A completely integrable system and parametric representation of solutions of the Wadati–Konno–Ichikawa hierarchy

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A finite-dimensional involutive system is presented, and the Wadati–Konno–Ichikawa (WKI) hierarchy of nonlinear evolution equations and their commutator representations are discussed in this article. By this finite-dimensional involutive system, it is proven that under the so-called Bargmann constraint between the potentials and the eigenfunctions, the eigenvalue problem (called the WKI eigenvalue problem) studied by Wadati, Konno, and Ichikawa [J. Phys. Soc. Jpn. **47**, 1698 (1979)] is nonlinearized as a completely integrable Hamiltonian system in the Liouville sense. Moreover, the parametric representation of the solution of each equation in the WKI hierarchy is obtained by making use of the solution of two compatible systems. © *1995 American Institute of Physics.*

I. INTRODUCTION

Since the beautiful "nonlinearization theory" came into use,[1,2] quite a few completely integrable Hamiltonian systems in the Liouville sense have been successively found in recent years.[3–10] This theory is used to not only search for new finite-dimensional involutive systems in explicit form, but also provide the representations of solutions for the soliton hierarchy. This article is devoted to the investigation of the Wadati–Konno–Ichikawa (WKI) hierarchy by the use of the "nonlinearization theory," and is a continuation of a previous work.[10] It is well-known that the hierarchy of nonlinear evolution equations (called the WKI hierarchy below) related to the eigenvalue problem (called the WKI eigenvalue problem below) presented by Wadati, Konno, and Ichikawa were obtained about eleven years ago.[11] But discussions on the representations of solutions of the WKI hierarchy were very few. In this article, we first give an involutive system of functions, which guarantees integrability of the nonlinearized WKI eigenvalue problem under the so-called Bargmann constraint. Then, giving the WKI hierarchy of nonlinear evolution equations by the spectral gradient method[12] and their commutator representations, and using the solution of two compatible equations, we present the parametric representation of the solution for each equation in the WKI hierarchy.

II. A FINITE-DIMENSIONAL INVOLUTIVE SYSTEM $\{E_k\}_{k=1}^N$ AND HAMILTONIAN SYSTEMS (F_m)

Denote the standard Poisson bracket of two functions F and G on the symplectic manifold $(R^{2N}, dp\wedge dq)$ by[13]

$$(F,G)=\sum_{j=1}^{N}\left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j}-\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right)=\left\langle\frac{\partial F}{\partial q},\frac{\partial G}{\partial p}\right\rangle-\left\langle\frac{\partial F}{\partial p},\frac{\partial G}{\partial q}\right\rangle,$$

which is skew symmetric, bilinear, and satisfies the Jacobi identity and the Leibnitz rule: $(EF,H)=E(F,H)+F(E,H)$, where $\langle\cdot,\cdot\rangle$ stands for the standard inner product in R^{2N}.

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Let $\lambda_1,\ldots,\lambda_N$ be N different constants and $\lambda_1<\cdots<\lambda_N$. Write

$$\Gamma_k=\sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\lambda_k\lambda_j B_{kj}^2}{\lambda_k-\lambda_j}, \quad B_{kj}=p_k q_j - q_k p_j. \tag{1}$$

Then through a lengthy calculation, we obtain the following proposition.

Proposition 1: A system of functions $E_1,\ldots,E_N$ defined as follows consists of a new finite-dimensional involutive system:

$$E_k=-\langle\Lambda p,q\rangle p_k q_k+\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}\,p_k q_k+\tfrac{1}{2}\Gamma k, \quad k=1,2,\ldots,N, \tag{2}$$

where $q=(q_1,\ldots,q_N)^T$, $p=(p_1,\ldots,p_N)^T$, $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$, and $\langle\xi,\eta\rangle=\Sigma_{j=1}^N \xi_j\eta_j$.

Define a bilinear function $Q_z(\xi,\eta)$ on R^N as follows:[14]

$$Q_z(\xi,\eta)=\langle(z-\Lambda)^{-1}\xi, \quad \eta\rangle=\sum_{k=1}^{N}(z-\lambda_k)^{-1}\xi_k\eta_k. \tag{3}$$

Then the generated functions of the involutive system $\{E_k\}$ are

$$-\langle\Lambda p,q\rangle Q_z(p,q)+\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}\,Q_z(p,q)+\frac{1}{2}\begin{vmatrix} Q_z(\Lambda q,q) & Q_z(\Lambda q,p)\\ Q_z(\Lambda p,q) & Q_z(\Lambda p,p)\end{vmatrix}=\sum_{k=1}^{N}\frac{E_k}{z-\lambda_k}. \tag{4}$$

Proposition 2: Let $F_m=\Sigma_{k=1}^N \lambda_k^m E_k$, $m=0,1,2,\ldots$. Then we have

$$F_0=-\langle\Lambda p,q\rangle\langle p,q\rangle+\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}\,\langle p,q\rangle, \tag{5}$$

$$F_m=-\langle\Lambda p,q\rangle\langle\Lambda^m p,q\rangle+\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}\,\langle\Lambda^m p,q\rangle+\frac{1}{2}\sum_{j=0}^{m-1}\begin{vmatrix}\langle\Lambda^{j+1}q,q\rangle & \langle\Lambda^{j+1}q,p\rangle\\ \langle\Lambda^{m-j}p,q\rangle & \langle\Lambda^{m-j}p,p\rangle\end{vmatrix} \tag{6}$$

and $(F_m,F_l)=0$, $\forall m,l\in Z^+$.

Proof: Obviously, $(F_m,F_l)=0$. When $|z|>\max\{|\lambda_1|,\ldots,|\lambda_N|\}$, we get

$$(z-\lambda_k)^{-1}=\sum_{m=0}^{\infty} z^{-m-1}\lambda_k^m, \quad Q_z(\xi,\eta)=\sum_{m=0}^{\infty}\langle\Lambda^m\xi, \quad \eta\rangle z^{-m-1}.$$

Hence, Eqs. (5) and (6) are deduced by virtue of substituting the Laurent expansion of $Q_z(\xi,\eta)$ and the power series of $(z-\lambda_k)^{-1}$ into both sides of Eq. (5), and sorting terms with the power of z.

By virtue of this proposition, we promptly have

Theorem 1: The Hamiltonian systems

$$(F_m):\ q_{t_m}=\frac{\partial F_m}{\partial p}, \quad p_{t_m}=-\frac{\partial F_m}{\partial q}, \quad m=0,1,2,\ldots, \tag{7}$$

are completely integrable in the Liouville sense.

III. THE WKI HIERARCHY AND COMMUTATOR REPRESENTATIONS

In this section, we first give the WKI hierarchy of nonlinear evolution equations by the spectral gradient $\nabla\lambda$ of eigenvalue λ with respect to the potentials u and v, then present the commutator representation of each equation in the WKI hierarchy.

Consider the WKI eigenvalue problem[15]

$$y_x = My, \quad M = \begin{pmatrix} -i\lambda & \lambda u \\ \lambda v & i\lambda \end{pmatrix}, \quad i^2 = -1, \tag{8}$$

i.e.,

$$Ly = \lambda y, \quad L = L(u,v) = \frac{1}{1-uv}\begin{pmatrix} i & -u \\ -v & -i \end{pmatrix}\partial, \quad \partial = \frac{\partial}{\partial x}. \tag{9}$$

Let $y=(y_1,y_2)^T$ satisfy Eq. (9) and $\nabla\lambda = (\lambda y_2^2, -\lambda y_1^2)^T$. Then $\nabla\lambda$ satisfies

$$K\nabla\lambda = \lambda \cdot J\nabla\lambda, \tag{10}$$

where K and J are two skew-symmetric operators ($\partial = \partial/\partial x$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$)

$$K = \frac{1}{2i}\begin{pmatrix} -\frac{1}{2}\partial^2 \frac{u}{w}\partial^{-1}\frac{u}{w}\partial^2 & \partial^3 + \frac{1}{2}\partial^2\frac{u}{w}\partial^{-1}\frac{v}{w}\partial^2 \\ \partial^3 + \frac{1}{2}\partial^2\frac{v}{w}\partial^{-1}\frac{u}{w}\partial^2 & -\frac{1}{2}\partial^2\frac{v}{w}\partial^{-1}\frac{v}{w}\partial^2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -\partial^2 \\ \partial^2 & 0 \end{pmatrix}, \quad w = \sqrt{1-uv}, \tag{11}$$

which are called the pair of Lenard's operators of Eq. (8).

Now, recursively define the Lenard's gradient sequence $\{G_m\}_{m=-1}^{\infty}$ as follows:

$$\begin{cases} KG_{m-1} = JG_m, \quad (m=0,1,2,\ldots,) \\ G_{-1} = (1,1)^T \in \ker J, \quad G_0 = \left(i\frac{v}{w}, i\frac{u}{w} \right)^T, \end{cases} \tag{12}$$

which exactly generate the WKI hierarchy of the soliton equation

$$(u,v)_t^T = J(J^{-1}K)^m G_0, \quad m = 0,1,2,\ldots \tag{13}$$

Particularly, as $v=-1$, $(u,v)_t^T = KG_0 = JG_1$ can be reduced to the well-known Harry–Dyn equation $s_t = -(1/\sqrt{s})_{xxx}$ with $s = 1+u$. The WKI hierarchy of soliton equations (13) possess the following commutator (or Lax) representations:

$$L_t = [W_m, L], \quad m = 0,1,2,\ldots,$$

$$W_m = \sum_{j=0}^{m} \left\{ \frac{1}{2i}\begin{pmatrix} 0 & (G_{j-1,x}^{(2)} - iuA(G_{j-1}))_x \\ (G_{j-1,x}^{(1)} + ivA(G_{j-1}))_x & 0 \end{pmatrix} L^{m+1-j} + iA(G_{j-1})\begin{pmatrix} -i & u \\ v & i \end{pmatrix} L^{m+2-j} \right\}, \tag{14}$$

with $A(G_{j-1}) = (1/2iw)\partial^{-1}((u/w)G_{j-1,xx}^{(1)} - (v/w)G_{j-1,xx}^{(2)})$, $w = \sqrt{1-uv}$, where $G_{j-1} = (G_{j-1}^{(1)}, G_{j-1}^{(2)})^T$ $(j=0,1,2,\ldots,m)$ are determined by Eq. (12), L is defined by Eq. (9).

Remark: In Eq. (14), the operators W_m ($m=0,1,2,\ldots,$) are a set of isospectral Lax operators[16] of the WKI spectral problem (9), which can compose an operator Lie algebra. These results are going to be reported in another article.

IV. INTEGRABILITY OF THE NONLINEARIZED WKI SPECTRAL PROBLEM (8)

Let $\lambda_1,\ldots,\lambda_N$ be N different eigenvalues of Eq. (8), and $y=(q_j,p_j)^T$ be the eigenfunctions corresponding to λ_j ($j=1,2,\ldots,N$). The so-called Bargmann constraint (Ref. 3): $G_0=\Sigma_{j=1}^N \nabla\lambda_j$ reads

$$u=\frac{i\langle\Lambda q,q\rangle}{\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}},\quad v=\frac{-i\langle\Lambda p,p\rangle}{\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}}. \tag{15}$$

Proposition 3: Under the Bargmann constraint (15), the nonlinearized system of the WKI eigenvalue problem (9) can be expressed as the Hamiltonian structure

$$\begin{cases} q_x=-i\Lambda q+\dfrac{i\langle\Lambda q,q\rangle}{\sqrt{1+\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle}}\Lambda p=\dfrac{\partial H}{\partial p} \\[2ex] p_x=i\Lambda p-\dfrac{i\langle\Lambda p,p\rangle}{\sqrt{1+\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle}}\Lambda q=-\dfrac{\partial H}{\partial q}, \end{cases} \tag{16}$$

with the Hamiltonian function

$$H=-i\langle\Lambda p,q\rangle+i\sqrt{1+\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle}. \tag{17}$$

Proof: Obvious.

Proposition 4: The Hamiltonian function H and F_m are involutive, i.e.,

$$(H,F_m)=0,\quad \forall m\in Z^+. \tag{18}$$

Proof: Equation (18) is obtained through substituting Eqs. (17) and (6) into the Poisson bracket

$$(H,F_m)=\left\langle\frac{\partial H}{\partial q},\frac{\partial F_m}{\partial p}\right\rangle-\left\langle\frac{\partial H}{\partial p},\frac{\partial F_m}{\partial q}\right\rangle$$

and carefully calculating it.

According to this proposition, we get

Theorem 2: The Hamiltonian system (16) is completely integrable in the Liouville sense.

V. PARAMETRIC REPRESENTATIONS OF SOLUTIONS

$(H,F_m)=0$ implies that the Hamiltonian systems (H) and (F_m) are compatible, and thus their corresponding phase flow $g_0^x\, g_m^{t_m}$ commute (see Ref. 13). Define

$$\begin{pmatrix} q(x,t_m) \\ p(x,t_m)\end{pmatrix}=g_0^x g_m^{t_m}\begin{pmatrix} q(0,0) \\ p(0,0)\end{pmatrix},$$

which is an analytic function of (x,t_m).

Theorem 3: Let $(q(x,t_m),p(x,t_m))^T$ be a solution of the compatible system (H) and (F_m). Then the nonlinear WKI evolution equations

$$4i\begin{pmatrix} u \\ v \end{pmatrix}_{t_{m+2}} = J(J^{-1}K)^m G_0, \quad m=0,1,2,..., \tag{19}$$

possess the parametric representations of solutions as follows:

$$\begin{aligned} u(x,t_m) &= \frac{i\langle \Lambda q(x,t_m), q(x,t_m)\rangle}{\sqrt{1+\langle \Lambda q(x,t_m), q(x,t_m)\rangle\langle \Lambda p(x,t_m), p(x,t_m)\rangle}}, \\ v(x,t_m) &= \frac{-i\langle \Lambda p(x,t_m), p(x,t_m)\rangle}{\sqrt{1+\langle \Lambda q(x,t_m), q(x,t_m)\rangle\langle \Lambda p(x,t_m), p(x,t_m)\rangle}}, \end{aligned} \tag{20}$$

where J, K are expressed by Eq. (11).

Proof: Let $Q = \sqrt{1+\langle \Lambda p,p\rangle\langle \Lambda q,q\rangle}$. Then, from Eq. (20) we have

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_{m+2}} = iQ^{-3}\langle \Lambda q, q_{t_{m+2}}\rangle \begin{pmatrix} 2+\langle \Lambda p,p\rangle\langle \Lambda q,q\rangle \\ \langle \Lambda p,p\rangle^2 \end{pmatrix} - iQ^{-3}\langle \Lambda p, \quad p_{t_{m+2}}\rangle \begin{pmatrix} \langle \Lambda q,q\rangle^2 \\ 2+\langle \Lambda p,p\rangle\langle \Lambda q,q\rangle \end{pmatrix}. \tag{21}$$

In addition, letting the operator $(J^{-1}K)^m$ operate upon the equality $G_0 = \Sigma_{j=1}^N \nabla\lambda_j$, and noticing Eqs. (10) and (12), we get

$$(J^{-1}K)^m G_0 = \sum_{j=1}^N \lambda_j^m \nabla\lambda_j. \tag{22}$$

After Eqs. (6) and (7) are substituted into Eq. (21), through a series of careful calculations, Eq. (21) reads as

$$4i\begin{pmatrix} u \\ v \end{pmatrix}_{t_{m+2}} = \begin{pmatrix} 0 & -\partial^2 \\ \partial^2 & 0 \end{pmatrix} \begin{pmatrix} \langle \Lambda^{m+1}p,p\rangle \\ -\langle \Lambda^{m+1}q,q\rangle \end{pmatrix} = J\sum_{j=1}^N \lambda_j^m \nabla\lambda_j = J(J^{-1}K)^m G_0,$$

where Eq. (22) is used.

Remark: The C. Neumann constraint of the WKI hierarchy and its corresponding integrability, have already been discussed in another article.[17]

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