Proof Using Contrapositive

One more example

EX Let $x, y \in \mathbb{Z}$. Prove that if xy is odd, then x and y are both odd.

Proof. We will show that the contrapositive is true. Suppose x and y are not both odd. Then x is even or y is even.

Case ① If x is even, then $x = 2k$ for some $k \in \mathbb{Z}$. Then

$$xy = (2k)y = 2(ky).$$

Since $ky \in \mathbb{Z}$, this shows xy is even.

Case ② If y is even, then $y = 2\ell$ for some $\ell \in \mathbb{Z}$. Then

$$xy = x(2\ell) = 2(x\ell).$$

Since $x\ell \in \mathbb{Z}$, this shows xy is even.

Therefore, in either case xy is even ∎

OR Notice that Case 1 + Case 2 were almost identical.

Proof. We will show that the contrapositive is true. Suppose x and y are not both ~~even~~ odd. So either x is even or y is even.

This enables us to do both cases at the same time

Assume without loss of generality, that x is even. Then $x = 2k$ for some $k \in \mathbb{Z}$.

. . .

So xy is even. ■

"Same parity"

x and y have the same parity if they are both even or both odd.

4.1 $\boxed{\text{Divisibility of Integers}}$

2 divides 18 $\quad \frac{18}{2} = 9 \in \mathbb{Z} \quad 18 = 2 \cdot 9$

2 does not divide 7 $\quad \frac{7}{2} \notin \mathbb{Z} \quad 7 \neq 2 \cdot k$

<u>Definition</u> For integers a and b with $a \neq 0$, we say that <u>a divides b</u> if there exists an integer k such that $b = k \cdot a$.

In this case, we write $a \mid b$.

<u>Ex</u> (a) $2 \mid 10 \quad 10 = k \cdot 2 \quad k \in \mathbb{Z}$ True

(b) $3 \mid 27 \quad 27 = k \cdot 3 \quad k \in \mathbb{Z}$ True

(c) $7 \mid -28 \quad -28 = \ell \cdot 7 \quad \ell \in \mathbb{Z}$ True

(d) $7 \mid 15 \quad 15 = c \cdot 7 \quad c \in \mathbb{Z}$ False.

Note: $7 \nmid 15$

<u>Note</u> If $a \mid b$, we will also say that <u>b is a multiple of a</u>. Also <u>a is a divisor (integer) of b</u>.

<u>Result 1</u> Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$ and $b \neq 0$. If $a | b$ and $b | c$, then $a | c$.

Aside: $2|4$ and $4|8$ then $\overset{?}{} 2|8$

Proof. Suppose $a, b, c \in \mathbb{Z}$ with $a \neq 0$ and $b \neq 0$, and $a|b$ and $b|c$. So

$$b = k \cdot a \ , \quad c = l \cdot b$$

for some $k, l \in \mathbb{Z}$. Then

$$c = l \cdot b = l \cdot ka = (lk)\, a.$$

Since $lk \in \mathbb{Z}$, so $a | c$. ∎

<u>Result 2</u> Let $a, b, c,$ and d be integers with $a \neq 0$ and $b \neq 0$. If $a \mid c$ and $b \mid d$, then $ab \mid cd$.

Aside: $\frac{c}{a} \in \mathbb{Z}$ $\frac{d}{b} \in \mathbb{Z}$ then$\overset{?}{}$ $\frac{(cd)}{(ab)} \in \mathbb{Z}$.

$\frac{10}{2} \in \mathbb{Z}$ $\frac{8}{4} \in \mathbb{Z}$ then$\overset{?}{}$ $\frac{80}{8} \in \mathbb{Z}$

Proof Let $a, b, c,$ and d be integers with $a \neq 0$ and $b \neq 0$, and $a \mid c$ and $b \mid d$. So

$$c = ka \quad , \quad d = \ell \cdot b$$

for some integers k and ℓ. Then

$$cd = (ka)(\ell b) = (k\ell)(ab)$$

where $k\ell \in \mathbb{Z}$. So $ab \mid cd$. ■

Result 4 Let $x \in \mathbb{Z}$. If $2 \mid (x^2-1)$, then $4 \mid (x^2-1)$.

Aside: If $2 \mid 6$ then $4 \mid 6$ False

If $2 \mid 15$ then $4 \mid 15$ True

If $2 \mid 8$ then $4 \mid 8$ True

Proof. Suppose $x \in \mathbb{Z}$ and $2 \mid (x^2-1)$.

Then $x^2 - 1 = k \cdot 2$ for some $k \in \mathbb{Z}$.

So

$$x^2 = 2k + 1$$

showing that x^2 is odd. Then x is also odd, so

$$x = 2\ell + 1$$

for some $\ell \in \mathbb{Z}$. Then

$$\begin{aligned} x^2 - 1 &= (2\ell+1)^2 - 1 \\ &= 4\ell^2 + 4\ell + 1 - 1 \\ &= 4\ell^2 + 4\ell \\ &= 4(\ell^2 + \ell). \end{aligned}$$

Since $\ell^2 + \ell \in \mathbb{Z}$, so $4 \mid (x^2-1)$. $\blacksquare$

Result 5 Let $x, y \in \mathbb{Z}$. If $3 \nmid xy$, then $3 \nmid x$ and $3 \nmid y$.

Aside: If $3 \nmid (8)$ then $3 \nmid 2$ and $3 \nmid 4$.

Proof. We will show the contrapositive statement is true. Let $x, y \in \mathbb{Z}$ and suppose it is not the case that $3 \nmid x$ and $3 \nmid y$. Then either $3 \mid x$ or $3 \mid y$. Assume without loss of generality, that $3 \mid x$. So

$$x = c \cdot 3$$

for some $c \in \mathbb{Z}$. Then

$$xy = (3c)y = 3(cy)$$

where $cy \in \mathbb{Z}$. So $3 \mid xy$. ■

Result 6 Let $x \in \mathbb{Z}$. If $3 \nmid (x^2-1)$, then $3 \mid x$.

(C.P. If $3 \nmid x$, then $3 \mid (x^2-1)$.)

Proof. ~~So.~~ We will show that the contrapositive statement is true. Let $x \in \mathbb{Z}$ and suppose $3 \nmid x$. So either

$$\cancel{3}\ x = 3q+1 \quad \text{or} \quad x = 3q+2$$

for some $q \in \mathbb{Z}$.

Case ① If $x = 3q+1$, then

$$\begin{aligned} x^2-1 &= (3q+1)^2 - 1 \\ &= 9q^2 + 6q + 1 - 1 \\ &= 9q^2 + 6q \\ &= 3(3q^2+2q). \end{aligned}$$

Since $3q^2+2q \in \mathbb{Z}$, this shows $3 \mid x^2-1$.

Case ② If $x = 3q+2$, then

$$\begin{aligned} x^2-1 &= (3q+2)^2 - 1 \\ &= 9q^2 + 12q + \overbrace{4-1}^{3} \\ &= 3(3q^2 + 4q + 1) \end{aligned}$$

Since $3q^2+4q+1 \in \mathbb{Z}$, $3 \mid (x^2-1)$.

Therefore, in either case $3 \mid (x^2-1)$ ∎