5.4 $\boxed{\text{Existence and Uniqueness}}$

<u>Result</u> Let $S = \{a+b\sqrt{2} : a,b \in \mathbb{Q}\}$. For each $x \in S$, there exist <u>unique</u> $a,b \in \mathbb{Q}$ such that $x = a+b\sqrt{2}$.

Proof. Let $x \in S$. By definition of S,

there are rational numbers $a,b \in \mathbb{Q}$ such

that $x = a+b\sqrt{2}$.

Next, suppose c and d are rational

numbers such that $x = c+d\sqrt{2}$. Then

$$a+b\sqrt{2} = c+d\sqrt{2}$$

So

$$a-c = (d-b)\sqrt{2}.$$

$\frac{a-c}{d-b} = \sqrt{2}$
Can't do if $d=b$
Can do if $d \neq b$, but
$\sqrt{2}$ is irrational

Case (1) Suppose $d \neq b$. Then

$$\frac{a-c}{d-b} = \sqrt{2}.$$

But $\sqrt{2}$ is irrational, and $\frac{a-c}{d-b}$ is rational.
This is a contradiction, so $d \neq b$ is not possible.

case ② Suppose $d=b$. Then

$$a-c = (\underbrace{d-b}_{0})\sqrt{2} = 0$$

Then $a=c$.

We must have $a=c$ and $b=d$. This shows there are no other choices for c and d other than $c=a$, $d=b$. This shows the choice of a and b is <u>unique</u>. ∎

($\mathbb{Q}$ is a <u>field</u> in modern algebra.

$S = \{a+b\sqrt{2} : a,b \in \mathbb{Q}\}$ is a field extension.)

$$\frac{1}{2} + \frac{7}{2}\sqrt{2}$$

5.5 $\boxed{\text{Disproving Existence Statements}}$

Suppose we have an existence statement

There exist $x \in S$ such that $P(x)$.

To show this is false (disprove), we show the negation is true:

For all $x \in S$, $\sim P(x)$

<u>EX</u> Disprove: There exists odd integers a, b such that $3a^2 + 7b^2 \equiv 0 \pmod 4$.

Disproof: Let a and b be any odd integers. "For all $x \in S$"

Then $a = 2k+1$ and $b = 2\ell+1$ for some $k, \ell \in \mathbb{Z}$.

$$\begin{aligned} 3a^2 + 7b^2 &= 3(2k+1)^2 + 7(2\ell+1)^2 \\ &= 3(4k^2+4k+1) + 7(4\ell^2+4\ell+1) \\ &= 12k^2 + 12k + 3 + 28\ell^2 + 28\ell + 7 \\ &= 12k^2 + 12k + 28\ell^2 + 28\ell + 10 \\ &= 4(3k^2 + 3k + 7\ell^2 + 7\ell + 2) + 2 \end{aligned}$$

Since $3k^2+3k+7\ell^2+7\ell+2 \in \mathbb{Z}$, this shows $4 \nmid 3a^2+7b^2$.

So $3a^2 + 7b^2 \not\equiv 0 \pmod{4}$. ∎

What clue(s) could we give about a set to conclude that the set is $\{1,2,3,4,\dots\} = \mathbb{N}$?

Integers only

1 is in the set

If n is in the set, $n+1$ is in the set.

$\boxed{\text{Proof By Induction}}$

A style of proof for statements of the form

For all $n \in \mathbb{N}$, $P(n)$.

EX (a) For all $n \in \mathbb{N}$, $1+2+\cdots+n = \dfrac{n(n+1)}{2}$

(b) For all $n \in \mathbb{N}$, $3 \mid (7^n - 4^n)$.

(c) For all $n \in \mathbb{N}$, $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$

(d) For all $n \in \mathbb{N}$, $(1+a)^n \geq 1 + na$. $(a \geq 0)$

Result ~~Use the math~~ For all $n \in N$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof. First, it will be shown that this true when $n=1$.

Let $n=1$. Then

Base Step

$$1 + 2 + \cdots + n = 1$$

$$\frac{n(n+1)}{2} = \frac{1(2)}{2} = 1$$

So when $n=1$, $1 + \cdots + n = \frac{n(n+1)}{2}$.

Next, it will be shown that if $k \in N$ ~~is an~~ such that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$, then the statement is true when $n = k+1$.

~~Let~~ Suppose k is an integer such that the statement is true when $n = k$. Then

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Now when ~~k=~~ $n = k+1$,

$$
\begin{aligned}
1+2+\cdots+k+(k+1) &= (1+2+\cdots+k)+(k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}. \\
&= \frac{n(n+1)}{2}.
\end{aligned}
$$

So by induction, the statement is true for all $n \in \mathbb{N}$. $\blacksquare$