2.8 $\boxed{\text{Logical Equivalence}}$

EX $P \Rightarrow Q \equiv (\sim P) \vee Q$

EX If 17 is prime, then 25 is a perfect square.

~~IS~~ has the same truth value as

17 is not prime or 25 is a perfect square.

This statement is true because 25 is a perfect square.

So the original If... then statement is true.

EX Show that $P \Rightarrow Q$ is not equivalent to $Q \Rightarrow P$.

$P \Rightarrow Q$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$Q \Rightarrow P$

P	Q	$Q \Rightarrow P$
T	T	T
T	F	T
F	T	F
F	F	T

The truth values for $P \Rightarrow Q$ are not the same as $Q \Rightarrow P$.

2.9 $\boxed{\text{Other Important Logical Equivalences}}$

For any statements P, Q, and R

1. $\sim(\sim P) \equiv P$

2. $P \vee Q \equiv Q \vee P$, $P \wedge Q \equiv Q \wedge P$

3. $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$ $\quad P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$

4. $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

   $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

Negation of AND, OR

For any statements P, Q

1. $\sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q)$

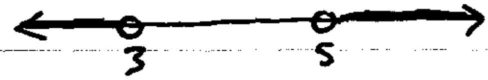
2. $\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$

Justifying 1

P	Q	$(P \vee Q)$	$\sim(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

P	Q	$\sim P$	$\sim Q$	$(\sim P) \wedge (\sim Q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Ex Write negations of the following.

(a) $X > 5$ or $X < 3$ 

$X \leq 5$ and $X \geq 3$ 

(b) a is even and b is odd

a is odd or b is even

2.10 $\boxed{\text{Quantified Statements}}$

Quantifiers → For every $x \in \mathbb{Z}$, $2x$ is even ← Open Sentences

There exists $x \in \mathbb{R}$ such that $2x - 1 = 5$.

<u>The Universal Quantifier</u> "For every"

"For every..." "For all" "For each" $\forall$

Suppose $P(x)$ is an open sentence over domain S.

$$\text{For every } x \in S,\ P(x).$$

is called a universally quantified statement.

- True if $P(x)$ is a true statement for every $x \in S$.
- False if $P(x)$ is a false statement for at least one $x \in S$.

<u>Ex</u> (a) For every $n \in \mathbb{N}$, $2n$ is even.

$2n$ is even for every $n \in \mathbb{N}$, so this statement is true.

(b) $\forall\, x \in \mathbb{R},\ |x| = x$. This is false.

$x = -4 \quad |-4| = 4 \neq -4$

The Existential Quantifier "There exists"

"There exists..." "There is..." "For some.." $\exists$

Suppose $P(x)$ is an open sentence over domain S.

There exists $x \in S$, such that $P(x)$

$$\exists\ x \in S,\ P(x)$$

is called an ~~ext~~ existentially quantified statement.

- True if $P(x)$ is true for at least one $x \in S$
- False if $P(x)$ is false for all $x \in S$.

EX (a) There exists $x \in \{1,3,5,7\}$ such that $\frac{2n^2+5+(-1)^n}{2}$ is prime.

$x=1 \qquad \frac{2(1)^2+5+(-1)^1}{2} = \frac{6}{2} = 3$ prime.

So this is a true statement.

(b) $\exists\ x \in \mathbb{R},\ |x| = -x.$

$x = -2 \qquad |-2| = 2 = -(-2)$

So this is a true statement.

The Negation of Quantifiers

Prelimary Examples:

(a) For every $n \in \mathbb{Z}$, $2n$ is even.
There exists $n \in \mathbb{Z}$, $2n$ is odd.

(b) There exists $x \in \mathbb{R}$ such that $x^2 = -1$.
For every $x \in \mathbb{R}$, $x^2 \neq -1$.

Suppose $P(x)$ is an open sentence over domain S.

$$\sim\left(\forall\, x \in S,\ P(x)\right) \equiv \exists\, x \in S,\ \sim P(x).$$

$$\sim\left(\exists\, x \in S,\ P(x)\right) \equiv \forall\, x \in S,\ \sim P(x)$$

<u>EX</u> (a) For every $x > 0$, $x^2 + 1 > 0$.

Negation There exist $x > 0$, $x^2 + 1 \leq 0$

(b) Suppose $A = \{1,2,3\}$. Let $P(A)$ be the power set of A. Consider

For every $B \in P(A)$, $A - B = \emptyset$.

Negation

There exists $B \in P(A)$, such that $A - B \neq \emptyset$.

Consider $B = \{1\}$. $A - B = \{1,2,3\} - \{1\} = \{2,3\} \neq \emptyset$

So this is true.

~~So t~~

So the original statement is false.

(c) There exists a natural number x such that $x^2 = 2$.

Negation

For every natural number x, $x^2 \neq 2$.

This is true,

So the original statement is false.

Universal Quantifiers as Implications

For every $x \in S$, $P(x)$.

is equivalent to

If $x \in S$, then $P(x)$.

<u>EX</u> For every $n \in \mathbb{N}$, $2n$ is even.

Equivalent: If $n \in \mathbb{N}$, then $2n$ is even.

Quantified Statements with More than One Variable

<u>EX</u> <u>For every</u> $s \in \{1,3\}$ and $t \in \{3,9\}$, $st+2$ is prime.

$s=1, t=3$ $\quad st+2 = 5$ is prime.
$s=1, t=9$ $\quad st+2 = 11$ is prime.
$s=3, t=3$ $\quad st+2 = 11$ is prime
$s=3, t=9$ $\quad st+2 = 29$ is prime.

This statement is true. The negation is

There exists $s \in \{1,3\}$ and $t \in \{3,9\}$ such that $st+2$ is not prime.