2.4 Exact Equations

In general, the differential form of a $1^{st}$ order D.E. is

$$M(x, y)dx + N(x, y)dy = 0$$

**EX**

$$(x + y)dx + 3ydy = 0$$

In derivative form,

$$3y(dy) = -(x + y)(dx)$$

$$3y \frac{dy}{dx} = -(x + y)$$

$$\frac{dy}{dx} = -\frac{(x + y)}{3y}$$
Partial Derivatives of $f(x, y)$

If $f(x, y)$ is a function that depends on $x$ and $y$, then

$$\frac{\partial f}{\partial x} = \text{the derivative of } f, \text{ with respect to } x,$$

treating $y$ as a constant.

$$\frac{\partial f}{\partial y} = \text{the derivative of } f, \text{ with respect to } y,$$

treating $x$ as a constant.

**EX**

(a) $f(x, y) = x^2 + y^3$

$$\frac{\partial f}{\partial x} = 2x + 0 = 2x$$

$$\frac{\partial f}{\partial y} = 0 + 3y^2 = 3y^2$$

(b) $g(x, y) = x^2 e^{y^2} - \tan(y)$

$$\frac{\partial g}{\partial x} = 2x \cdot e^{y^2} - 0 = 2x e^{y^2}$$

$$\frac{\partial g}{\partial y} = x^2 \cdot e^{y^2} \cdot 2y - \sec^2(y)$$
The (Total) Differential of $f(x,y)$

$$df = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy$$

If $df = 0$

$$\left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy = 0$$

$$\downarrow \quad \downarrow$$

$$\frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial y} = 0$$

$$\downarrow \quad \downarrow$$

$f$ does not depend on $x$, but may depend on $y$

$$f(x, y) = C \quad (a \quad true \quad constant)$$
Exact Differential Equation

Suppose a first order D.E. has been written in differential form

\[ M(x, y)dx + N(x, y)dy = 0 \]

If it is possible to find a \( F(x, y) \) so that

\[ M = \frac{\partial F}{\partial x}, \quad N = \frac{\partial F}{\partial y} \]

Then

\[ M \, dx + N \, dy = 0 \]
\[
\frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy = 0
\]
\[ dF = 0 \]
\[ F(x, y) = C \]

\( F(x,y)=C \) is an implicit solution of the D.E.
\[ dy \over dx = -{(2xy^2 + 1) \over 2x^2y} \]

Not separable  \[ dy \over dx \neq g(x)h(y) \]

\[ 2x^2y{dy \over dx} = -(2xy^2 + 1) \]

\[ 2x^2y{dy \over dx} + 2xy^2 = -1 \]

Not linear.

\[ dy \over dx = -{(2xy^2 + 1) \over 2x^2y} \]

\[ 2x^2ydy = -(2xy^2 + 1)dx \]

\[(2xy^2 + 1)dx + (2x^2y)dy = 0 \]

Differential Form

Is there an \( F(x,y) \) so that

\[ \partial F \over \partial x = 2xy^2 + 1 \Rightarrow F(x, y) = x^2y^2 + x + f(y) \]

\[ \partial F \over \partial y = 2x^2y \Rightarrow F(x, y) = x^2y^2 + g(x) \]

Let’s pick \( g(x) = x \), \( f(y) = 0 \)
Then \( F(x, y) = x^2y^2 + x \) is a function so that

\[
dF = (2xy^2 + 1)dx + (2x^2y)dy = 0 \text{ (by D.E.)}
\]

So

\[
F(x, y) = C
\]

\[
x^2y^2 + x = C
\]

is an implicit soln. to D.E.
A Test for Exactness

\[ M(x, y)dx + N(x, y)dy = 0 \]

If \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \), then this equation is exact, meaning there is a \( F(x, y) \) so that

\[ dF = M(x, y)dx + N(x, y)dy \]

Note: Suppose

\[ Mdx + Ndy = 0 \]

We can find \( F \) so that

\[ \frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N \]

Then (by calc III)

\[ \frac{\partial }{\partial y} \left( \frac{\partial F}{\partial x} \right) = \frac{\partial M}{\partial y} = \]

\[ \frac{\partial }{\partial x} \left( \frac{\partial F}{\partial y} \right) = \frac{\partial N}{\partial x} = \]
Solving an Exact DE

Solve

\[ 2xydx + (x^2 - 1)dy = 0 \]

Solution: With \( M(x, y) = 2xy \) and \( N(x, y) = x^2 - 1 \) we have

\[ \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \]

Thus the equation is exact, and so, by Theorem 2.1, there exists a function \( f(x, y) \) such that

\[ \frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 1. \]

From the first of these equations we obtain, after integrating

\[ f(x, y) = x^2y + g(y). \]

Taking the partial derivative of the last expression with respect to \( y \) and setting the result equal to \( N(x, y) \) gives

\[ \frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \quad \leftarrow N(x, y) \]
It follows that \( g'(y) = -1 \) and \( g(y) = -y \). Hence \( f(x, y) = x^2y - y \), and so the solution of the equation in implicit form is \( x^2y - y = C \). The explicit form of the solution is easily seen to be \( y = c/(1 - x^2) \) and is defined on any interval not containing either \( x = 1 \) or \( x = -1 \).

**An Initial-Value Problem**

Solve

\[
\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2.
\]

Solution: By writing the differential equation in the form

\[
(\cos x \sin x - xy^2)dx + y(1 - x^2)dy = 0
\]

we recognize that the equation is exact because

\[
\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}
\]

Now

\[
\frac{\partial f}{\partial y} = y(1 - x^2)
\]

\[
f(x, y) = \frac{y^2}{2} (1 - x^2) + h(x)
\]

\[
\frac{\partial f}{\partial x} = -xy^2 + h'(x)
\]

\[
= \cos x \sin x - xy^2
\]
The last equation implies that \( h'(x) = \cos x \sin x \). Integrating gives

\[
h(x) = - \int (\cos x)(- \sin x) \, dx = - \frac{1}{2} \cos^2 x.
\]

Thus \( \frac{y^2}{2}(1 - x^2) - \frac{1}{2} \cos^2 x = C_1 \) or

\[
y^2(1 - x^2) - \cos^2 x = C \quad (7)
\]

where \( 2C_1 \) has been replaced by \( C \). The initial condition \( y = 2 \) when \( x = 0 \) demands that \( 4(1) - \cos^2(0) = C \), and so \( C = 3 \). An implicit solution of the problem is then \( y^2(1 - x^2) - \cos^2 x = 3 \).

The solution curve of the IVP is the curve drawn in color in Figure 2.29; it is part of an interesting family of curves. The graphs of the members of the one-parameter family of solutions given in (7) can be obtained in several ways, two of which are using software to graph level curves (as discussed in Section 2.2) and using a graphing utility to carefully graph the explicit functions obtained for various values of \( C \) by solving \( y^2 = (C + \cos^2 x)/(1 - x^2) \) for \( y \).