A hierarchy of nonlinear evolution equations and finite-dimensional involutive systems

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A spectral problem and an associated hierarchy of nonlinear evolution equations are presented in this article. In particular, the reductions of the two representative equations in this hierarchy are given: one is the nonlinear evolution equation $r_t=-\alpha r_x-2i\alpha\beta|r^2|r$ which looks like the nonlinear Schrödinger equation, the other is the generalized derivative nonlinear Schrödinger equation $r_t=\frac{1}{2}i\alpha r_{xx}-i\alpha|r|^2r-\alpha\beta(|r|^2r)_x-\alpha\beta|r|^2r_x-2i\alpha\beta^2|r|^4r$ which is just a combination of the nonlinear Schrödinger equation and two different derivative nonlinear Schrödinger equations [D. J. Kaup and A. C. Newell, J. Math. Phys. **19**, 789 (1978); M. J. Ablowitz, A. Ramani, and H. Segur, J. Math. Phys. **21**, 1006 (1980)]. The spectral problem is nonlinearized as a finite-dimensional completely integrable Hamiltonian system under a constraint between the potentials and the spectral functions. At the end of this article, the involutive solutions of the hierarchy of nonlinear evolution equations are obtained. Particularly, the involutive solutions of the reductions of the two representative equations are developed.

I. INTRODUCTION

It is well known that the inverse scattering transformation[1] (IST) method plays an important role in solving many nonlinear evolution equations[2–9] such as Korteweg–de Vries (KdV), nonlinear Schrödinger, sine-Gordon, and other nonlinear differential equations which have great application in physics. But the main difficulty of the IST method lies in finding an appropriate spectral problem for a given nonlinear evolution equation. Hence, it is interesting for us to search for as many new spectral problems and corresponding nonlinear evolution equations as possible in soliton theory. In 1989, Tu[10] proposed a so-called loop algebra scheme to generate integrable evolution equations from the spectral problem. According to this method, many integrable nonlinear evolution equations, their trace identities, and Hamiltonian structures are successively obtained.[11–15]

In this article we introduce the spectral problem

$$y_x=My,\quad y=\begin{pmatrix} y_1\\ y_2\end{pmatrix},\quad M=\begin{pmatrix} -iu & \lambda+v+\beta(u^2-v^2)\\ -\lambda+v-\beta(u^2-v^2) & iu\end{pmatrix},\quad i=\sqrt{-1},\tag{1.1}$$

which is a simple extension of the Dirac eigenvalue problem[16]

$$y_x=My,\quad M=\begin{pmatrix} -iu & \lambda+v\\ -\lambda+v & iu\end{pmatrix},$$

where u and v are two scalar potentials, λ is a constant spectral parameter and β is a constant. Using the spectral gradient of spectral value λ with respect to the potentials u and v, which was

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cited earliest by Fokas[17] in 1982 to obtain hereditary symmetries for Hamiltonian systems, we derive a new hierarchy of nonlinear evolution equations associated with Eq. (1.1). The two representative systems of equations in this hierarchy are as follows:

$$\begin{cases} u_t=-\alpha u_x+2i\alpha\beta v(u^2-v^2), & (1.2a)\\ v_t=-\alpha v_x+2i\alpha\beta u(u^2-v^2), & (1.2b)\end{cases}$$

and

$$\begin{cases} u_t=\tfrac{1}{2}i\alpha v_{xx}+i\alpha v(u^2-v^2)+\alpha\beta(u(u^2-v^2))_x+\alpha\beta(u^2-v^2)u_x-2i\alpha\beta^2 v(u^2-v^2)^2, & (1.3a)\\ v_t=\tfrac{1}{2}i\alpha u_{xx}+i\alpha u(u^2-v^2)+\alpha\beta(v(u^2-v^2))_x+\alpha\beta(u^2-v^2)v_x-2i\alpha\beta^2 u(u^2-v^2)^2. & (1.3b)\end{cases}$$

As α, β are two real numbers and $u=i\,\mathrm{Im}\,r$, $v=\mathrm{Re}\,r$, Eqs. (1.2) and (1.3) are reduced to a nonlinear evolution equation

$$r_t=-\alpha r_x-2i\alpha\beta|r|^2r \tag{1.4}$$

and a generalized derivative nonlinear Schrödinger equation (GDNSE)

$$r_t=\tfrac{1}{2}i\alpha r_{xx}-i\alpha|r|^2r-\alpha\beta(|r|^2r)_x-\alpha\beta|r|^2r_x-2i\alpha\beta^2|r|^4r. \tag{1.5}$$

respectively. The former looks like the well-known nonlinear Schrödinger equation, the latter is just a combination of the nonlinear Schrödinger equation and two derivative nonlinear Schrödinger equations.[18,19] Through the nonlinearization[20] of the spectral problem (1.1), which has already been applied successfully to generate completely integrable systems in the Liouville sense and obtain the involutive solutions of nonlinear evolution equations,[21–33] we give a finite-dimensional completely integrable Hamiltonian system in the Liouville sense. Furthermore, the involutive solutions of this hierarchy of nonlinear evolution equations are obtained. In particular, the involutive solutions of the nonlinear evolution equation (1.4) and the GDNSE (1.5) are given.

II. THE HIERARCHY OF NLEEs ASSOCIATED WITH EQ. (1.1)

Consider the spectral problem (1.1). Let λ and $y=(y_1,y_2)^T$ be the spectral value and the associated spectral function of Eq. (1.1). It is easy to calculate the spectral gradient $\nabla\lambda=(\delta\lambda/\delta u,\delta\lambda/\delta v)^T$ of spectral value λ with respect to the potentials u and v

$$\nabla\lambda=\begin{pmatrix}\dfrac{\delta\lambda}{\delta u}\\ \dfrac{\delta\lambda}{\delta v}\end{pmatrix}=\begin{pmatrix}-2iy_1y_2+2\beta u(y_1^2+y_2^2)\\ -y_1^2+y_2^2-2\beta v(y_1^2+y_2^2)\end{pmatrix}. \tag{2.1}$$

Noting

$$\frac{1}{2}(y_1^2+y_2^2)_x=i\left(v\frac{\delta\lambda}{\delta u}+u\frac{\delta\lambda}{\delta v}\right), \tag*{(2.2)$_1$}$$

$$\tfrac{1}{2}(y_2^2-y_1^2)_x=iu(y_1^2+y_2^2)-2\lambda y_1y_2-2\beta(u^2-v^2)y_1y_2, \tag*{(2.2)$_2$}$$

$$(y_1y_2)_x=\lambda(y_2^2-y_1^2)+v(y_1^2+y_2^2)+\beta(u^2-v^2)(y_2^2-y_1^2), \tag*{(2.2)$_3$}$$

we have the following proposition:

Proposition 2.1:

$$K\nabla\lambda=\lambda\cdot J\nabla\lambda, \tag{2.3}$$

where K and J are two operators ($\partial=\partial/\partial x, \partial\partial^{-1}=\partial^{-1}\partial=1$)

$$K=\begin{pmatrix} -\partial+4i\beta\partial u\partial^{-1}v+4v\partial^{-1}v+8\beta^2v(u^2-v^2)\partial^{-1}v & 4i\beta\partial u\partial^{-1}u+4v\partial^{-1}u+8\beta^2v(u^2-v^2)\partial^{-1}u-2i\beta(u^2-v^2) \\ 4i\beta\partial v\partial^{-1}v+4u\partial^{-1}v+8\beta^2u(u^2-v^2)\partial^{-1}v+2i\beta(u^2-v^2) & \partial+4i\beta\partial v\partial^{-1}u+4u\partial^{-1}u+8\beta^2u(u^2-v^2)\partial^{-1}u \end{pmatrix},$$

$$J=\begin{pmatrix} -8\beta v\partial^{-1}v & 2i-8\beta v\partial^{-1}u \\ -2i-8\beta u\partial^{-1}v & -8\beta u\partial^{-1}u \end{pmatrix}, \quad J^*=-J,$$

which are called the pair of Lenard's operators of Eq. (1.1).

Proof. Directly calculate.

Consider the Lenard's gradient sequences G_j defined by

$$KG_{j-1}=JG_j, \quad j=1,2,\ldots, \quad G_0=\alpha(u,-v)^T\in\operatorname{Ker} J, \tag{2.4}$$

where α is an arbitary constant. The vector fields $X_m\triangleq KG_m$ yield the hierarchy of nonlinear evolution equations (NLEEs) associated with Eq. (1.1)

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_m}=X_m(u,v)=K(J^{-1}K)^mG_0, \quad m=0,1,2,\ldots . \tag{2.5}$$

The first and second systems of evolution equations in the hierarchy (2.5) are

$$\begin{cases} u_t=-\alpha u_x+2i\alpha\beta v(u^2-v^2), \\ v_t=-\alpha v_x+2i\alpha\beta u(u^2-v^2), \end{cases}$$

and

$$\begin{cases} u_t=\frac{1}{2}i\alpha v_{xx}+i\alpha v(u^2-v^2)+\alpha\beta(u(u^2-v^2))_x+\alpha\beta(u^2-v^2)u_x-2i\alpha\beta^2v(u^2-v^2)^2, \\ v_t=\frac{1}{2}i\alpha u_{xx}+i\alpha u(u^2-v^2)+\alpha\beta(v(u^2-v^2))_x+\alpha\beta(u^2-v^2)v_x-2i\alpha\beta^2u(u^2-v^2)^2, \end{cases}$$

respectively. As $\alpha,\beta\in R$ and $u=i\operatorname{Im} r$, $v=\operatorname{Re} r$, the former can be reduced to the nonlinear equation (1.4) whose physical property is to be known, and the latter can be reduced to the GDNSE (1.5).

III. FINITE-DIMENSIONAL INVOLUTIVE SYSTEMS AND AN INTEGRABLE SYSTEM

The Poisson bracket of two functions in the symplectic space $(R^{2N},dp\wedge dq)$ is defined by[34]

$$(F,G)=\sum_{j=1}^{N}\left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j}-\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right)=\langle F_q,G_p\rangle-\langle F_p,G_q\rangle. \tag{3.1}$$

F, G are called involutive, if $(F,G)=0$.

Let $\lambda_1<\cdots<\lambda_N$ and define

$$\Gamma_k=\sum_{\substack{j=1\\ j\neq k}}^{N} \frac{B_{kj}^2}{\lambda_k-\lambda_j}, \quad B_{kj}=p_kq_j-q_kp_j. \tag{3.2}$$

By virtue of the equalities

$$((\langle p,p\rangle,p_k^2)=(\langle q,q\rangle,q_k^2)=0, \quad (p_k^2,p_l^2)=(q_k^2,q_l^2)=0, \tag{3.3$_1$}$$

$$((\langle p,p\rangle,\Gamma_k)=(\langle q,q\rangle,\Gamma_k)=(\Gamma_k,\Gamma_l)=0, \tag{3.3$_2$}$$

$$((\langle q,q\rangle,p_k^2)=(q_k^2,\langle p,p\rangle)=4p_kq_k, \tag{3.3$_3$}$$

$$(q_l^2,p_k^2)=(q_k^2,p_l^2)=4q_kp_l\delta_{kl},((\langle q,q\rangle,\langle p,p\rangle)=4\langle q,p\rangle, \tag{3.3$_4$}$$

$$(\Gamma_k,p_l^2)=4(\lambda_k-\lambda_l)^{-1}p_kp_lB_{kl}, \quad (\Gamma_k,q_l^2)=4(\lambda_k-\lambda_l)^{-1}q_kq_lB_{kl} \tag{3.3$_5$}$$

we easily verify the following proposition:

Proposition 3.1: Let

$$E_k=\tfrac{1}{2}(1-2(\langle p,p\rangle+\langle q,q\rangle))(p_k^2+q_k^2)+\Gamma_k, \quad k=1,2,\ldots,N. \tag{3.4}$$

Then $E_1,\ldots,E_N$ compose an N-involutive system, that is $(E_k,E_1)=0$.

Define a bilinear function $Q_z(\xi,\eta)$ on R^N

$$Q_z(\xi,\eta)\equiv\langle(z-\Lambda)^{-1}\xi,\eta\rangle=\sum_{k=1}^{N}(z-\lambda_k)^{-1}\xi_k\eta_k=\sum_{m=0}^{\infty}z^{-m-1}\langle\Lambda^m\xi,\eta\rangle. \tag{3.5}$$

The generating function of Γ_k is given by (see Refs. 35,36)

$$\begin{vmatrix} Q_z(q,q) & Q_z(q,p)\\ Q_z(p,q) & Q_z(p,p)\end{vmatrix}=\sum_{k=1}^{N}\frac{\Gamma_k}{z-\lambda_k}. \tag{3.6}$$

Thus the generating function of E_k is

$$\frac{1}{2}(1-2(\langle q,q\rangle+\langle p,p\rangle))(Q_z(p,p)+Q_z(q,q))+\begin{vmatrix} Q_z(q,q) & Q_z(q,p)\\ Q_z(p,q) & Q_z(p,p)\end{vmatrix}=\sum_{k=1}^{N}\frac{E_k}{z-\lambda_k}. \tag{3.7}$$

Substituting Eq. (3.5) and $(z-\lambda_k)^{-1}=\Sigma_{m=0}^{\infty}z^{-m-1}\lambda_k^m$ into both sides of Eq. (3.7), respectively, we get

Proposition 3.2: Let

$$F_m=\sum_{k=1}^{N}\lambda_k^mE_k, \quad m=0,1,2,\ldots, \tag{3.8}$$

then

$$F_0=\tfrac{1}{2}(1-2(\langle q,q\rangle+\langle p,p\rangle))(\langle p,p\rangle+\langle q,q\rangle), \tag{3.9}$$

$$F_m=\frac{1}{2}(1-2(\langle q,q\rangle+\langle p,p\rangle))(\langle\Lambda^m p,p\rangle+\langle\Lambda^m q,q\rangle)+\sum_{j=1}^{m}\begin{vmatrix}\langle\Lambda^{j-1}q,q\rangle & \langle\Lambda^{j-1}q,p\rangle\\ \langle\Lambda^{m-j}p,q\rangle & \langle\Lambda^{m-j}p,p\rangle\end{vmatrix} \tag{3.10}$$

and $(F_k,F_1)=0$, $\forall k,1\in Z^+$.

Proposition 3.3: The Hamiltonian systems

$$(F_m):q_{t_m}=\frac{\partial F_m}{\partial p},\quad p_{t_m}=\frac{\partial F_m}{\partial q} \tag{3.11}$$

are completely integrable in the Liouville sense.

Let λ_j $(j=1,2,\ldots,N)$ and $y=(q_j,p_j)^T$ be N different spectral parameters and corresponding solutions of Eq. (1.1). Consider the Bargmann constraint (Ref. 21): $G_0|_{\alpha=1}=\Sigma_{j=1}^{N}\nabla\lambda_j$, that is,

$$u=\frac{2i\langle p,q\rangle}{1-2\beta(\langle q,q\rangle+\langle p,p\rangle)},\quad v=\frac{\langle p,p\rangle-\langle q,q\rangle}{1-2\beta(\langle q,q\rangle+\langle p,p\rangle)}. \tag{3.12}$$

The nonlinearization of Eq. (1.1) under Eq. (3.12) gives the Hamiltonian system

$$(H):\begin{cases} q_x=\Lambda p+\dfrac{2\langle p,q\rangle q+(\langle p,p\rangle-\langle q,q\rangle)p}{1-2\beta(\langle q,q\rangle+\langle p,p\rangle)}-\beta\,\dfrac{4\langle p,q\rangle^2+(\langle p,p\rangle-\langle q,q\rangle)^2}{(1-2\beta(\langle q,q\rangle+\langle p,p\rangle))^2}\,p=\dfrac{\partial H}{\partial p},\\ p_x=-\Lambda q+\dfrac{-2\langle p,q\rangle p+(\langle p,p\rangle-\langle q,q\rangle)q}{1-2\beta(\langle q,q\rangle+\langle p,p\rangle)}+\beta\,\dfrac{4\langle p,q\rangle^2+(\langle p,p\rangle-\langle q,q\rangle)^2}{(1-2\beta(\langle q,q\rangle+\langle p,p\rangle))^2}\,q=-\dfrac{\partial H}{\partial q},\end{cases} \tag{3.13}$$

with the Hamiltonian function

$$H=\tfrac{1}{2}\langle\Lambda p,p\rangle+\tfrac{1}{2}\langle\Lambda q,q\rangle-\beta\,\frac{4\langle p,q\rangle^2+(\langle p,p\rangle-\langle q,q\rangle)^2}{4-8\beta(\langle p,p\rangle+\langle q,q\rangle)}. \tag{3.14}$$

Proposition 3.4: The Hamiltonian system (3.13) is completely integrable in the Liouville sense in the symplectic manifold $(R^{2N},dp\wedge dq)$ and its involutive system is $\{F_m\}$.

Proof. Through some careful calculations, we obtain $(H,F_m)=0$, $m=0,1,2,\ldots$. So, the above result is correct.

IV. THE INVOLUTIVE SOLUTIONS OF THE HIERARCHY (2.5)

Denote the solution operators of the initial-value problems of the integrable Hamiltonian systems (H) and (F_m) by g_0^x and $g_m^{t_m}$, respectively. $(H,F_m)=0$ implies their own flows g_0^x, $g_m^{t_m}$ commute (see Ref. 34). Then the involutive solution of the consistent equations (H) and (F_m)

$$\begin{pmatrix}q(x,t_m)\\ p(x,t_m)\end{pmatrix}=g_0^x g_m^{t_m}\begin{pmatrix}q(0,0)\\ p(0,0)\end{pmatrix} \tag{4.1}$$

is a smooth function of (x,t_m).

Proposition 4.1: Let $(q(x,t_m),p(x,t_m))^T$ be an involutive solution of the compatible systems (H) and (F_m). Then

$$u(x,t_m)=\frac{2i\langle p,q\rangle}{1-2\beta(\langle q,q\rangle+\langle p,p\rangle)},\quad v(x,t_m)=\frac{\langle p,p\rangle-\langle q,q\rangle}{1-2\beta(\langle q,q\rangle+\langle p,p\rangle)} \tag{4.2}$$

satisfy the NLEEs

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = K(J^{-1}K)^m G_0|_{\alpha=1}, \quad G_0|_{\alpha=1} = (u, -v)^T, \quad m = 0,1,2,\dots . \tag{4.3}$$

Proof. Substituting Eqs. (3.11) and (3.10) into the t_m derivative formula of Eq. (4.2), and noticing $G_0|_{\alpha=1} = \Sigma_{j=1}^N \nabla\lambda_j$ and Eq. (2.3), through a lengthy calculations we can know that Eq. (4.3) holds.

If we choose $m=0,1$ in Proposition 4.1, then we can obtain the solution representations of Eqs. (1.4) and (1.5), respectively. Hence, we have

Corollary 4.2: Let $(q(x,t_0),p(x,t_0))^T$ and $(q(x,t_1),p(x,t_1))^T$ be an involutive solution of the compatible systems (H), (F_0) and (H), (F_1), respectively. Let

$$u(x,t_j) = \frac{2i\langle p(x,t_j),q(x,t_j)\rangle}{1-2\beta(\langle q(x,t_j),q(x,t_j)\rangle + \langle p(x,t_j),p(x,t_j)\rangle)}, \tag{4.4}$$

$$v(x,t_j) = \frac{\langle p(x,t_j),p(x,t_j)\rangle - \langle q(x,t_j),q(x,t_j)\rangle}{1-2\beta(\langle q(x,t_j),q(x,t_j)\rangle + \langle p(x,t_j),p(x,t_j)\rangle)}, \tag{4.5}$$

$$r(x,t_j) = v(x,t_j) + u(x,t_j) = \frac{\langle p(x,t_j)+iq(x,t_j),p(x,t_j)+iq(x,t_j)\rangle}{1-2\beta(\langle q(x,t_j),q(x,t_j)\rangle + \langle p(x,t_j),p(x,t_j)\rangle)}, \quad (j=0,1). \tag{4.6}$$

Then $r(x,t_0)$ and $r(x,t_1)$ satisfy the nonlinear equation

$$r_{t_0} = -r_x - 2i\beta|r|^2 r \tag{4.7}$$

and the GDNSE

$$r_{t_1} = \tfrac{1}{2} i r_{xx} - i|r|^2 r - \beta(|r|^2 r)_x - \beta|r|^2 r_x - 2i\beta^2 |r|^4 r, \tag{4.8}$$

respectively.

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