ∎

<u>Result</u> Let A and B be sets. Then $A \cup B = A$ if and only if $B \subseteq A$.



Proof. First, we will show that if $B \subseteq A$, then $A \cup B = A$. Suppose $B \subseteq A$.

Part ① We will show $A \cup B \subseteq A$. Suppose $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then we are done. If $x \in B$, then $x \in A$ because $B \subseteq A$. So $x \in A$.

Part ② Next we will show $A \subseteq A \cup B$.

~~be~~ Suppose $x \in A$. Then $x \in A \cup B$.

Therefore, $A = A \cup B$.

Next, we will show that if $A \cup B = A$,

then $B \subseteq A$. This will be done by

showing the contrapositive that if $B \nsubseteq A$, then $A \cup B \neq A$.

Suppose $B \nsubseteq A$.

So there must be an $x \in B$ but $\underline{\underline{x \notin A}}$.

Since $x \in B$, it must be that $\underline{\underline{x \in A \cup B}}$.

So it is not the case that every element in $A \cup B$

is an element of A. So $A \cup B \nsubseteq A$.

Therefore $A \cup B \neq A$.

So therefore, $A \cup B = A$ if and only if $B \subseteq A$. ∎

<u>General Set Properties</u>

Suppose $A, B, C,$ are sets. Then

(a) $A \cup B = B \cup A$, $A \cap B = B \cap A$

(b) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$

(c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(d) $\overline{A \cup B} = \bar{A} \cap \bar{B}$, $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Proof of $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

First, we will show that $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Suppose $x \in \overline{A \cup B}$. So $x \notin A \cup B$. Then

$x \notin A$ and ~~or~~ $x \notin B$. Then $x \in \bar{A}$ and $x \in \bar{B}$.

Therefore $x \in \bar{A} \cap \bar{B}$.

Next, we will show that $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

Suppose $x \in \bar{A} \cap \bar{B}$. Then $x \in \bar{A}$ and $x \in \bar{B}$.
So $x \notin A$ and $x \notin B$. Then $x \notin A \cup B$, so $x \in \overline{A \cup B}$.
Therefore, $\overline{A \cup B} = \bar{A} \cap \bar{B}$.