Comments on Exam 1

1(b) $\{2z+1 : z \in \mathbb{Z}\}$

The elements of this set are those numbers $2z+1$ where $z = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

$$\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$$

2(b) $\{\ldots, -4, -2, 0, 2, 4 \ldots\} = \{x \in \mathbb{Z} : P(x)\}$

$\{x \in \mathbb{Z} : 2x\}$ has no meaning. $2x$ is not a "property". We want to pick those integers that have the form $\ldots, -4, -2, 0, 2, \ldots$

$$\{x \in \mathbb{Z} : x \text{ is even}\} \text{ or } \{x \in \mathbb{Z} : x = 2k \text{ for some } k \in \mathbb{Z}\}$$

are acceptable ways to write the set.

7. "$P(1) = 5(1)+3 = 8$ False" has no meaning.

- First, $P(1)$ is the statement "$P(1)$: $5(1)+3$ is prime".
- "$5(1)+3 = 8$" is a true statement.
  "$5(1)+3$ is prime" is the false statement
- Don't use equal signs to write statements.

4.3 Proofs with Real Numbers

<u>Result</u> If $x^3+2x+1 \leq 0$, then $x \leq 0$.
Let $x \in \mathbb{R}$.

Proof. ~~Suppose $x \in \mathbb{R}$ with $x>0$~~. We will show that the contrapositive is true. Suppose $x \in \mathbb{R}$ with $x>0$. Since $x>0$, we have $x^3>0$, $2x>0$, and $1>0$. So

$$x^3+2x+1>0$$

as well. ■

<u>Result</u> Let $x \in \mathbb{R}$. Then $|2x-1|<7$ if and only if $-3<x<4$.

(If $|2x-1|<7$, then $-3<x<4$. $(\Rightarrow)$
If $-3<x<4$, then $|2x-1|<7$. $(\Leftarrow)$)

Proof. $(\Rightarrow)$ Suppose $x \in \mathbb{R}$ with $|2x-1|<7$. So

$$-7<2x-1<7$$

Then

$$-6<2x<8$$

So $-3<x<4$.

($\Leftarrow$) Suppose $x \in \mathbb{R}$ with $-3 < x < 4$. Then

$$-6 < 2x < 8$$

So

$$-7 < 2x-1 < 7.$$

Then $|2x-1| < 7$. $\blacksquare$

Therefore, $|2x-1| < 7$ if and only if $-3 < x < 4$. $\blacksquare$

<u>Result</u> For all positive real numbers a and b, $\sqrt{ab} \overset{?}{\leq} \frac{a+b}{2}$.

Aside: "If a and b are positive real numbers then $\sqrt{ab} \leq \frac{a+b}{2}$."

$$\sqrt{ab} \leq \frac{a+b}{2}$$

$$2\sqrt{ab} \leq a+b$$

$$4ab \leq (a+b)^2$$

$$4ab \leq a^2+2ab+b^2$$

$$0 \leq a^2+2ab+b^2-4ab$$

$$0 \leq a^2-2ab+b^2$$

$$0 \leq (a-b)^2$$

Proof. Suppose a and b are positive real numbers. Then $(a-b)^2 \geq 0$. So

$$a^2 - 2ab + b^2 \geq 0.$$

Then

$$a^2 - 2ab + b^2 + 4ab \geq 4ab$$

So

$$a^2 + 2ab + b^2 \geq 4ab.$$

Then

$$(a+b)^2 \geq 4ab$$

$$\sqrt{a^2} = |a|$$

So

$$|a+b| \geq 2\sqrt{ab}$$

Since $a > 0$ and $b > 0$, we know $a+b > 0$. So

$$a+b \geq 2\sqrt{ab}$$

Therefore $$\frac{a+b}{2} \geq \sqrt{ab}.$$ ∎

<u>Triangle Inequality</u> For any real numbers a and b,

$$|a+b| \leq |a| + |b|.$$

Aside: $a = 4 \quad b = -2$

$$\begin{array}{ccc} |4+(-2)| & & |4|+|-2| \\ 2 & \leq & 4+2 \end{array}$$

$a = 5 \quad b = 7$

$$\begin{array}{ccc} |5+7| & & |5|+|7| \\ 12 & = & 5+7 \end{array}$$

Note that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \qquad \text{Two Cases}$$

has two cases ($x \geq 0$ and $x < 0$). So the Triangle Inequality

$$|a+b| \leq |a| + |b|$$

has multiple cases

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} \qquad |b| = \begin{cases} b & \text{if } b \geq 0 \\ -b & \text{if } b < 0 \end{cases}$$

$$|a+b| = \begin{cases} a+b & \text{if } a+b \geq 0 \\ -(a+b) & \text{if } a+b < 0 \end{cases}$$

There are eight total cases:

$$\boxed{a \geq 0 \qquad b \geq 0 \qquad a+b \geq 0}$$

$\cancel{a \geq 0 \qquad b \geq 0 \qquad a+b < 0}$ ← This case will never happen

$a \geq 0 \qquad b < 0 \qquad a+b \geq 0$

These two are essentially the same. We can combine with a "Assume without loss of generality"

$a \geq 0 \qquad b < 0 \qquad a+b < 0$

$a < 0 \qquad b \geq 0 \qquad a+b \geq 0$

Same here

$a < 0 \qquad b \geq 0 \qquad a+b < 0$

$\cancel{a < 0 \qquad b < 0 \qquad a+b \geq 0}$ ← This case will never happen

$$\boxed{a < 0 \qquad b < 0 \qquad a+b < 0}$$

Proof. Suppose $a, b \in \mathbb{R}$.

Case 1 If $a \geq 0$, $b \geq 0$...

Case 2 If $a < 0$, $b < 0$...

Case 3 If $a \geq 0$, $b < 0$, and $a+b \geq 0$...

Case 4 If $a \geq 0$, $b < 0$, and $a+b < 0$...

See back ~~for~~ to finish proof

4.5 Proofs Involving Sets

Proof That One Set is a Subset of Another

Suppose A and B are sets. $A \subseteq B$

means "For every $x \in A$, $x \in B$."

OR "If $x \in A$, then $x \in B$."

So to show $A \subseteq B$, we start with "Suppose $x \in A$."

EX $A = \{x \in \mathbb{Z} : 12 \mid x\}$

$B = \{x \in \mathbb{Z} : 3 \mid x\}$

Prove that $A \subseteq B$.

Proof. ~~Let~~ Suppose $x \in A$. Then $12 \mid x$, so

$$x = 12k$$

for some $k \in \mathbb{Z}$. So $x = 3(4k)$ where $4k \in \mathbb{Z}$.

This shows $3 \mid x$, therefore $x \in B$. ■

Proofs that Two Sets are Equal

Suppose A and B are two sets. $A = B$ means

$$A \subseteq B \quad \text{and} \quad B \subseteq A.$$

To show this, we can prove

"If $x \in A$, then $x \in B$."

"If $x \in B$, then $x \in A$."

EX $A = \{x \in \mathbb{Z} : 10 \mid x\}$

$B = \{x \in \mathbb{Z} : 5 \mid x\}$

$C = \{x \in \mathbb{Z} : 2 \mid x\}$

Prove that $A = B \cap C$.

Aside: We need to show

If $x \in A$, then $x \in B \cap C$

If $x \in B \cap C$, then $x \in A$.