Problem # 1.28

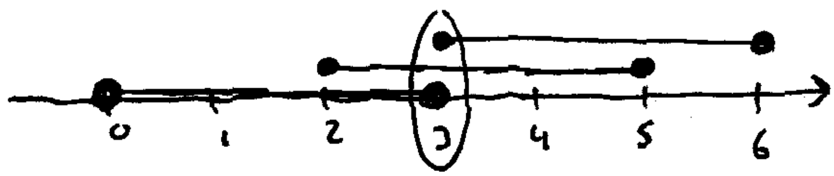
$$S_r = [r-1, r+2]$$

$$A = [1,3,4]$$

$$\bigcup_{\alpha \in A} S_\alpha = S_1 \cup S_3 \cup S_4$$

$$= [0,3] \cup [2,5] \cup [3,6]$$

$$= [0,6]$$



$$\bigcap_{\alpha \in A} S_\alpha = S_1 \cap S_3 \cap S_4 = \{3\}$$

$$\{1,2,3\} \cap \{4,5,6\} \stackrel{?}{=} \cancel{\{\emptyset\}}\ \emptyset$$

Chapter 3 - Proofs!

A <u>proof</u> a statement is a logically valid argument showing that the statement is true.

<u>Prelimary Example</u> Prove that the sum of two even integers is even.

Proof. Suppose x and y are two even integers. Then there exists integers n and m such that

$$x = 2n, \quad y = 2m.$$

Then

$$x + y = 2n + 2m = 2(n+m).$$

Since $(n+m)$ is also an integer, this shows $x+y$ is also even. ∎

1. Use complete sentences.
2. Explain all symbols/variables.
3. ~~Never~~ Alway begin a senteme with a word, not a symbol.
4. Avoid symbols $\exists$, $\forall$, $\wedge$, $\vee$, $\sim$, $\Rightarrow$ in the writing.

3.2 Direct Proofs

Suppose we wish to show that a statement

$$\text{For all } x \in S, \; P(x) \Rightarrow Q(x)$$

is true. So we need to verify that

$$P(x) \Rightarrow Q(x)$$

is a true statement for each $x \in S$. Recall that

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- If $x \in S$ is an element such that $P(x)$ is false, then $P(x) \Rightarrow Q(x)$ is <u>true</u>.

  So proofs don't need to consider this case.

- If $x \in S$ is an element such that $P(x)$ is true, then we have work to do to show that $Q(x)$ is true as well.

<u>Definition</u> An integer n is <u>even</u> if there exists an integer K such that $n = 2K$.

An integer m is <u>odd</u> if there exists an integer l such that $m = 2l+1$.

<u>Ex</u> Prove that: If (n is an odd integer) then $3n+7$ is an even integer.

Proof. Suppose n is an odd integer. Then there exists an integer k such that

$$n = 2k+1.$$

Then

$$3n+7 = 3(2k+1)+7 = 6k+3+7 = 2(3k+5).$$

Since $(3k+5)$ is an integer, $3n+7$ is even. ∎

<u>Ex</u> Prove that: If (a and b are odd integers), then ab is also odd.

Proof. Suppose a and b are odd integers. Then there exist integers $\underline{k}$ and $\underline{l}$ such that

$$a = 2k+1, \quad b = 2l+1.$$

Then

$$\begin{aligned} ab &= (2k+1)(2l+1) \\ &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1. \end{aligned}$$

Since $(2kl+k+l)$ is another integer, ab must be odd. ∎

EX Prove: If $x \in \mathbb{R}$ with $x^2-4x=-4$,

then $13x$ is an even integer.

Proof. Suppose $x \in \mathbb{R}$ with $x^2-4x=-4$. Then

$$x^2-4x+4=0$$

so

$$(x-2)^2=0.$$

This tells us that $x=2$. Then

$$13x = 13\cdot 2 = 2\cdot 13$$

and so $13x$ is an even number. ■

3.1 $\boxed{\text{Trivial and Vacuous Proofs}}$

<u>Trivial Proof</u> Suppose we wish to prove

For $x \in S$, $P(x) \Rightarrow Q(x)$,

In some (rare), $Q(x)$ is already true

for all $x \in S$, regardless if $P(x)$ is true or not.

Then the statement is automatically true

<u>EX</u> Prove: Let $x \in \mathbb{R}$. If $x < 0$, then $x^2 + 1 > 0$.

Proof. For any $x \in \mathbb{R}$, $x^2 \geq 0$. So

$$x^2 + 1 \geq 0 + 1$$

then $x^2 + 1 > 0$. ■

Vacuous Proofs Suppose we wish to prove

For all $x \in S$, $P(x) \Rightarrow Q(x)$.

In some cases (rare), $P(x)$ is false for all $x \in S$.

Then the statement is true.

EX Prove: Let $x \in \mathbb{R}$. If $x^2 - 2x + 2 \le 0$, then $x^3 > 8$.

Proof. First note

$$(x^2 - 2x + 1) + 1 \le 0$$

So

$$(x-1)^2 \le -1.$$

There is no value of $x \in \mathbb{R}$ for which $x^2 - 2x + 2 \le 0$.

So this statement is true vacuously. ■