Logical Connectives

not P $\quad\quad$ $\sim P$

P	$\sim P$
T	F
F	T

P and Q $\quad\quad$ $P \wedge Q$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P or Q $\quad\quad$ $P \vee Q$

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P implies Q

If P, then Q $\quad\quad$ $P \Rightarrow Q$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(Example Continued from Tuesday: For which $x \in \{0,1,2,3\}$ is $(x>1) \Rightarrow (3x-8 \geq 0)$ a true statement?

$P(x): \; x>1 \qquad Q(x): \; 3x-8 \geq 0$

$x=2$. $\quad P(2): \; 2>1 \qquad Q(2): \; -2 \geq 0$

If $\underbrace{2>1}_{\text{True}}$, then $\underbrace{-2 \geq 0}_{\text{False}}$. False

$x=3 \quad P(3): \; 3>1 \qquad Q(3): \; 1 \geq 0$

If $\underbrace{3>1}_{\text{True}}$, then $\underbrace{1 \geq 0}_{\text{True}}$. True

$P(x) \Rightarrow Q(x)$ is True when $x \in \{0,1,3\}$.

Premise and Conclusion

In an implication,

$$P \Rightarrow Q \qquad P(x) \Rightarrow Q(x)$$

P is called the premise or hypothesis.

Q is called the conclusion.

2.6 $\boxed{\text{The Biconditional}}$

The Converse of a Implication

The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.

Ex If $2x-1>0$, then $x>0$.

Converse: If $x>0$, then $2x-1>0$.

The Biconditional

Suppose P and Q are two statements (open sentences)

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

is called a biconditional statement, written

$P \Leftrightarrow Q$
P is equivalent to Q
P if and only if Q

Ex P: 10 is even $\qquad Q$: $10 = 2 \cdot 5$

$$P \Leftrightarrow Q$$

If 10 is even, then $10=2\cdot 5$ and If $10=2\cdot5$, then 10 is even.

10 is even if and only if $10=2\cdot5$.

Truth Table $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

So the truth table for $P \Leftrightarrow Q$ is

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$P \Leftrightarrow Q$ is True exactly when Q and P have the same truth value. (both T or both F).

EX (a) (3 is even) if and only if (9 is prime)
False False

This statement is true.

(b) $x = -3 \Leftrightarrow |x| = 3$ domain $\mathbb{Z}$

True for which values of x?

when $x = -3$ $\quad \underset{T}{-3 = -3} \Leftrightarrow \underset{T}{|-3| = 3}$ True

Also true when x is any integer other than 3.

2.7 Compound Statements, Tautologies, Contraditions

<u>EX</u> If $(3^2=9)$ and (7 is odd) then (5 is prime) or (7 is even)

This is a <u>compound statement</u>, that is, a statement made up of smaller component statements, and one or more of $\sim, \vee, \wedge, \Rightarrow, \Leftrightarrow$

$P: 3^2=9$ $\quad Q$: 7 is odd $\quad R$: 5 is prime

$\sim Q$: 7 is even

$$(P \wedge Q) \Rightarrow (R \vee \sim Q)$$

Is this statement True or False?

P is True, Q is True, so $P \wedge Q$ is True.

R is True, $\sim Q$ is False, so $R \vee \sim Q$ is True.

So

$$\underset{T}{(P \wedge Q)} \Rightarrow \underset{T}{(R \vee \sim Q)}$$

is TRUE.

A compound statement is a tautology if it is true for all possible truth values of its component statements.

EX $P \vee (\sim P)$ is a tautology.

P	$\sim P$	$P \vee (\sim P)$
T	F	T
F	T	T

All possible truth values of P

Always True

EX The 1000^{th} digit of π is 7 or the 1000^{th} digit of π is not 7.

is True, even if we don't know the 1000^{th} digit of π.

A compound statement is a <u>contradiction</u> if it is false for every possible truth value of its component statements.

<u>EX</u> $P \wedge (\sim P)$ is a contradiction.

P	$\sim P$	$P \wedge (\sim P)$
T	F	F
F	T	F

Always False

EX $\big((P \Rightarrow Q) \wedge (Q \Rightarrow R)\big) \Rightarrow (P \Rightarrow R)$

[See truth table on next page]

This is a tautology. It is always true regardless of the truth values of P, Q, R.

Note: If $x > 3$, then $x > 2$. $\quad P \Rightarrow Q$

If $x > 2$, then $x > 1$. $\quad Q \Rightarrow R$

So

If $x > 3$, then $x > 1$. $\quad P \Rightarrow R$.

This is called a <u>Syllogism</u>.

EX (A HW problem) $\Big(\big((P \Rightarrow Q) \wedge P\big)\Big) \Rightarrow Q$

Note: If $x > 2$, then $x > 1$. $\quad P \Rightarrow Q$

$x > 2 \quad P$

So

$x > 1$. $\quad Q$

This is called modus pones.

Note If P is a tautology, then $\neg P$ is a contradiction.

and

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q, \wedge Q \Rightarrow R$	$P \Rightarrow R$	$((P \Rightarrow Q \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

All possible truth values

Always true

2.8 Logical Equivalence

Ex $R: P \Rightarrow Q$ $\qquad$ $S: (\sim P) \vee Q$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\sim P$	$(\sim P) \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Same

In this case, we say R and S are logically equivalent, written

$$R \equiv S$$