An involutive system and integrable C. Neumann system associated with the modified Korteweg–de Vries hierarchy

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In this article, a system of finite-dimensional involutive functions is presented and proven to be integrable in the Liouville sense. By using the nonlinearization method, the C. Neumann system associated with the modified Korteweg–de Vries (mKdV) hierarchy is obtained. Thus, the C. Neumann system is shown to be completely integrable via a gauge transformation between it and an integrable Hamiltonian system. Finally, the solution of a stationary mKdV equation and the involutive solutions of the mKdV hierarchy are secured. As two examples, the involutive solutions are given for the mKdV equation: $v_t+\frac{1}{4}v_{xxx}-\frac{3}{2}v^2v_x=0$ and the 5th mKdV equation $v_t-\frac{1}{16}v_{xxxxx}+\frac{5}{8}v^2v_{xxx}+\frac{5}{2}vv_xv_{xx}+\frac{5}{8}v_x^3-\frac{3}{40}v^4v_x=0$.

I. INTRODUCTION

Recently, an investigation on completely integrable systems is fascinating in soliton theory. Many people have devoted themselves to doing studies in this field. In particular, since the nonlinearization method[1,2] about the spectral problem and Lax pair came into use, many classical completely integrable Liouville's systems[3–12] have been successively found. These integrable systems include the C. Neumann system, Bargmann system, and others, which depend on the existence of N-involutive system F_m $(m=0,1,2,...,)$ of Hamiltonian functions; it naturally gives rise to a problem: Are there some relations among those completely integrable systems or not? From the view point of geometry or algebra, what roles do the terms of polynomials included in the various kinds of constructions of functions F_0 and F_m $(m=1,2,3,\ldots,)$ play? The general method has not been looked for yet. In the present article, a gauge transformation between the C. Neumann system associated with the modified Korteweg–de Vries (mKdV) hierarchy and an integrable Hamiltonian system whose involutive system F_m exist (see Sec. II) is found, and from this the C. Neumann system is proved to be completely integrable in the Liouville sense.

In a previous article,[11] completely integrable Hamiltonian systems associated with the Kaup–Newell hierarchy and Levi hierarchy were discussed under the so-called *Bargmann constraints*.[2] This article deals with an integrable C. Neumann system and the involutive solutions of the mKdV hierarchy under the so-called *C. Neumann constraint*,[2] i.e., the present work is an extension of Ref. 11. The article is organized as follows: in the next section, a set of finite-dimensional involutive functions F_m which guarantees the existence of the first integral of Hamiltonian system (F_0) is presented in explicit form and the Hamiltonian systems (F_m) are shown to be integrable in the Liouville sense first. Then by the use of the nonlinearization method, under the C. Neumann constraint, the spectral problem associated with the mKdV hierarchy is nonlinearized as an integrable C. Neumann system, which is proven through making a gauge transformation between the C. Neumann system and an integrable Hamiltonian system. Section IV gives a description of the solution of a stationary mKdV system and the involutive solutions of the mKdV hierarchy. Particularly, the involutive solutions of the well-known mKdV equation $v_t+\frac{1}{4}v_{xxx}-\frac{3}{2}v^2v_x=0$ and the 5th mKdV equation $v_t-\frac{1}{16}v_{xxxxx}+\frac{5}{8}v^2v_{xxx}+\frac{5}{2}vv_xv_{xx}+\frac{5}{8}v_x^3-\frac{3}{40}v^4v_x=0$ are obtained.

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II. AN INVOLUTIVE SYSTEM, C. NEUMANN SYSTEM, AND GAUGE TRANSFORMATION

The Poisson bracket of two functions F, G in the symplectic space $(R^{2N},dp\wedge dq)$ is defined by[13]

$$(F,G)=\sum_{j=1}^{N}\left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j}-\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right)=\left\langle\frac{\partial F}{\partial q},\frac{\partial G}{\partial p}\right\rangle-\left\langle\frac{\partial F}{\partial p},\frac{\partial G}{\partial q}\right\rangle. \tag{1}$$

The functions F, G are called involutive if $(F,G)=0$.

Now, we construct a set of functions $\{F_m\}$ as follows:

$$F_m=-i\langle\Lambda^{2m+1}p,q\rangle+\frac{1}{4}\sum_{j=0}^{m}(\langle\Lambda^{2j}p,p\rangle-\langle\Lambda^{2j}q,q\rangle)(\langle\Lambda^{2(m-j)}p,p\rangle-\langle\Lambda^{2(m-j)}q,q\rangle)$$

$$-\frac{1}{4}\sum_{j=1}^{m}\begin{vmatrix}(\langle\Lambda^{2j-1}p,p\rangle+\langle\Lambda^{2j-1}q,q\rangle) & 2\langle\Lambda^{2(m-j)+1}q,p\rangle\\ 2\langle\Lambda^{2j-1}p,q\rangle & (\langle\Lambda^{2(m-j)+1}p,p\rangle+\langle\Lambda^{2(m-j)+1}q,q\rangle)\end{vmatrix}, \tag{2}$$

where $\lambda_1,...,\lambda_N$ are N different constants, $\Lambda=\mathrm{diag}(\lambda_1,...,\lambda_N)$, $q=(q_1,...,q_N)^T$, $p=(p_1,...,p_N)^T$, and $\langle\cdot,\cdot\rangle$ is the standard inner product in R^N. In particular, one has

$$F_0=-i\langle\Lambda p,q\rangle+\tfrac{1}{4}(\langle p,p\rangle-\langle q,q\rangle)^2. \tag{3}$$

Through a lengthy calculations, it is not difficult to get the following results.

Lemma 1:

$$\left\langle\frac{\partial F_k}{\partial q},\frac{\partial F_l}{\partial p}\right\rangle=\left\langle\frac{\partial F_l}{\partial q},\frac{\partial F_k}{\partial p}\right\rangle,\quad \forall k,l\in Z^+. \tag{4}$$

Hence, $(F_k,F_l)=\langle\partial F_k/\partial q,\partial F_l/\partial p\rangle-\langle\partial F_k/\partial p,\partial F_l/\partial q\rangle=0$. Following this result, we have the following.

Theorem 2: The Hamiltonian systems (F_m) determined by Eq. (2)

$$(F_m): q_{t_m}=\frac{\partial F_m}{\partial p},\quad p_{t_m}=-\frac{\partial F_m}{\partial q},\quad m=0,1,2,... \tag{5}$$

are completely integrable in the Liouville sense.

Particularly, the Hamiltonian system $(t_0=x)$

$$(F_0):\begin{cases} q_x=\dfrac{\partial F_0}{\partial p}=-i\Lambda q+(\langle p,p\rangle-\langle q,q\rangle)p\\ p_x=-\dfrac{\partial F_0}{\partial q}=i\Lambda p+(\langle p,p\rangle-\langle q,q\rangle)q\end{cases} \tag{6}$$

is integrable. Here (F_0) is defined by Eq. (3).

Consider the spectral problem

$$y_x=\begin{pmatrix} v & \lambda^2\\ -1 & -v\end{pmatrix}y, \tag{7}$$

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where v is a potential function, λ is a spectral parameter, $y=(y_1,y_2)^T$. Let $\lambda_1,\ldots,\lambda_N$ be N different spectral parameters of Eq. (7), and $y_j=(Q_j,P_j)^T$ be eigenfunctions corresponding to λ_j. Then it is easy to calculate the functional gradient $\delta\lambda_j/\delta v$ of λ_j with regard to v

$$\delta\lambda_j/\delta v=P_jQ_j,\quad j=1,2,\ldots,N, \tag{8}$$

which satisfies the linear equation

$$\mathscr{L}\delta\lambda_j/\delta v=\lambda_j^2\delta\lambda_j/\delta v,\quad \mathscr{L}=-\tfrac{1}{4}\partial^2+v\partial^{-1}v\partial,\quad \partial=\partial/\partial x. \tag{9}$$

Gu[14] has proven that under the Bargmann constraint (Ref. 2) $G_0=\Sigma_{j=1}^N\ \delta\lambda_j/\delta v$, i.e., $v=\langle P,Q\rangle$, Eq. (7) is nonlinearized as a completely integrable system in the Liouville sense. Here $G_0=v$ is the second element of the Lenard's recursive sequence $\{G_j|G_j=\mathscr{L}G_{j-1},\ G_{-1}=0\ (\partial^{-1}0=1,\ G_0=\mathscr{L}G_{-1}=v),\ j=0,1,2,\ldots,\}$. Now, we consider the C. Neumann constraint[2]

$$G_{-1}=-4i\sum_{k=1}^N \lambda_k^{-1}\delta\lambda_k/\delta v,\quad i^2=-1. \tag{10}$$

Acting with the operator $\mathscr{L}$ upon Eq. (10) and noting Eq. (9), we get

$$v=-4i\langle P,\Lambda Q\rangle. \tag{11}$$

Under Eq. (11), Eq. (7) is nonlinearized as

$$\begin{aligned} Q_x&=-4i\langle P,\Lambda Q\rangle Q+\Lambda^2P,\\ P_x&=-Q+4i\langle P,\Lambda Q\rangle P, \end{aligned} \tag{12}$$

which is called the C. Neumann system of Eq. (7).

A basic problem is whether the C. Neumann system (12) is completely integrable in the Liouville sense or not. In order to prove the integrability of Eq. (12), we make the transformation

$$Q=\tfrac{1}{2}(p+q),\quad P=\tfrac{1}{2}i\Lambda^{-1}(p-q). \tag{13}$$

Thus, Eq. (12) becomes

$$q_x=-i\Lambda q+(\langle p,p\rangle-\langle q,q\rangle)p,$$

$$p_x=i\Lambda p+(\langle p,p\rangle-\langle q,q\rangle)q,$$

which is exactly the Hamiltonian system (6). On the contrary, Eq. (6) can be changed into Eq. (12) via the inverse transformation of Eq. (13)

$$q=Q+i\Lambda p,\quad p=Q-i\Lambda P. \tag{14}$$

By the integrability of Eq. (6), from Eq. (13) we immediately know that Eq. (12) is completely integrable. The transformation (13) is called the gauge transformation between the C. Neumann system (12) and Hamiltonian system (6).

Theorem 3: The C. Neumann system (12) is completely integrable.

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III. A STATIONARY MKDV SYSTEM AND INVOLUTIVE SOLUTIONS OF MKDV HIERARCHY

Theorem 4: Let (Q,P) be a solution of the C. Neumann system (12), then $v=-4i\langle P,\Lambda Q\rangle$ satisfies a stationary mKdV equation

$$J\mathscr{L}^N v+\sum_{k=0}^{N-1}\alpha_{N-k}J\mathscr{L}^k v=0, \tag{15}$$

where $J=\partial$, α_j are determined by $\lambda_1,\dots,\lambda_N$.

Proof: Letting the operator $\mathscr{L}^l$ act upon Eq. (11) and noticing Eq. (9), $G_{j+1}=\mathscr{L}G_j$ ($j=0,1,2,\dots$), we have

$$\mathscr{L}^l v=-4i\sum_{k=1}^{N}\lambda_k^{2l+1}\delta\lambda_k/\delta v. \tag{16}$$

Introduce the polynomial ($\alpha_0=1$)

$$p(\lambda)=\sum_{k=1}^{N}\lambda(\lambda-\lambda_k^2)=\alpha_0\lambda^{N+1}+\alpha_1\lambda^N+\cdots+\alpha_N\lambda. \tag{17}$$

Acting with the operator $J\Sigma_{l=0}^{N}\alpha_{N-l}$ upon Eq. (16) and using Eq. (17), we get Eq. (15).

We denote by $g_m^{t_m}$ the solution operator of the initial-value problem of (F_m). Since $(F_0,F_m)=0$, the two Hamiltonian systems (F_0), (F_m) are compatible, and their phase flows g_0^x, $g_m^{t_m}$ ($t_0=x$) commute.[13] Define

$$\begin{pmatrix} q(x,t_m)\\ p(x,t_m)\end{pmatrix}=g_0^x g_m^{t_m}\begin{pmatrix} q(0,0)\\ p(0,0)\end{pmatrix}, \tag{18}$$

which is called the involutive solution of the consistent equations (F_0) and (F_m), and is a smooth function of (x,t_m).

Theorem 5: Let $(q(x,t_m),\ p(x,t_m))$ be an involutive solution of the commutative flows (F_0) and (F_m). Then

$$v=-4i\langle P(x,t_m),\Lambda Q(x,t_m)\rangle,\quad P(x,t_m)=\tfrac{1}{2}i\Lambda^{-1}(p-q),\quad Q(x,t_m)=\tfrac{1}{2}(p+q) \tag{19}$$

is a solution of the higher-order mKdV equation

$$v_{t_m}=J\mathscr{L}^m v,\quad m=0,1,2,\dots \tag{20}$$

Proof: $v=-4i\langle P(x,t_m),\Lambda Q(x,t_m)\rangle=\langle p(x,t_m),p(x,t_m)\rangle-\langle q(x,t_m),q(x,t_m)\rangle$

$$v_{t_m}=2(\langle p,p_{t_m}\rangle-\langle q,q_{t_m}\rangle)=-2\left(\left\langle p,\frac{\partial F_m}{\partial q}\right\rangle+\left\langle q,\frac{\partial F_m}{\partial p}\right\rangle\right)=2i(\langle\Lambda^{2m+1}p,p\rangle+\langle\Lambda^{2m+1}q,q\rangle)$$

$$=\partial(\langle\Lambda^{2m}p,p\rangle-\langle\Lambda^{2m}q,q\rangle)=\partial(-4i\langle\Lambda^{2m+1}P,Q\rangle)=J\left(-4i\sum_{k=1}^{N}\lambda_k^{2m+1}\delta\lambda_k/\delta v\right)$$

$$=J\mathscr{L}^m G_0=J\mathscr{L}^m v.$$

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In the above lengthy calculations, Eqs. (5), (6), (14), and (16) are used. The proof is completed.

Choosing $m=1$ and 2 in Theorem 5, we can obtain the involutive solution of the mKdV equation $v_t+\frac{1}{4}v_{xxx}-\frac{3}{2}v^2v_x=0$ and 5th mKdV equation $v_t-\frac{1}{16}v_{xxxxx}+\frac{5}{8}v^2v_{xxx}+\frac{5}{2}vv_xv_{xx}+\frac{5}{8}v_x^3-\frac{3}{40}v^4v_x=0$, respectively. Thus, we have the following corollary.

Corollary 6: Let $(q(x,t_1),p(x,t_1))$ be an involutive solution of the compatible Eqs. (F_0) and (F_1). Then

$$v=-4i\langle P(x,t_1),\Lambda Q(x,t_1)\rangle, \quad P(x,t_1)=\tfrac{1}{2}i\Lambda^{-1}(p-q), \quad Q(x,t_1)=\tfrac{1}{2}(p+q) \tag{21}$$

satisfies the well-known mKdV equation

$$v_t+\tfrac{1}{4}v_{xxx}-\tfrac{3}{2}v^2v_x=0, \quad t=t_1. \tag{22}$$

Proof: In virtue of $v=\langle p,p\rangle-\langle q,q\rangle$, and Eqs. (2), (14), and (16), we get

$$v_t=2(\langle p,p_t\rangle)-\langle q,q_t\rangle=-2\left(\left\langle p,\frac{\partial F_1}{\partial q}\right\rangle+\left\langle q,\frac{\partial F_1}{\partial p}\right\rangle\right)=2i(\langle\Lambda^3p,p\rangle+\langle\Lambda^3q,q\rangle)$$

$$=\partial(\langle\Lambda^2p,p\rangle-\langle\Lambda^2q,q\rangle)=\partial(-4i\langle\Lambda^3P,Q\rangle)$$

$$=J\left(-4i\sum_{k=1}^{N}\lambda^3\delta\lambda_k/\delta v\right)=J\mathscr{L}G_0=J\mathscr{L}v=-\tfrac{1}{4}v_{xxx}+\tfrac{3}{2}v^2v_x.$$

In analogy to the proof of Corollary 6, we also have the next corollary.

Corollary 7:

$$v=-4i\langle P,(x,t_2),\Lambda Q(x,t_2)\rangle, \quad P(x,t_2)=\tfrac{1}{2}i\Lambda^{-1}(p-q), \quad Q(x,t_2)=\tfrac{1}{2}(p+q) \tag{23}$$

satisfies the 5th mKdV equation

$$v_t-\tfrac{1}{16}v_{xxxxx}+\tfrac{5}{8}v^2v_{xxx}+\tfrac{5}{2}vv_xv_{xx}+\tfrac{5}{8}v_x^3-\tfrac{3}{40}v^4v_x=0, \quad t=t_2, \tag{24}$$

where $(q(x,t_2),p(x,t_2))$ is an involutive solution of the consistent systems (F_0) and (F_2).

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