3.1 Linear Models

Growth and decay

\[
\frac{dp}{dt} = kp, \quad p(t_0) = p_0
\]

R: constant >0 growth < 0. decay

Solution

\[
\frac{dp}{dt} = (b - d)p
\]

\[
\frac{dp}{dt} = kp, \quad p(0) = p_0
\]

Can solve as seapndle eq., linear eq.,

Solving as a linear equation,

\[
\frac{dp}{dt} - kp = 0
\]

Integrations factor \( \mu = e^{\int -kd t} = e^{-kt} \)
\[
e^{-kt} \frac{dp}{dt} - ke^{-kt} p = 0 \cdot e^{-kt}
\]
\[
(e^{-kt} \cdot p)' = 0
\]
\[
e^{-kt} \cdot p = C
\]
\[
p = C e^{kt}
\]

At \( t = 0, p = p_0 : p_0 = C e^{k \cdot 0} = C \)

The population at time is \( p = p_0 e^{kt} \)

Example: 1 Bacterial Growth.

\[
\frac{dp}{dt} = kp. \quad p(0) = p_0
\]

\[
t = 1 \text{how} \quad p = \frac{3}{2} p_0
\]

\[
t = ? \quad p(t) = 3p_0
\]

Solution.

\[
p = p_0 e^{k \cdot t}
\]

\[
\frac{3}{2} p_0 = p_0 e^{k \cdot 1} \quad k = \ln \frac{3}{2} = 0.4055
\]

\[
3p_0 = p_0 e^{0.4055 \cdot t}
\]

\[
t = \frac{\ln 3}{0.4055} \approx 2.71h
\]
Radioactive Decay: \[
\frac{dA}{dt} = kA. \quad A(0) = A_0
\]

half-life: the time it takes for one-half of the radioactive atoms to transmute into the atoms of another element.

Uranium U-238. \(4.5 \times 10^9\) years. 4.5 billion.

Radium Ra-226 \(1700\) years

Isotope carbon C-14 \(5600\) years

**Carbon dating:**

**EX:** Age of a Fossil

A fossilized bone is found to contain one-thousands of the C-14 level found in living matter:

Estimate the age of the fossil.
\[ A(t) = A_0 e^{kt} \]
\[ \frac{1}{2} A_0 = A_0 e^{k5600} \]
\[ 5600k = -\ln 2, \]
\[ k = \frac{-(ln2)}{5600} = -0.00012378 \]
\[ A(t) = A_0 e^{-0.00012378t} \]
\[ \frac{1}{1000} A_0 = A_0 e^{-0.00012378t} \]
\[ t = \frac{\ln 1000}{0.00012378} \approx 55,800 \text{ y.} \]

Series Circuits.

LR circuit

![LR series circuit](image)

Fig 3.5 on page 96 Kirchhoff’s second law:

\[ L \frac{di}{dt} + Ri = E(t) \]
RC: Fig 3.6 on page 96

\[
Ri + \frac{1}{C}q = E(t) \\
R \frac{dq}{dt} + \frac{1}{C}q = E(t)
\]

Solution to \( L \frac{di}{dt} + Ri = E(t) \)

\[
i(t) = e^{-\frac{R}{L}t} \int e^{\left(\frac{R}{L}\right)t} E(t) dt + Ce^{-\left(\frac{R}{L}\right)+} \\
E = E_0 \\
i(t) = \frac{E_0}{R} + Ce^{-\left(\frac{R}{L}\right)+} \\
\left/ \right/ \quad \left/ \right/ \\
steady\ state\ transient\ term
\]

\[
\lim_{t \to \infty} i(t) = \frac{E_0}{R}
\]