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ALGEBRO-GEOMETRIC SOLUTIONS FOR THE DEGASPERIS–PROCESI HIERARCHY*

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Abstract. Though the completely integrable Camassa–Holm (CH) equation and Degasperis–Procesi (DP) equation are cast in the same peakon family, they possess the second- and third-order Lax operators, respectively. From the viewpoint of algebro-geometrical study, this difference lies in hyper-elliptic and non-hyper-elliptic curves. The non-hyper-elliptic curves lead to great difficulty in the construction of algebro-geometric solutions of the DP equation. In this paper, we derive the DP hierarchy with the help of Lenard recursion operators. Based on the characteristic polynomial of a Lax matrix for the DP hierarchy, we introduce a third order algebraic curve $\mathcal{K}_{r-2}$ with genus $r-2$, from which the associated Baker–Akhiezer functions, meromorphic function and Dubrovin-type equations are established. Furthermore, the theory of algebraic curve is applied to derive explicit representations of the theta function for the Baker–Akhiezer functions and the meromorphic function. In particular, the algebro-geometric solutions are obtained for all equations in the whole DP hierarchy.

Key words. Degasperis–Procesi hierarchy, algebro-geometric solutions, third order algebraic curve, Baker–Akhiezer function, Riemann-theta function

AMS subject classifications. 35Q53, 58F07, 35Q51

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1. Introduction.

The Degasperis–Procesi (DP) equation

$$u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0, \tag{1.1}$$

was first discovered in a search for asymptotically integrable PDEs [2]. It arose as a model equation in the study of the two-dimensional water waves propagating in an irrotational flow over a flat bed [20], [26], [31]. Given the intricate structure of the full governing equations for water waves, it is natural to seek simpler approximate model equations in various physical regimes. The DP equation may be derived in the moderate amplitude regime: introducing the wave-amplitude parameter ε and the long-wave parameter δ. In this regime we assume that $\delta \ll 1$ and $\varepsilon \sim \delta$. This regime is more appropriate for the study of nonlinear waves than dispersive waves, the stronger nonlinearity of which could allow for the occurrence of wave-breaking. The other regime studied most is the shallow water system for which $\delta \ll 1$ and $\varepsilon \sim \delta^2$. In the parameter δ range, due to a balance between nonlinearity and dispersion, various

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integrable systems like the Korteweg–de Vries (KdV) equation arose as approximations to the governing equations. However, among the models of moderate amplitude regime, only the Camassa–Holm (CH) equation and the DP equation are integrable in the peakon family [3] in the sense that they admit a bi-Hamiltonian structure and a Lax pair. Also, they are two integrable equations from a family, corresponding to parameters $b = 2$ and $b = 3$, respectively, of the following b-family of equations:

$$u_t - u_{txx} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \tag{1.2}$$

where b is a constant.

Quasi-periodic solutions (also called algebro-geometric solutions or finite gap solutions) of nonlinear equations were originally studied on the KdV equation based on the inverse spectral theory and algebro-geometric method developed by pioneers such as the authors in [1], [4], [5], [6], [7], [8], [9], [10] in the late 1970s. This theory has been extended to the whole hierarchies of nonlinear integrable equations by Gesztesy and Holden using polynomial recursion method [13], [14], [15], [16], [17], [18]. As a degenerated case of algebro-geometric solution, the multisoliton solution and elliptic function solution may be obtained [4], [7], [28]. It is well known that the algebro-geometric solutions of the CH hierarchy have been obtained with different techniques, see Gesztesy and Holden [14] and Qiao [30]. However, to the authors' knowledge, the algebro-geometric solutions of the DP hierarchy are still not presented yet.

Before turning to each section, it seems appropriate to review some related literature as usual. Over the past three decades soliton equations associated with 2×2 matrix spectral problems have widely been studied. Various methods were developed to construct algebro-geometric solutions for integrable equations such as KdV, modified KdV, Kadomtsev–Petviashvili equation, Schrödinger, CH equations, sine-Gordon, Ablowitz–Kaup–Newell–Segur, Ablowitz–Ladik lattice, Toda lattice, etc. [4], [5], [6], [7], [8], [9], [10], [13], [14], [15], [16], [17], [18], [21], [22], [32], [33]. But it is very difficult to extend these methods to soliton equations associated with 3×3 matrix spectral problems. The main reasons for this complexity get traced back to the associated algebraic curve, which is the second-order hyper-elliptic in the 2×2 matrix spectral problems while it is non-hyper-elliptic of the third order one typically arising in the 3×3 case.

In [29], Qiao proposed the DP hierarchy through the procedure of recursion operator and connected the DP hierarchy (including the DP equation as a special negative member) to finite-dimensional integrable systems and gave its parametric solution on a symplectic submanifold by using the Neumann constraint under the nonlinearization technique. In [27], the N-soliton of the DP equation is obtained by Hirota's method. In [19], the inverse scattering method for the DP equation is studied based on a 3×3 matrix Riemann–Hilbert (RH) problem, where the solution of the DP equation is extracted from the large-k behavior of the solution of the RH problem. In [23], [24], Dickson, Gesztesy, and Unterkofler proposed an unified framework, which yields all algebro-geometric solutions of the entire Boussinesq (Bsq) hierarchy. Geng, Wu, and He further investigated the algebro-geometric solutions of the modified Bsq hierarchy in a recent paper [25].

The purpose of this paper is to construct the algebro-geometric solutions for the DP hierarchy which contains the DP equation (1.1) as a special member. The outline of the present paper is as follows. In section 2, based on the Lenard recursion operators and the stationary zero-curvature equation, we derive the DP hierarchy associated with a 3×3 matrix spectral problem. An algebraic curve $\mathcal{K}_{r-2}$ of arithmetic genus

$r-2$ is introduced with the help of the characteristic polynomial of Lax matrix for the stationary DP hierarchy.

In section 3, we study the meromorphic function ϕ satisfying a second-order nonlinear differential equation. Moreover, the stationary DP equations are decomposed into a system of Dubrovin-type equations.

In section 4, we present the explicit theta function representations for the Baker–Akhiezer function and the meromorphic function. In particular, we give the algebro-geometric solutions of the entire stationary DP hierarchy.

In sections 5 and 6, we extend all the Baker–Akhiezer function, the meromorphic function, the Dubrovin-type equations, and the theta function representations dealt with in sections 3 and 4 to the time-dependent cases. Each equation in the time-dependent DP hierarchy is permitted to evolve in terms of an independent time parameter t_p. We use a stationary solution of the nth equation of the DP hierarchy as an initial data to construct a time-dependent solution of the pth equation of the DP hierarchy.

2. The DP hierarchy. In this section, we derive the DP hierarchy and the corresponding sequence of zero-curvature pairs by using a Lenard recursion formalism (see [29] for more details). Throughout this section let us make the following assumption.

HYPOTHESIS 2.1. *In the stationary case we assume that* $u : \mathbb{C} \to \mathbb{C}$ *satisfies*

$$u \in C^\infty(\mathbb{C}),\ \partial_x^k u \in L^\infty(\mathbb{C}),\ k \in \mathbb{N}_0. \tag{2.1}$$

In the time-dependent case we suppose $u : \mathbb{C}^2 \to \mathbb{C}$ *satisfies*

$$\begin{gathered} u(\cdot,t) \in C^\infty(\mathbb{C}),\ \partial_x^k u(\cdot,t) \in L^\infty(\mathbb{C}),\ k \in \mathbb{N}_0, t \in \mathbb{C}, \\ u(x,\cdot), u_{xx}(x,\cdot) \in C^1(\mathbb{C}),\ x \in \mathbb{C}. \end{gathered} \tag{2.2}$$

We start by the following 3×3 matrix isospectral problem:

$$\psi_x = U\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -mz^{-1} & 1 & 0 \end{pmatrix}, \tag{2.3}$$

where $m = u - u_{xx}$, the function u is a potential, and z is a constant spectral parameter independent of variable x. Next, we introduce two Lenard operators

$$K = 4\partial - 5\partial^3 + \partial^5, \tag{2.4}$$

$$J = 3(2m\partial + \partial m)(\partial - \partial^3)^{-1}(m\partial + 2\partial m). \tag{2.5}$$

Obviously, K and J are two skew-symmetric operators. A direct calculation shows that

$$\begin{aligned} K^{-1} &= (\partial - \partial^3)^{-1}(4 - \partial^2)^{-1}, \\ J^{-1} &= \frac{1}{27} m^{-2/3}\partial^{-1} m^{-1/3}(\partial - \partial^3) m^{-1/3}\partial^{-1} m^{-2/3}, \end{aligned}$$

and we further define an operator

$$\mathscr{L} = K^{-1}J = 3(\partial - \partial^3)^{-1}(4 - \partial^2)^{-1}(2m\partial + \partial m)(\partial - \partial^3)^{-1}(m\partial + 2\partial m).$$

Choose $G_0 = \frac{1}{6} \in \ker K$; the Lenard's recursive sequence is defined as follows:

$$G_{j-1} = \mathscr{L}^{-1} G_j, \quad j = 1, 2, \ldots. \tag{2.6}$$

Hence G_j are uniquely determined, for example, the first two elements read as

$$G_0 = \frac{1}{6}, \quad G_1 = (\partial - \partial^3)^{-1} u u_x.$$

In order to obtain the DP hierarchy associated with the spectral problem (2.3), we first solve the stationary zero-curvature equation

$$V_x - [U, V] = 0, \quad V = (V_{ij})_{3\times 3} \tag{2.7}$$

with

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}, \tag{2.8}$$

where each entry V_{ij} is a Laurent expansion in z,

$$V_{ij} = \sum_{\ell=0}^{n} V_{ij}^{(\ell)}(G_\ell) z^{2(n-\ell+1)} \quad i, j = 1, \ldots, 3, \quad \ell = 0, \ldots, n. \tag{2.9}$$

Equation (2.7) can be rewritten as

$$\begin{aligned} V_{11,x} &= V_{21} + z^{-1} m V_{13}, \\ V_{12,x} &= V_{22} - V_{11} - V_{13}, \\ V_{13,x} &= V_{23} - V_{12}, \\ V_{21,x} &= V_{31} + z^{-1} m V_{23}, \\ V_{22,x} &= V_{32} - V_{21} - V_{23}, \\ V_{23,x} &= V_{33} - V_{22}, \\ V_{31,x} &= z^{-1} m (V_{33} - V_{11}) + V_{21}, \\ V_{32,x} &= -z^{-1} m V_{12} + V_{22} - V_{31} - V_{33}, \\ V_{33,x} &= -z^{-1} m V_{13} + V_{23} - V_{32}. \end{aligned} \tag{2.10}$$

Inserting (2.9) into (2.10) yields

$$\begin{aligned} V_{11}^{(\ell)} &= z^{-1}(4 - \partial^2) G_\ell + 3z^{-2} \partial (\partial - \partial^3)^{-1} (m\partial + 2\partial m) G_\ell, \\ V_{12}^{(\ell)} &= 3z^{-1} G_{\ell,x} - 3z^{-2} (\partial - \partial^3)^{-1} (m\partial + 2\partial m) G_\ell, \\ V_{13}^{(\ell)} &= -6z^{-1} G_\ell, \\ V_{21}^{(\ell)} &= z^{-1}(4 - \partial^2) G_{\ell,x} + 3z^{-2} (\partial^2 (\partial - \partial^3)^{-1} (m\partial + 2\partial m) G_\ell + 2m G_\ell), \\ V_{22}^{(\ell)} &= -2z^{-1} (G_\ell - G_{\ell,xx}), \\ V_{23}^{(\ell)} &= -3z^{-1} G_{\ell,x} - 3z^{-2} (\partial - \partial^3)^{-1} (m\partial + 2\partial m) G_\ell, \\ V_{31}^{(\ell)} &= z^{-1}(4 - \partial^2) G_{\ell,xx} + 3z^{-2} (\partial + z^{-1} m)(\partial - \partial^3)^{-1} (m\partial + 2\partial m) G_\ell, \\ V_{32}^{(\ell)} &= -z^{-1} (\partial - \partial^3) G_\ell - 3z^{-2} (\partial^{-1} (m\partial + 2\partial m) G_\ell - 2m G_\ell), \\ V_{33}^{(\ell)} &= z^{-1} (-2G_\ell - G_{\ell,xx}) - 3z^{-2} (\partial (\partial - \partial^3)^{-1} (m\partial + 2\partial m) G_\ell). \end{aligned} \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.7), we can show that Lenard sequence G_ℓ satisfies the Lenard equation

$$K G_\ell = z^{-2} J G_\ell, \ \ell = 0, 1, \ldots. \tag{2.12}$$

For our use in Theorem 6.2, we introduce the following notation:

$$
\begin{aligned}
&V_{11}^{(\ell,0)} = (4-\partial^2)G_\ell, \qquad V_{11}^{(\ell,1)} = 3\partial(\partial-\partial^3)^{-1}(m\partial+2\partial m)G_\ell,\\
&V_{12}^{(\ell,0)} = G_{\ell,x}, \qquad V_{12}^{(\ell,1)} = -3(\partial-\partial^3)^{-1}(m\partial+2\partial m)G_\ell,\\
&V_{13}^{(\ell,0)} = -6G_\ell, \qquad V_{13}^{(\ell,1)} = 0,\\
&V_{21}^{(\ell,0)} = (4-\partial^2)G_{\ell,x},\\
&V_{21}^{(\ell,1)} = 3(\partial^2(\partial-\partial^3)^{-1}(m\partial+2\partial m)G_\ell+2mG_\ell),\\
&V_{22}^{(\ell,0)} = -2(G_\ell-G_{\ell,xx}), \qquad V_{22}^{(\ell,1)} = 0,\\
&V_{23}^{(\ell,0)} = -3G_{\ell,x}, \qquad V_{23}^{(\ell,1)} = -3(\partial-\partial^3)^{-1}(m\partial+2\partial m)G_\ell,\\
&V_{31}^{(\ell,0)} = (4-\partial^2)G_{\ell,xx},\\
&V_{31}^{(\ell,1)} = 3(\partial+z^{-1}m)(\partial-\partial^3)^{-1}(m\partial+2\partial m)G_\ell,\\
&V_{32}^{(\ell,0)} = -(\partial-\partial^3)G_\ell, \quad V_{32}^{(\ell,1)} = -3\partial^{-1}(m\partial+2\partial m)G_\ell-2mG_\ell),\\
&V_{33}^{(\ell,0)} = -2G_\ell-G_{\ell,xx}, \qquad V_{33}^{(\ell,1)} = -3(\partial(\partial-\partial^3)^{-1}(m\partial+2\partial m)G_\ell).
\end{aligned}
$$

Let ψ satisfy the spectral problem (2.3) and an auxiliary problem

$$\psi_{t_n} = V\psi, \tag{2.13}$$

where V is defined by (2.8) and (2.9). The compatibility condition between (2.3) and (2.13) yields the zero-curvature equation

$$U_{t_n} - V_x + [U,V] = 0,$$

which is equivalent to the DP hierarchy

$$\mathrm{DP}_n(u) = m_{t_n} - X_n = 0, \qquad n \geq 0, \tag{2.14}$$

where the vector fields are given by

$$X_n = JG_n = J\mathscr{L}^n G_0, \quad n \geq 0.$$

By definition, the set of solutions of (2.14), with n ranging in $\mathbb{N}_0$, represents the class of algebro-geometric DP solutions. At times it is convenient to abbreviate algebro-geometric stationary DP solutions u simply as DP potentials.

The system of equations $\mathrm{DP}_0(u) = 0$ represents the DP equation.

In order to derive the corresponding plane algebraic curve, we consider the stationary zero-curvature equation

$$z^{1/2}V_x = [U, z^{1/2}V], \tag{2.15}$$

which is equivalent to (2.7), but the term $z^{1/2}V$ can ensure that the following algebraic curve is in positive powers of z.

A direct calculation shows that the matrix $yI - z^{1/2}V$ also satisfies the stationary zero-curvature equation; then we conclude that

$$\frac{d}{dx}(\det(yI - z^{1/2}V)) = 0,$$

which implies that the characteristic polynomial $\det(yI - z^{1/2}V)$ of Lax matrix $z^{1/2}V$ is independent of the variable x. Therefore we define the algebraic curve

$$\mathcal{F}_r(z,y) = \det(yI - z^{1/2}V) = y^3 + yS_r(z) - T_r(z), \tag{2.16}$$

where $S_r(z)$ and $T_r(z)$ are polynomials with constant coefficients of z,

$$S_r(z) = z\left(\begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} + \begin{vmatrix} V_{22} & V_{23} \\ V_{32} & V_{33} \end{vmatrix} + \begin{vmatrix} V_{11} & V_{13} \\ V_{31} & V_{33} \end{vmatrix} \right), \tag{2.17}$$

$$T_r(z) = z^{3/2} \begin{vmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{vmatrix}. \tag{2.18}$$

In order to ensure the polynomials with integer powers, we introduce $z = \tilde{z}^2$, and the algebraic curve becomes

$$\mathcal{F}_r(\tilde{z},y) = y^3 + yS_r(\tilde{z}) - T_r(\tilde{z}), \tag{2.19}$$

where $S_r(\tilde{z})$ and $T_r(\tilde{z})$ are polynomials with constant coefficients of $\tilde{z}$,

$$\begin{aligned} S_r(\tilde{z}) &= \tilde{z}^2 \left(\begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} + \begin{vmatrix} V_{22} & V_{23} \\ V_{32} & V_{33} \end{vmatrix} + \begin{vmatrix} V_{11} & V_{13} \\ V_{31} & V_{33} \end{vmatrix} \right) \\ &= \sum_{j=0}^{4n+2} S_{r,j} \tilde{z}^{8n+6-2j}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} T_r(\tilde{z}) &= \tilde{z}^3 \begin{vmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{vmatrix} \\ &= \sum_{j=0}^{6n+4} T_{r,j} \tilde{z}^{12n+9-2j}. \end{aligned} \tag{2.21}$$

We note that $T_r(\tilde{z})$ is a polynomial of degree r ($r = 3(4n+3)$) with respect to $\tilde{z}$, and then $\mathcal{F}_r(\tilde{z},y) = 0$ naturally leads to the plane third-order algebraic curve $\mathcal{K}_{r-2}$ of genus $r-2 \in \mathbb{N}$ (see Remark 2.2 and Remark 2.3),

$$\mathcal{K}_{r-2} : \mathcal{F}_r(\tilde{z},y) = y^3 + yS_r(\tilde{z}) - T_r(\tilde{z}) = 0, \quad r = 12n+9. \tag{2.22}$$

The algebraic curve $\mathcal{K}_{r-2}$ in (2.22) is compactified by joining three points at infinity

$$P_{\infty_i},\ i = 1,2,3,$$

but for notational simplicity the compactification is also denoted by $\mathcal{K}_{r-2}$. Points on

$$\mathcal{K}_{r-2} \setminus \{P_{\infty_i}\},\ i = 1,2,3$$

are represented as pairs $P = (\tilde{z}, y(P))$, where $y(\cdot)$ is the meromorphic function on $\mathcal{K}_{r-2}$ satisfying

$$\mathcal{F}_r(\tilde{z}, y(P)) = 0.$$

The complex structure on $\mathcal{K}_{r-2}$ is defined in the usual way by introducing local coordinates

$$\zeta_{Q_0} : P \to \zeta = \tilde{z} - \tilde{z}_0$$

near points

$$Q_0 = (\tilde{z}_0, y(Q_0)) \in \mathcal{K}_{r-2} \backslash \{P_0 = (0,0)\},$$

which are neither branch nor singular points of $\mathcal{K}_{r-2}$; near $P_0 = (0,0)$, the local coordinate is

$$\zeta_{P_0} : P \to \zeta = \tilde{z}^{\frac{1}{3}}, \tag{2.23}$$

and similarly at branch and singular points of $\mathcal{K}_{r-2}$; near the points $P_{\infty_i} \in \mathcal{K}_{r-2}$, the local coordinates are

$$\zeta_{P_{\infty_i}} : P \to \zeta = \tilde{z}^{-1}, \quad i = 1, 2, 3. \tag{2.24}$$

The holomorphic map $*$, changing sheets, is defined by

$$* : \begin{cases} \mathcal{K}_{r-2} \to \mathcal{K}_{r-2}, \\ P = (\tilde{z}, y_j(\tilde{z})) \to P^* = (\tilde{z}, y_{j+1 (\mathrm{mod}\ 3)}(\tilde{z})), \quad j = 0, 1, 2, \end{cases}$$
$$P^{**} := (P^*)^*, \quad \text{etc.}, \tag{2.25}$$

where $y_j(\tilde{z})$, $j = 0, 1, 2$ denote the three branches of $y(P)$ satisfying $\mathcal{F}_r(\tilde{z}, y) = 0$.

Finally, positive divisors on $\mathcal{K}_{r-2}$ of degree $r-2$ are denoted by

$$\mathcal{D}_{P_1,\ldots,P_{r-2}} : \begin{cases} \mathcal{K}_{r-2} \to \mathbb{N}_0, \\ P \to \mathcal{D}_{P_1,\ldots,P_{r-2}}(P) = \begin{cases} k \text{ if } P \text{ occurs } k \text{ times in } \{P_1, \ldots, P_{r-2}\}, \\ 0 \text{ if } P \notin \{P_1, \ldots, P_{r-2}\}. \end{cases} \end{cases} \tag{2.26}$$

In particular, the divisor $(\phi(\cdot))$ of a meromorphic function $\phi(\cdot)$ on $\mathcal{K}_{r-2}$ is defined by

$$(\phi(\cdot)) : \mathcal{K}_{r-2} \to \mathbb{Z}, \quad P \mapsto \omega_\phi(P), \tag{2.27}$$

where $\omega_\phi(P) = m_0 \in \mathbb{Z}$ if $(\phi \circ \zeta_P^{-1})(\zeta) = \sum_{n=m_0}^{\infty} c_n(P)\zeta^n$ for some $m_0 \in \mathbb{Z}$ by using a chart (U_P, ζ_P) near $P \in \mathcal{K}_{r-2}$.

Remark 2.2. In this paper, we make the following two assumptions about the curve $\mathcal{K}_{r-2}$:

(i) The affine plane algebraic curve $\mathcal{K}_{r-2}$ is nonsingular.

(ii) The leading coefficients of $S_r(z)$, $T_r(z)$ satisfy

$$\pm \frac{2\sqrt{-3}}{9} S_{r,0}^{3/2} - T_{r,0} \neq 0. \tag{2.28}$$

Multiplying the polynomial $T_r(\tilde{z})$ by a constant $\hbar \in \mathbb{R}$ (or $\mathbb{C}$), one easily finds that the curve (2.22) changes into

$$\widetilde{\mathcal{F}}_r(\tilde{z}, y) = y^3 + y S_r(\tilde{z}) - \hbar T_r(\tilde{z}) = 0.$$

Since there exists a constant $\hbar$ such that

$$\frac{2\sqrt{-3}}{9} S_{r,0}^{3/2} - \hbar T_{r,0} \neq 0,$$

we assume (2.28) is always true for the curve $\mathcal{K}_{r-2}$ (2.22) without loss of generality.

Next, we offer a few words about computing the genus of the curve (2.22) under the two assumptions in Remark 2.2.

Remark 2.3. In this paper, we denote by $\overline{\mathcal{K}}$ the associated projective curve of an affine curve $\mathcal{K}$. There are two approaches to compute the genus g of $\overline{\mathcal{K}}_{r-2}$. One of them is to use the formula

$$g = (n-1)(n-2)/2, \tag{2.29}$$

where n is the degree of corresponding homogeneous polynomial of $\overline{\mathcal{K}}_{r-2}$, if the curve $\mathcal{K}_{r-2}$ is nonsingular (smooth). The Fermat curve is a celebrated example of smooth projective curves. In general, the projective curve $\overline{\mathcal{K}}_{r-2}$ may be singular even if the associated affine curve $\mathcal{K}_{r-2}$ is nonsingular. In this case one has to account for the singularities at infinity and properly amend the genus formula (2.29) according to the results of Clebsch, Noether, and Plücker. An alternative and more efficient way is to use a special case of the Riemann–Hurwitz formula. The g-number g [34] of $\mathcal{K}_{r-2}$ and hence the genus of $\mathcal{K}_{r-2}$ if $\mathcal{K}_{r-2}$ is nonsingular (smooth) is

$$g = 1 - N + B/2 \quad \text{with} \quad B = \sum_{P\in\mathcal{K}_{r-2}} (k(P)-1), \tag{2.30}$$

where N is the number of sheets of $\mathcal{K}_{r-2}$, B is the total branching number of sheets of $\mathcal{K}_{r-2}$, and $k(P)-1$ is the branching order of $P \in \mathcal{K}_{r-2}$. In the current DP case, one easily finds $N = 3$. Next, one accounts for the computation of B. The discriminant $\Delta(\tilde{z})$ of the curve (2.22) is defined by $\Delta(\tilde{z}) = 27T_r^2(\tilde{z}) + 4S_r^3(\tilde{z}) = \tilde{z}^2\Delta_1(\tilde{z})$, where $\Delta_1(\tilde{z})$ is a polynomial of degree $2r-2$ with $\Delta_1(0) \neq 0$. Hence the Riemann surface defined by the compactification of (2.22) can have at most $2r$ double points. However, since $\tilde{z} = 0$ is a triple root of (2.22), there are at most $2r-2$ double points on $\mathcal{K}_{r-2}$. Then if all branch points except P_0 are distinct double points, one obtains (taking into account the triple point at P_0)

$$\begin{aligned} B &= \sum_{P\in\mathcal{K}_{r-2}} (k(P)-1) = \sum_{P\in\mathcal{K}_{r-2}\setminus\{P_0\}} (k(P)-1) + (k(P_0)-1) \\ &= (2r-2) + 2 = 24n + 18. \end{aligned}$$

Substituting the value of N, B into the Riemann–Hurwitz formula (2.30), we derive $g = 12n + 7 = r - 2$.

Obviously, the DP-type curve $\mathcal{K}_{r-2}$ differs from other kinds of algebraic curves (such as KdV-type, AKNS-type, Bsq-type, etc.) in the sense that it is compactified by three distinct points P_{∞_i} $(i = 1, 2, 3)$ at infinity. Moreover, the genus of $\mathcal{K}_{r-2}$ is not $r-2$ if we remove the assumption (ii) in Remark 2.2. In the KdV (or AKNS, Bsq) case, the topological genus is uniquely determined as long as the given affine curve is nonsingular. However, in the DP case, the assumption that the affine curve is nonsingular cannot ensure that its topological genus is of one type. Thus we add a condition (2.28) to the curve $\mathcal{K}_{r-2}$.

Remark 2.4. We investigate what happens at the point infinity on our DP-type curve $\mathcal{K}_{r-2}$. Following the treatment in [11] we substitute the variable $v = \tilde{z}^{-1}$ into (2.22), which yields

$$\begin{aligned} &(v^{4n+3}y)^3 + (S_{r,0} + S_{r,1}v^2 + \cdots + S_{r,4n+2}v^{8n+4})v^{4n+3}y \\ &\quad - (T_{r,0} + \cdots + T_{r,6n+4}v^{12n+8}) = 0. \end{aligned} \tag{2.31}$$

Let $v_1 = v^{4n+3}y$, and (2.31) becomes

$$v_1^3 + S_{r,0}v_1 - T_{r,0} = 0 \tag{2.32}$$

as $v \to 0$ (corresponding to $\tilde{z} \to \infty$). This corresponds to three distinct points P_{∞_j}, $j = 1, 2, 3$ at infinity (each with multiplicity one), given by the three points $(0, \aleph_j)$ for $j = 1, 2, 3$, where $\aleph_j$ $(j = 1, 2, 3)$ are the three distinct roots of (2.32). As each point at infinity has multiplicity one, none are branch points, and consequently each admits the local coordinate (2.24) for $|\tilde{z}|$ sufficiently large.

Similarly, near point $P_0 = (0, 0) \in \mathcal{K}_{r-2}$, one finds $y^3 = 0$ by taking $\tilde{z} \to 0$ in (2.22). This corresponds to one point of multiplicity three at $\tilde{z} = 0$. We therefore use the coordinate (2.23) at the branch point P_0.

3. The stationary DP formalism. In this section, we are devoted to a detailed study of the stationary DP hierarchy. Our principle tools are derived from a fundamental meromorphic function ϕ on the algebraic curve $\mathcal{K}_{r-2}$. With the help of ϕ we study the Baker–Akhiezer vector ψ and Dubrovin-type equations.

First, we give a brief description about the Baker–Akhiezer functions. The exponential e^z is analytic in $\mathbb{C}$ and has an essential singularity at the point $z = \infty$. If $q(z)$ is a rational function, then $f(z) = e^{q(z)}$ is analytic in $\bar{\mathbb{C}} = \mathbb{CP}^1$ everywhere except at the poles of $q(z)$, where $f(z)$ has essential singular points. In the last century Clebsh and Gordan considered generalizing functions of exponential type to Riemann surfaces of higher genus. Baker noted that such functions of exponential type can be expressed in terms of theta functions of Riemann surfaces. Akhiezer first directed attention to the fact that under certain conditions functions of exponential type on hyper-elliptic Riemann surfaces are eigenfunctions of second-order linear differential operators. Following the established tradition, we call functions of exponential type on Riemann surfaces Baker–Akhiezer functions.

Next, we introduce the stationary vector Baker–Akhiezer function $\psi = (\psi_1, \psi_2, \psi_3)^t$

$$\begin{aligned} &\psi_x(P, x, x_0) = U(u(x), \tilde{z}(P))\psi(P, x, x_0), \\ &\tilde{z}V(u(x), \tilde{z}(P))\psi(P, x, x_0) = y(P)\psi(P, x, x_0), \\ &\psi_2(P, x_0, x_0) = 1, \quad P = (\tilde{z}, y) \in \mathcal{K}_{r-2} \setminus \{P_{\infty_i}, P_0\},\ i = 1, 2, 3,\ x \in \mathbb{C}. \end{aligned} \tag{3.1}$$

Closely related to $\psi(P, x, x_0)$ is the meromorphic function $\phi(P, x)$ on $\mathcal{K}_{r-2}$ defined by

$$\phi(P, x) = \tilde{z}\frac{\psi_{2,x}(P, x, x_0)}{\psi_2(P, x, x_0)}, \quad P \in \mathcal{K}_{r-2},\ x \in \mathbb{C} \tag{3.2}$$

such that

$$\psi_2(P, x, x_0) = \exp\left(\tilde{z}^{-1}\int_{x_0}^{x} \phi(P, x')dx'\right), \quad P \in \mathcal{K}_{r-2} \setminus \{P_{\infty_i}, P_0\}, \quad i = 1, 2, 3. \tag{3.3}$$

Since ϕ is the fundamental ingredient for the construction of algebro-geometric solutions of the stationary DP hierarchy, we next seek its connection with the recursion formalism of section 2. By using (3.1), a direct calculation gives

$$\phi = \tilde{z}\frac{yV_{31} + C_r}{yV_{21} + A_r} = \frac{\tilde{z}F_r}{y^2V_{31} - yC_r + D_r} = \tilde{z}\frac{y^2V_{21} - yA_r + B_r}{E_r}, \tag{3.4}$$

where

$$\begin{aligned} A_r &= \tilde{z}(V_{23}V_{31} - V_{33}V_{21}) \\ &= \tilde{z}[V_{23}V_{31} + V_{21}(V_{22} + V_{11})], \\ B_r &= \tilde{z}^2[V_{22}(V_{11}V_{21} + V_{23}V_{31}) - V_{21}(V_{12}V_{21} + V_{23}V_{32})], \\ C_r &= \tilde{z}(V_{21}V_{32} - V_{22}V_{31}) \\ &= \tilde{z}[V_{21}V_{32} + V_{31}(V_{11} + V_{33})], \end{aligned} \tag{3.5}$$

$$\begin{aligned} D_r &= \tilde{z}^2[V_{31}(V_{11}V_{33} - V_{13}V_{31}) + V_{32}(V_{21}V_{33} - V_{23}V_{31})], \\ E_r &= \tilde{z}^2[V_{23}(V_{21}V_{33} - V_{11}V_{21} - V_{23}V_{31}) + V_{13}V_{21}^2], \\ F_r &= \tilde{z}^2[V_{31}(V_{22}V_{32} - V_{11}V_{32} + V_{12}V_{31}) - V_{21}V_{32}^2]. \end{aligned} \tag{3.6}$$

The quantities $A_r, \ldots, F_r$ in (3.5) and (3.6) are of course not independent of each other. There exist various interrelationships between them and S_r, T_r, some of which are summarized below.

LEMMA 3.1. *Let* $(\tilde{z}, x) \in \mathbb{C}^2$. *Then*

$$\begin{aligned} V_{21}F_r &= V_{31}D_r - C_r^2 - V_{31}^2 S_r, \\ A_r F_r &= T_r V_{31}^2 + C_r D_r, \end{aligned} \tag{3.7}$$

$$\begin{aligned} V_{31}E_r &= V_{21}B_r - A_r^2 - V_{21}^2 S_r, \\ E_r C_r &= T_r V_{21}^2 + A_r B_r, \end{aligned} \tag{3.8}$$

$$\begin{aligned} V_{21}D_r + V_{31}B_r - V_{21}V_{31}S_r + A_r C_r &= 0, \\ T_r V_{21}V_{31} + S_r C_r V_{21} + S_r A_r V_{31} - A_r D_r - B_r C_r &= 0, \\ E_r F_r &= -T_r C_r V_{21} - T_r A_r V_{31} + B_r D_r, \end{aligned} \tag{3.9}$$

$$\begin{aligned} E_{r,x} &= -2S_r V_{21} + 3B_r, \\ V_{31}F_{r,x} &= -2V_{31}^2 S_r + 3V_{31}D_r + 2\tilde{z}^{-2} m V_{33} F_r + V_{31} m J_r, \end{aligned} \tag{3.10}$$

where

$$J_r = V_{22}^2 V_{32} - V_{11}V_{22}V_{32} - V_{13}V_{31}V_{32} - V_{23}V_{32}^2 + 2V_{21}V_{32}V_{12} + V_{31}V_{33}V_{12}.$$

Proof. Using (2.22) and (3.4), we have

$$\begin{aligned} F_r V_{21} y + F_r A_r &= V_{31}^2 y^3 + (V_{31}D_r - C_r^2) y + C_r D_r \\ &= (V_{31}D_r - C_r^2 - V_{31}^2 S_r) y + T_r V_{31}^2 + C_r D_r, \end{aligned}$$

$$\begin{aligned} E_r V_{31} y + E_r C_r &= V_{21}^2 y^3 + (V_{21}B_r - A_r^2) y + A_r B_r \\ &= (V_{21}B_r - A_r^2 - V_{21}^2 S_r) y + T_r V_{21}^2 + A_r B_r, \end{aligned}$$

$$\begin{aligned} E_r F_r &= (y^2 V_{21} - y A_r + B_r)(y^2 V_{31} - y C_r + D_r) \\ &= (V_{21}D_r + V_{31}B_r - V_{21}V_{31}S_r + A_r C_r) y^2 \\ &\quad + (T_r V_{21}V_{31} + S_r C_r V_{21} + S_r A_r V_{31} - A_r D_r - B_r C_r) y. \\ &\quad - T_r C_r V_{21} - T_r A_r V_{31} + B_r D_r. \end{aligned}$$

By comparing the same powers of y, we arrive at (3.7)–(3.9). With the help of (3.6) and the stationary zero-curvature equation (2.10), we have

$$\begin{aligned}
E_{r,x} &= \tilde{z}^2[V_{21}(V_{33}^2 - V_{11}V_{33} - V_{22}V_{33} + V_{22}V_{11}) - V_{33}V_{23}V_{31} \\
&\quad + 2V_{22}V_{23}V_{31} - V_{31}V_{23}V_{11} + 2V_{31}V_{13}V_{21} - V_{23}V_{21}V_{32} - V_{12}V_{21}^2] \\
&= -2S_rV_{21} + 3B_r, \\
V_{31}F_{r,x} &= \tilde{z}^2[2V_{31}^2(V_{21}V_{12} - V_{11}V_{22} + V_{13}V_{31} - V_{11}V_{33} + V_{32}V_{23} - V_{22}V_{33}) \\
&\quad + V_{31}^2V_{11}V_{22} - 3V_{31}^3V_{13} + 3V_{31}^2V_{11}V_{33} - 3V_{31}^2V_{23}V_{32} + V_{31}^2V_{22}V_{33}] \\
&\quad + 2\tilde{z}^{-2}mV_{33}F_r + mV_{31}J_r \\
&= -2V_{31}^2S_r + 3V_{31}D_r + 2\tilde{z}^{-2}mV_{33}F_r + mV_{31}J_r,
\end{aligned}$$

which is just (3.10). □

By inspection of (2.11) and (3.6), one infers that E_r and $\tilde{z}^2F_r$ are polynomials with respect to $\tilde{z}$ of degree $r-5$ and $r-3$, respectively. Let $\{\mu_j(x)\}_{j=1,\dots,r-5}$ and $\{\nu_j(x)\}_{j=1,\dots,r-3}$ denote the zeros of $E_r(x)$ and $\tilde{z}^2F_r(x)$, respectively. Hence we may write

$$E_r = u\prod_{j=1}^{r-5}(\tilde{z} - \mu_j(x)), \tag{3.11}$$

$$F_r = -uu_x^2\tilde{z}^{-2}\prod_{j=1}^{r-3}(\tilde{z} - \nu_j(x)). \tag{3.12}$$

Defining

$$\hat{\mu}_j(x) = \left(\mu_j(x), -\frac{A_r(\mu_j(x), x)}{V_{21}(\mu_j(x), x)}\right) \in \mathcal{K}_{r-2},\ j = 1, \dots, r-5,\ x \in \mathbb{C}, \tag{3.13}$$

$$\hat{\nu}_j(x) = \left(\nu_j(x), -\frac{C_r(\nu_j(x), x)}{V_{31}(\nu_j(x), x)}\right) \in \mathcal{K}_{r-2},\ j = 1, \dots, r-3,\ x \in \mathbb{C}. \tag{3.14}$$

One infers from (3.4) that the divisor $(\phi(P,x))$ of $\phi(P,x)$ is given by

$$(\phi(P,x)) = \mathcal{D}_{P_0,\underline{\hat{\nu}}(x)}(P) - \mathcal{D}_{P_{\infty_1},\underline{\hat{\mu}}(x)}(P), \tag{3.15}$$

where

$$\underline{\hat{\nu}}(x) = \{\hat{\nu}_1(x), \dots, \hat{\nu}_{r-3}(x)\}, \quad \underline{\hat{\mu}}(x) = \{P_{\infty_2}, P_{\infty_3}, \hat{\mu}_1(x), \dots, \hat{\mu}_{r-5}(x)\}.$$

That is, $P_0, \hat{\nu}_1(x), \dots, \hat{\nu}_{r-3}(x)$ are the $r-2$ zeros of $\phi(P,x)$ and $P_{\infty_1}, P_{\infty_2}, P_{\infty_3}, \hat{\mu}_1(x), \dots, \hat{\mu}_{r-5}(x)$ its $r-2$ poles.

Since from (2.25), $y_j(\tilde{z})$, $j = 0,1,2$ satisfy $\mathcal{F}_r(\tilde{z}, y) = 0$, that is,

$$(y - y_0(\tilde{z}))(y - y_1(\tilde{z}))(y - y_2(\tilde{z})) = y^3 + yS_r(\tilde{z}) - T_r(\tilde{z}) = 0, \tag{3.16}$$

we can easily get

$$\begin{aligned}
y_0 + y_1 + y_2 &= 0, \\
y_0y_1 + y_0y_2 + y_1y_2 &= S_r(\tilde{z}), \\
y_0y_1y_2 &= T_r(\tilde{z}), \\
y_0^2 + y_1^2 + y_2^2 &= -2S_r(\tilde{z}), \\
y_0^3 + y_1^3 + y_2^3 &= 3T_r(\tilde{z}), \\
y_0^2y_1^2 + y_0^2y_2^2 + y_1^2y_2^2 &= S_r^2(\tilde{z}).
\end{aligned} \tag{3.17}$$

Further properties of $\phi(P,x)$ and $\psi_2(P,x,x_0)$ are summarized as follows.

THEOREM 3.2. *Assume* (3.1), (3.2), $P = (\tilde{z}, y) \in \mathcal{K}_{r-2} \setminus \{P_{\infty_i}, P_0\}$, $i = 1, 2, 3$, *and let* $(\tilde{z}, x, x_0) \in \mathbb{C}^3$. *Then*

$$\begin{aligned}\phi_{xx}(P,x) + 3\tilde{z}^{-1}\phi(P,x)\phi_x(P,x) + \tilde{z}^{-2}\phi^3(P,x) - \frac{m_x(x)}{m(x)}\phi_x(P,x)\\ - \phi(P,x) - \tilde{z}^{-1}\frac{m_x(x)}{m(x)}\phi^2(P,x) + m(x)\tilde{z}^{-1} + \frac{m_x(x)}{m(x)}\tilde{z} = 0,\end{aligned} \tag{3.18}$$

$$\phi(P,x)\phi(P^*,x)\phi(P^{**},x) = -\tilde{z}^3\frac{F_r(\tilde{z},x)}{E_r(\tilde{z},x)}, \tag{3.19}$$

$$\phi(P,x) + \phi(P^*,x) + \phi(P^{**},x) = \tilde{z}\frac{E_{r,x}(\tilde{z},x)}{E_r(\tilde{z},x)}, \tag{3.20}$$

$$\begin{aligned}\frac{1}{\phi(P,x)} + \frac{1}{\phi(P^*,x)} + \frac{1}{\phi(P^{**},x)} = \frac{F_{r,x}(\tilde{z},x)}{\tilde{z}F_r(\tilde{z},x)} - \frac{m(x)J_r(\tilde{z},x)}{\tilde{z}F_r(\tilde{z},x)}\\ - \frac{2m(x)V_{33}(\tilde{z},x)}{\tilde{z}^3V_{31}(\tilde{z},x)},\end{aligned} \tag{3.21}$$

$$\begin{aligned}y(P)\phi(P,x) + y(P^*)\phi(P^*,x) + y(P^{**})\phi(P^{**},x)\\ = \tilde{z}\frac{3T_r(\tilde{z})V_{21}(\tilde{z},x) + 2S_r(\tilde{z})A_r(\tilde{z},x)}{E_r(\tilde{z},x)},\end{aligned} \tag{3.22}$$

$$\psi_2(P,x,x_0)\psi_2(P^*,x,x_0)\psi_2(P^{**},x,x_0) = \frac{E_r(\tilde{z},x)}{E_r(\tilde{z},x_0)}, \tag{3.23}$$

$$\psi_{2,x}(P,x,x_0)\psi_{2,x}(P^*,x,x_0)\psi_{2,x}(P^{**},x,x_0) = -\frac{F_r(\tilde{z},x)}{E_r(\tilde{z},x_0)}, \tag{3.24}$$

(3.25)
$$\begin{aligned}\psi_2(P,x,x_0) = \left[\frac{E_r(\tilde{z},x)}{E_r(\tilde{z},x_0)}\right]^{1/3}\\ \times \exp\left(\int_{x_0}^{x}\frac{y(P)^2V_{21}(\tilde{z},x') - y(P)A_r(\tilde{z},x') + \frac{2}{3}S_r(\tilde{z})V_{21}(\tilde{z},x')}{E_r(\tilde{z},x')}dx'\right).\end{aligned}$$

Proof. A straightforward calculation shows that (3.18) holds. Next, we prove (3.19)–(3.25). From (3.2), (3.4), (3.7)–(3.10), and (3.17), we have

$$\begin{aligned}&\phi(P,x)\phi(P^*,x)\phi(P^{**},x)\\ &\quad= \tilde{z}\frac{y_0V_{31} + C_r}{y_0V_{21} + A_r} \times \tilde{z}\frac{y_1V_{31} + C_r}{y_1V_{21} + A_r} \times \tilde{z}\frac{y_2V_{31} + C_r}{y_2V_{21} + A_r}\\ &\quad= \tilde{z}^3\frac{y_0y_1y_2(V_{31})^3 + C_r(V_{31})^2(y_0y_1 + y_0y_2 + y_1y_2) + C_r^2V_{31}(y_0 + y_1 + y_2) + C_r^3}{y_0y_1y_2(V_{21})^3 + A_r(V_{21})^2(y_0y_1 + y_0y_2 + y_1y_2) + A_r^2V_{21}(y_0 + y_1 + y_2) + A_r^3}\\ &\quad= \tilde{z}^3\frac{T_r(V_{31})^3 + C_r(V_{31})^2S_r + C_r^3}{T_r(V_{21})^3 + A_r(V_{21})^2S_r + A_r^3}\\ &\quad= \tilde{z}^3\frac{T_r(V_{31})^3 + C_r(V_{31}D_r - V_{21}F_r - C_r^2) + C_r^3}{T_r(V_{21})^3 + A_r(V_{21}B_r - V_{31}E_r - A_r^2) + A_r^3}\\ &\quad= -\tilde{z}^3\frac{F_r(\tilde{z},x)}{E_r(\tilde{z},x)},\end{aligned}$$

$$\begin{aligned}\phi(P,x)+\phi(P^*,x)+\phi(P^{**},x) &= \tilde{z}\frac{V_{21}(y_0^2+y_1^2+y_2^2)-A_r(y_0+y_1+y_2)+3B_r}{E_r}\\ &= \tilde{z}\frac{-2S_rV_{21}+3B_r}{E_r}\\ &= \tilde{z}\frac{E_{r,x}(\tilde{z},x)}{E_r(\tilde{z},x)},\end{aligned}$$

$$\begin{aligned}\frac{1}{\phi(P,x)}+\frac{1}{\phi(P^*,x)}+\frac{1}{\phi(P^{**},x)} &= \frac{V_{31}(y_0^2+y_1^2+y_2^2)-C_r(y_0+y_1+y_2)+3D_r}{\tilde{z}F_r}\\ &= \frac{-2S_rV_{31}+3D_r}{\tilde{z}F_r}\\ &= \frac{F_{r,x}(\tilde{z},x)}{\tilde{z}F_r(\tilde{z},x)}-\frac{mJ_r(\tilde{z},x)}{\tilde{z}F_r(\tilde{z},x)}-2\frac{mV_{33}}{\tilde{z}^3V_{31}},\end{aligned}$$

$$\begin{aligned}&y(P)\phi(P,x)+y(P^*)\phi(P^*,x)+y(P^{**})\phi(P^{**},x)\\ &\quad= \tilde{z}\frac{V_{21}(y_0^3+y_1^3+y_2^3)-A_r(y_0^2+y_1^2+y_2^2)+B_r(y_0+y_1+y_2)}{E_r}\\ &\quad= \tilde{z}\frac{3T_rV_{21}+2S_rA_r}{E_r},\end{aligned}$$

$$\begin{aligned}&\psi_2(P,x,x_0)\psi_2(P^*,x,x_0)\psi_2(P^{**},x,x_0)\\ &\quad= \exp\left(\tilde{z}^{-1}\int_{x_0}^{x}[\phi(P,x')+\phi(P^*,x')+\phi(P^{**},x')]dx'\right)\\ &\quad= \exp\left(\int_{x_0}^{x}\frac{E_{r,x'}}{E_r}dx'\right)\\ &\quad= \frac{E_r(\tilde{z},x)}{E_r(\tilde{z},x_0)},\end{aligned}$$

$$\begin{aligned}&\psi_{2,x}(P,x,x_0)\psi_{2,x}(P^*,x,x_0)\psi_{2,x}(P^{**},x,x_0)\\ &\quad= \tilde{z}^{-1}\psi_2(P,x,x_0)\phi(P,x)\times\tilde{z}^{-1}\psi_2(P^*,x,x_0)\phi(P^*,x)\\ &\qquad\times\tilde{z}^{-1}\psi_2(P^{**},x,x_0)\phi(P^{**},x)\\ &\quad= -\frac{F_r(\tilde{z},x)}{E_r(\tilde{z},x_0)}.\end{aligned}$$

Using (3.3), (3.4), and (3.10), we obtain

$$\begin{aligned}\psi_2(P,x,x_0) &= \exp\left(\tilde{z}^{-1}\int_{x_0}^{x}\phi(P,x')dx'\right)\\ &= \exp\left(\tilde{z}^{-1}\int_{x_0}^{x}\tilde{z}\frac{y^2V_{21}-yA_r+\frac{2S_rV_{21}+E_{r,x}}{3}}{E_r}dx'\right)\\ &= \exp\left(\int_{x_0}^{x}\frac{y^2V_{21}-yA_r+\frac{2}{3}S_rV_{21}}{E_r}dx'+\frac{1}{3}\int_{x_0}^{x}\frac{E_{r,x'}}{E_r}dx'\right)\\ &= \left[\frac{E_r(\tilde{z},x)}{E_r(\tilde{z},x_0)}\right]^{1/3}\exp\left(\int_{x_0}^{x}\frac{y^2V_{21}-yA_r+\frac{2}{3}S_rV_{21}}{E_r}dx'\right),\end{aligned}$$

which implies (3.25). ∎

Next, we derive Dubrovin-type equations which are first-order coupled systems of differential equations and govern the dynamics of the zeros $\mu_j(x)$ and $\nu_j(x)$ of $E_r(\tilde{z},x)$ and $F_r(\tilde{z},x)$ with respect to x.

Lemma 3.3. *Assume* (2.14) *to hold in the stationary case.*

(i) *Suppose the zeros* $\{\mu_j(x)\}_{j=1,\dots,r-5}$ *of* $E_r(\tilde{z},x)$ *remain distinct for* $x \in \Omega_\mu$, *where* $\Omega_\mu \subseteq \mathbb{C}$ *is open and connected. Then* $\{\mu_j(x)\}_{j=1,\dots,r-5}$ *satisfy the system of differential equations,*

$$\mu_{j,x}(x) = -\frac{[S_r(\mu_j(x)) + 3y(\hat{\mu}_j(x))^2]V_{21}(\mu_j(x),x)}{u\prod_{\substack{k=1\\k\neq j}}^{r-5}(\mu_j(x)-\mu_k(x))}, \quad j=1,\dots,r-5, \tag{3.26}$$

with initial conditions

$$\{\hat{\mu}_j(x_0)\}_{j=1,\dots,r-5} \in \mathcal{K}_{r-2} \tag{3.27}$$

for some fixed $x_0 \in \Omega_\mu$. *The initial value problem* (3.26), (3.27) *has a unique solution satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_{r-2}), \quad j=1,\dots,r-5. \tag{3.28}$$

(ii) *Suppose the zeros* $\{\nu_j(x)\}_{j=1,\dots,r-3}$ *of* $F_r(\tilde{z},x)$ *remain distinct for* $x \in \Omega_\nu$, *where* $\Omega_\nu \subseteq \mathbb{C}$ *is open and connected. Then* $\{\nu_j(x)\}_{j=1,\dots,r-3}$ *satisfy the system of differential equations,*

$$\nu_{j,x}(x) = \nu_j(x)^2\frac{[S_r(\nu_j(x)) + 3y(\hat{\nu}_j(x))^2]V_{31}(\nu_j(x),x) + m(x)J_r(\nu_j(x),x)}{uu_x^2\prod_{\substack{k=1\\k\neq j}}^{r-3}(\nu_j(x)-\nu_k(x))},$$
$$j=1,\dots,r-3 \tag{3.29}$$

with initial conditions

$$\{\hat{\nu}_j(x_0)\}_{j=1,\dots,r-3} \in \mathcal{K}_{r-2} \tag{3.30}$$

for some fixed $x_0 \in \Omega_\nu$. *The initial value problem* (3.29), (3.30) *has a unique solution satisfying*

$$\hat{\nu}_j \in C^\infty(\Omega_\nu, \mathcal{K}_{r-2}), \quad j=1,\dots,r-3. \tag{3.31}$$

Proof. From (3.7) and (3.8), substituting $\tilde{z} = \mu_j(x)$ and $\nu_j(x)$, respectively, we have

$$V_{21}^2(\mu_j(x),x)S_r(\mu_j(x)) - V_{21}(\mu_j(x),x)B_r(\mu_j(x),x) + A_r^2(\mu_j(x),x) = 0, \tag{3.32}$$
$$V_{31}^2(\nu_j(x),x)S_r(\nu_j(x)) - V_{31}(\nu_j(x),x)D_r(\nu_j(x),x) + C_r^2(\nu_j(x),x) = 0. \tag{3.33}$$

Then it is easy to get

$$\begin{aligned} B_r(\mu_j(x),x) &= V_{21}(\mu_j(x),x)S_r(\mu_j(x)) + \frac{A_r^2(\mu_j(x),x)}{V_{21}(\mu_j(x),x)} \\ &= [S_r(\mu_j(x)) + y(\hat{\mu}_j(x))^2]V_{21}(\mu_j(x),x), \end{aligned} \tag{3.34}$$

$$D_r(\nu_j(x),x) = V_{31}(\nu_j(x),x)S_r(\nu_j(x)) + \frac{C_r^2(\nu_j(x),x)}{V_{31}(\nu_j(x),x)}$$
$$= [S_r(\nu_j(x)) + y(\hat{\nu}_j(x))^2]V_{31}(\nu_j(x),x). \tag{3.35}$$

Inserting (3.34) and (3.35) into (3.10), respectively, we obtain

$$E_{r,x}(\mu_j(x),x) = [S_r(\mu_j(x)) + 3y(\hat{\mu}_j(x))^2]V_{21}(\mu_j(x),x), \tag{3.36}$$
$$F_{r,x}(\nu_j(x),x) = [S_r(\nu_j(x)) + 3y(\hat{\nu}_j(x))^2]V_{31}(\nu_j(x),x) + m(x)J_r(\nu_j(x),x). \tag{3.37}$$

On the other hand, derivatives of (3.11) and (3.12) with respect to x are

$$E_{r,x}|_{\tilde{z}=\mu_j(x)} = -u\mu_{j,x}(x)\prod_{\substack{k=1\\k\neq j}}^{r-5}(\mu_j(x)-\mu_k(x)), \tag{3.38}$$

$$F_{r,x}|_{\tilde{z}=\nu_j(x)} = uu_x^2\nu_j(x)^{-2}\nu_{j,x}(x)\prod_{\substack{k=1\\k\neq j}}^{r-3}(\nu_j(x)-\nu_k(x)). \tag{3.39}$$

Comparing (3.36)–(3.39) leads to (3.26) and (3.29). ∎

Remark 3.4. In Lemma 3.3, we assume that $\{\mu_j(x)\}_{j=1,\dots,r-5}$ are pairwise distinct. However, if two or more of $\{\mu_j(x)\}_{j=1,\dots,r-5}$ coincide at $x = x_0$, the Dubrovin-type equation (3.26) is ill-defined and the stationary algorithm breaks down at such value of x. Moreover, $\theta(\underline{\tilde{z}}(P,\underline{\hat{\mu}}(x))) = \theta(\underline{\tilde{z}}(P,\underline{\hat{\nu}}(x))) \equiv 0$. Therefore, when attempting to solve the Dubrovin-type equation (3.26), they must be augmented with appropriate divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0)} \in \sigma^{r-2}\mathcal{K}_{r-2}$ as initial conditions. The similar analysis holds for $\{\nu_j(x)\}_{j=1,\dots,r-3}$.

4. Stationary algebro-geometric solutions.

In this section we continue our study of the stationary DP hierarchy and will obtain explicit Riemann theta function representations for the meromorphic function ϕ, the Baker–Akhiezer function ψ_2, and the algebro-geometric solutions u for the stationary DP hierarchy.

LEMMA 4.1. *Let $x \in \mathbb{C}$.*

(i) *Near $P_{\infty_1} \in \mathcal{K}_{r-2}$, in terms of the local coordinate $\zeta = \tilde{z}^{-1}$, we have*

$$\phi(P,x) \underset{\zeta\to 0}{=} \frac{1}{\zeta}\sum_{j=0}^{\infty}\kappa_j(x)\zeta^j \quad \text{as } P \to P_{\infty_1}, \tag{4.1}$$

where

$$\kappa_0 = \frac{u_x(x)}{u(x)}, \qquad \kappa_1 = 0, \tag{4.2}$$

$$\kappa_{2,xx} + 3(\kappa_{0,x}\kappa_2 + \kappa_0\kappa_{2,x}) + 3\kappa_0^2\kappa_2 - \kappa_2 + m = \frac{m_x}{m}(\kappa_{2,x} + 2\kappa_0\kappa_2),$$
$$\kappa_3 = 0, \quad \dots\dots \tag{4.3}$$

$$\kappa_{2\varsigma,xx} + 3\sum_{i=0}^{\varsigma}\kappa_{2i}\kappa_{2\varsigma-2i,x} + \sum_{i=0}^{\varsigma}\sum_{\ell=0}^{\varsigma-i}\kappa_{2i}\kappa_{2\ell}\kappa_{2\varsigma-2i-2\ell} - \kappa_{2\varsigma}$$
$$= \frac{m_x}{m}\left(\sum_{i=0}^{\varsigma}\kappa_{2i}\kappa_{2\varsigma-2i}\right), \tag{4.4}$$

$$\kappa_{2\varsigma+1} = 0, \quad \varsigma \geq 2, \quad \varsigma \in \mathbb{N}. \tag{4.5}$$

(ii) *Near $P_0 \in \mathcal{K}_{r-2}$, in terms of the local coordinate $\zeta = \tilde{z}^{\frac{1}{3}}$, we have*

$$\phi(P,x) \underset{\zeta\to 0}{=} \sum_{j=0}^{\infty} \iota_j(x)\zeta^{j+1} \quad \text{as } P \to P_0, \tag{4.6}$$

where

$$\begin{aligned}
&\iota_0 = -m^{\frac{1}{3}}, \qquad \iota_1 = 0,\\
&\iota_2 = \frac{(m_x/m)\iota_0^2 - 3\iota_0\iota_{0,x}}{3\iota_0^2} = 0, \qquad \iota_3 = 0,\\
&\iota_4 = \frac{\frac{m_x}{m}\iota_{0,x} + \iota_0 - \iota_{0,xx}}{3\iota_0^2}, \qquad \iota_5 = 0,\\
&\iota_6 = \frac{\frac{m_x}{m}(2\iota_0\iota_4 - 1) - 3(\iota_{0,x}\iota_4 + \iota_0\iota_{4,x})}{3\iota_0^2}, \qquad \iota_7 = 0,\\
&\vdots\\
&\iota_{2\varsigma} = \frac{\frac{m_x}{m}(2\iota_0\iota_{2\varsigma-2} + \iota_{2\varsigma-4,x}) + \iota_{2\varsigma-4} - \iota_{2\varsigma-4,xx} - 3(\iota_0\iota_{2\varsigma-2,x} + \iota_{0,x}\iota_{2\varsigma-2})}{3\iota_0^2},
\end{aligned} \tag{4.7}$$

$$\iota_{2\varsigma+1} = 0, \qquad \varsigma \geq 4, \qquad \varsigma \in \mathbb{N}. \tag{4.8}$$

Proof. The existence of these asymptotic expansions (4.1) and (4.6) in terms of local coordinates $\zeta = \tilde{z}^{-1}$ near P_{∞_1} and $\zeta = \tilde{z}^{\frac{1}{3}}$ near P_0 is clear from the explicit form of ϕ in (3.4). Insertion of the polynomials V_{ij} $(i,j = 1,2,3)$ then, in principle, yields the explicit expansion coefficients in (4.1) and (4.6). For example, $\kappa_0 = u_x(x)/u(x)$ and $\kappa_1 = 0$ in (4.2). However, this is a cumbersome procedure, especially with regard to the next to leading coefficients in (4.1). Much more efficient is the actual computation of these coefficients utilizing the Riccati-type equation (3.18). Indeed, in terms of the local coordinate $\zeta = \tilde{z}^{\frac{1}{3}}$, (3.18) can be written as

$$\begin{aligned}
&\phi_{xx}(P,x) + 3\zeta^{-3}\phi(P,x)\phi_x(P,x) + \zeta^{-6}\phi^3(P,x) - \frac{m_x}{m}\phi_x(P,x)\\
&\quad - \phi(P,x) - \zeta^{-3}\frac{m_x}{m}\phi^2(P,x) + m\zeta^{-3} + \frac{m_x}{m}\zeta^3 = 0
\end{aligned} \tag{4.9}$$

near the point P_0. Substituting a power series ansatz

$$\phi \underset{\zeta\to 0}{=} \sum_{j=0}^{\infty} \iota_j(x)\zeta^{j+1} \quad \text{as } P \to P_0$$

into (4.9) and comparing the same powers of ζ then yields (4.7).

Similarly, in terms of the local coordinate $\zeta = \tilde{z}^{-1}$, (3.18) can be written as

$$\begin{aligned}
&\phi_{xx}(P,x) + 3\zeta\phi(P,x)\phi_x(P,x) + \zeta^2\phi^3(P,x) - \phi(P,x) - \frac{m_x(x)}{m(x)}\phi_x(P,x)\\
&\quad - \zeta\frac{m_x(x)}{m(x)}\phi^2(P,x) + m(x)\zeta + \frac{m_x(x)}{m(x)}\zeta^{-1} = 0
\end{aligned} \tag{4.10}$$

near the point P_{∞_1}. Substituting a power series ansatz

$$\phi \underset{\zeta\to 0}{=} \frac{1}{\zeta}\sum_{j=0}^{\infty} \kappa_j(x)\zeta^j \quad \text{as } P \to P_{\infty_1}$$

into (4.10) and comparing the same powers of ζ then yields the indicated Laurent series relations (4.3) and (4.4). Finally, (4.5) and (4.8) arise from the technical treatment in section 2 ($z = \tilde{z}^2$; see (2.19)). $\square$

Remark 4.2. We have derived the explicit expressions for $\kappa_0, \kappa_{2\varsigma+1}, \varsigma \in \mathbb{N}_0$ in Lemma 4.1. However, the coefficients $\kappa_{2\varsigma}, \varsigma \in \mathbb{N}$ in the high-energy expansion of ϕ are still implicit, since (4.3) and (4.4) involve the x-derivatives of $\kappa_{2\varsigma}, \varsigma \in \mathbb{N}$ and hence yield a series of second-order ODEs (or PDEs in the time-dependent case) with variable coefficients. In the process of solving other integrable evolution equations such as classical Thirring system (near the points $P_{0,\pm}$; see [15]), CH hierarchy (near the points $P_{\infty\pm}$; see [14], [15]), if we directly insert an ansatz into a Riccati-type equation, an analogous problem will arise. The DP hierarchy shares some similarities with the CH hierarchy at this point. Since the concrete expressions $\kappa_j, j \geq 2, j \in \mathbb{N}$ are useless in the process of finding the algebro-geometric solutions of DP hierarchy, we do not intend to write out their explicit forms from (3.4).

We assume $\mathcal{K}_{r-2}$ to be nonsingular for the remainder of this section. We now introduce the holomorphic differentials $\eta_l(P)$ on $\mathcal{K}_{r-2}$ defined by

$$\eta_l(P) = \frac{1}{3y(P)^2 + S_r(\tilde{z})} \begin{cases} \tilde{z}^{l-1} d\tilde{z}, & 1 \leq l \leq 8n+5, \\ y(P)\tilde{z}^{l-8n-6} d\tilde{z}, & 8n+6 \leq l \leq 12n+7, \end{cases} \tag{4.11}$$

and choose an appropriate fixed homology basis $\{a_j, b_j\}_{j=1}^{r-2}$ on $\mathcal{K}_{r-2}$ in such a way that the intersection matrix of cycles satisfies

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j,k = 1,\ldots,r-2.$$

Define an invertible matrix $E \in GL(r-2, \mathbb{C})$ as

$$\begin{aligned} E &= (E_{j,k})_{(r-2)\times(r-2)}, \quad E_{j,k} = \int_{a_k} \eta_j, \\ \underline{e}(k) &= (e_1(k), \ldots, e_{r-2}(k)), \quad e_j(k) = (E^{-1})_{j,k}, \end{aligned} \tag{4.12}$$

and the normalized holomorphic differentials

$$\omega_j = \sum_{l=1}^{r-2} e_j(l)\eta_l, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad \int_{b_k} \omega_j = \Gamma_{j,k}, \quad j,k = 1,\ldots,r-2. \tag{4.13}$$

One can see that the matrix $\Gamma = (\Gamma_{i,j})_{(r-2)\times(r-2)}$ is symmetric, and it has a positive-definite imaginary part.

Next, choosing a convenient base point $Q_0 \in \mathcal{K}_{r-2} \setminus \{P_{\infty_1}, P_0\}$, the vector of Riemann constants $\underline{\Xi}_{Q_0}$ is given by (A.45) [15], and the Abel maps $\underline{A}_{Q_0}(\cdot)$ and $\underline{\alpha}_{Q_0}(\cdot)$ are defined by

$$\begin{aligned} &\underline{A}_{Q_0} : \mathcal{K}_{r-2} \to J(\mathcal{K}_{r-2}) = \mathbb{C}^{r-2}/L_{r-2}, \\ &P \mapsto \underline{A}_{Q_0}(P) = (A_{Q_0,1}(P), \ldots, A_{Q_0,r-2}(P)) \\ &\qquad = \left(\int_{Q_0}^{P} \omega_1, \ldots, \int_{Q_0}^{P} \omega_{r-2} \right) (\mathrm{mod}\ L_{r-2}), \end{aligned}$$

and

$$\begin{aligned} &\underline{\alpha}_{Q_0} : \mathrm{Div}(\mathcal{K}_{r-2}) \to J(\mathcal{K}_{r-2}), \\ &\mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_{r-2}} \mathcal{D}(P) \underline{A}_{Q_0}(P); \end{aligned}$$

where $L_{r-2} = \{\underline{z} \in \mathbb{C}^{r-2} | \ \underline{z} = \underline{N} + \Gamma\underline{M}, \ \underline{N}, \ \underline{M} \in \mathbb{Z}^{r-2}\}$.

For brevity, define the function $\underline{z}:\mathcal{K}_{r-2}\times\sigma^{r-2}\mathcal{K}_{r-2}\to\mathbb{C}^{r-2}$ by[1]

$$\underline{z}(P,\underline{Q})=\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}),$$
$$P\in\mathcal{K}_{r-2},\ \underline{Q}=(Q_1,\ldots,Q_{r-2})\in\sigma^{r-2}\mathcal{K}_{r-2}; \tag{4.14}$$

here $\underline{z}(\cdot,\underline{Q})$ is independent of the choice of base point Q_0. The Riemann theta function $\theta(\underline{z})$ associated with $\mathcal{K}_{r-2}$ and the homology is defined by

$$\theta(\underline{z})=\sum_{\underline{n}\in\mathbb{Z}}\exp\left(2\pi i\langle\underline{n},\underline{z}\rangle+\pi i\langle\underline{n},\underline{n}\Gamma\rangle\right),\quad \underline{z}\in\mathbb{C}^{r-2},$$

where $\langle\underline{B},\underline{C}\rangle=\overline{\underline{B}}\cdot\underline{C}^t=\sum_{j=1}^{r-2}\overline{B}_jC_j$ denotes the scalar product in $\mathbb{C}^{r-2}$.

The normalized differential $\omega^{(3)}_{P_{\infty_1}P_0}(P)$ of the third kind is the unique differential holomorphic on $\mathcal{K}_{r-2}\setminus\{P_{\infty_1},P_0\}$ with simple poles at P_{∞_1} and P_0 with residues ±1, respectively, that is,

$$\begin{aligned}\omega^{(3)}_{P_{\infty_1}P_0}(P)&\underset{\zeta\to0}{=}(\zeta^{-1}+O(1))d\zeta,\quad \text{as } P\to P_{\infty_1},\\ \omega^{(3)}_{P_{\infty_1}P_0}(P)&\underset{\zeta\to0}{=}(-\zeta^{-1}+O(1))d\zeta,\quad \text{as } P\to P_0.\end{aligned}\tag{4.15}$$

In particular,

$$\int_{a_j}\omega^{(3)}_{P_{\infty_1}P_0}(P)=0,\quad j=1,\ldots,r-2.$$

Then

$$\begin{aligned}\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1}P_0}(P)&\underset{\zeta\to0}{=}\ln\zeta+e^{(3)}(Q_0)+O(\zeta),\quad \text{as } P\to P_{\infty_1},\\ \int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1}P_0}(P)&\underset{\zeta\to0}{=}-\ln\zeta+e^{(3)}(Q_0)+O(\zeta),\quad \text{as } P\to P_0,\end{aligned}\tag{4.16}$$

where $e^{(3)}(Q_0)$ is an integration constant.

The theta function representation of $\phi(P,x)$ then reads as follows.

THEOREM 4.3. *Assume that the curve $\mathcal{K}_{r-2}$ is nonsingular. Let $P=(\tilde{z},y)\in\mathcal{K}_{r-2}\setminus\{P_{\infty_1},P_0\}$ and let $x,x_0\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$, or equivalently $\mathcal{D}_{\underline{\hat{\nu}}(x)}$, is nonspecial[2] for $x\in\Omega_\mu$. Then*

$$\phi(P,x)=-m^{\frac{1}{3}}(x)\frac{\theta(\underline{\tilde{z}}(P,\underline{\hat{\nu}}(x)))\theta(\underline{\tilde{z}}(P_0,\underline{\hat{\mu}}(x)))}{\theta(\underline{\tilde{z}}(P_0,\underline{\hat{\nu}}(x)))\theta(\underline{\tilde{z}}(P,\underline{\hat{\mu}}(x)))}\exp\left(e^{(3)}(Q_0)-\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1}P_0}\right).\tag{4.17}$$

Proof. Let Φ be defined by the right-hand side of (4.17) with the aim to prove that $\phi=\Phi$. From (4.16) it follows that

$$\begin{aligned}\exp\left(e^{(3)}(Q_0)-\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1}P_0}\right)&\underset{\zeta\to0}{=}\zeta^{-1}+O(1),\quad \text{as } P\to P_{\infty_1},\\ \exp\left(e^{(3)}(Q_0)-\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1}P_0}\right)&\underset{\zeta\to0}{=}\zeta+O(\zeta^2),\quad \text{as } P\to P_0.\end{aligned}\tag{4.18}$$

[1] $\sigma^{r-2}\mathcal{K}_{r-2}=\underbrace{\mathcal{K}_{r-2}\times\cdots\times\mathcal{K}_{r-2}}_{r-2}$.

[2] For the definition of a nonspecial divisor, see [12].

Using (3.15) we immediately know that ϕ has simple poles at $\underline{\hat{\mu}}(x)$ and P_{∞_1} and simple zeros at P_0 and $\underline{\hat{\nu}}(x)$. By (4.17) and a special case of Riemann's vanishing theorem [12], [15], [16], we see that Φ shares the same properties. Hence, using the Riemann–Roch theorem [12], [15], [16] yields the holomorphic function $\Phi/\phi = \gamma$, where γ is a constant with respect to P. Finally, considering the asymptotic expansion of Φ and ϕ near P_0, we obtain

$$\frac{\Phi}{\phi} \underset{\zeta\to 0}{=} \frac{-m^{1/3}(1+O(\zeta))(\zeta+O(\zeta^2))}{-m^{1/3}\zeta+O(\zeta^2)} \underset{\zeta\to 0}{=} 1+O(\zeta), \quad \text{as } P \to P_0, \tag{4.19}$$

from which we conclude that $\gamma = 1$, where we used (4.18) and (4.6). Hence, we prove (4.17). □

Furthermore, let $\omega^{(2)}_{P_0,3}(P)$ denote the normalized differential of the second kind which is holomorphic on $\mathcal{K}_{r-2} \setminus \{P_0\}$ with a pole of order 3 at P_0,

$$\omega^{(2)}_{P_0,3}(P) = \frac{\tilde{z}^{-1}d\tilde{z}}{3(3y(P)^2+S_r(\tilde{z}))} + \sum_{j=1}^{r-2}\lambda_j\eta_j(P) \underset{\zeta\to 0}{=} (\zeta^{-3}+O(1))d\zeta, \quad \text{as } P \to P_0,$$

where the constants $\{\lambda_j\}_{j=1,\dots,r-2} \in \mathbb{C}$ are determined by the normalization condition

$$\int_{a_j} \omega^{(2)}_{P_0,3}(P) = 0, \quad j = 1, \dots, r-2,$$

and the differentials $\{\eta_j(P)\}_{j=1,\dots,r-2}$ (defined in (4.11)) form a basis for the space of holomorphic differentials. Moreover, we define the vector of b-periods of $\omega^{(2)}_{P_0,3}$,

$$\underline{\hat{U}}^{(2)}_3 = (\hat{U}^{(2)}_{3,1}, \dots, \hat{U}^{(2)}_{3,r-2}), \quad \hat{U}^{(2)}_{3,j} = \frac{1}{2\pi i}\int_{b_j}\omega^{(2)}_{P_0,3}, \quad j = 1, \dots, r-2. \tag{4.20}$$

Then

$$\int_{Q_0}^{P} \omega^{(2)}_{P_0,3}(P) \underset{\zeta\to 0}{=} -\frac{1}{2}\zeta^{-2} + e^{(2)}_3(Q_0) + O(\zeta), \quad \text{as } P \to P_0,$$

$$\int_{Q_0}^{P} \omega^{(2)}_{P_0,3}(P) \underset{\zeta\to 0}{=} e^{(2)}_3(Q_0) + f^{(2)}_3(Q_0)\zeta^2 + O(\zeta^4), \quad \text{as } P \to P_{\infty_1},$$

where $e^{(2)}_3(Q_0)$, $f^{(2)}_3(Q_0)$ are integration constants.

Similarly, the theta function representation of the Baker–Akhiezer function $\psi_2(P, x, x_0)$ is summarized in the following theorem.

THEOREM 4.4. *Assume that the curve $\mathcal{K}_{r-2}$ is nonsingular. Let $P = (\tilde{z}, y) \in \mathcal{K}_{r-2} \setminus \{P_{\infty_1}, P_0\}$ and let $x, x_0 \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$, or equivalently $\mathcal{D}_{\underline{\hat{\nu}}(x)}$, is nonspecial for $x \in \Omega_\mu$. Then*

$$\psi_2(P,x,x_0) = \frac{\theta(\underline{\tilde{z}}(P,\underline{\hat{\mu}}(x)))\theta(\underline{\tilde{z}}(P_0,\underline{\hat{\mu}}(x_0)))}{\theta(\underline{\tilde{z}}(P_0,\underline{\hat{\mu}}(x)))\theta(\underline{\tilde{z}}(P,\underline{\hat{\mu}}(x_0)))} \times \exp\left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x')dx' \left(\int_{Q_0}^{P}\omega^{(2)}_{P_0,3} - e^{(2)}_3(Q_0)\right)\right). \tag{4.21}$$

Proof. Assume temporarily that

$$\mu_j(x) \neq \mu_k(x) \quad \text{for } j \neq k \text{ and } x \in \widetilde{\Omega}_\mu \subseteq \Omega_\mu, \tag{4.22}$$

where $\widetilde{\Omega}_\mu$ is open and connected. For the Baker–Akhiezer function ψ_2 we will use the same strategy as was used in the previous proof. Let Ψ denote the right-hand side of

(4.21). We intend to prove $\psi_2 = \Psi$. For that purpose we first investigate the local zeros and poles of ψ_2. Since

$$\psi_2(P,x,x_0) = \exp\left(\tilde{z}^{-1}\int_{x_0}^{x}\phi(P,x')dx'\right), \tag{4.23}$$

we can see that the zeros and poles of ψ_2 can come only from simple poles in the integrand (with positive and negative residues, respectively). By using the definition (3.4) of ϕ, (3.10), and the Dubrovin equations (3.26), we obtain

$$\begin{aligned}\phi(P,x) &= \tilde{z}\frac{y^2V_{21} - yA_r + B_r}{E_r}\\ &= \tilde{z}\frac{y^2V_{21} - yA_r + \frac{2}{3}V_{21}S_r + \frac{1}{3}E_{r,x}}{E_r}\\ &= \tilde{z}\left(\frac{1}{3}V_{21}\frac{3y^2+S_r}{E_r} + \frac{1}{3}\frac{E_{r,x}}{E_r} + \frac{1}{3}\frac{-3yA_r + V_{21}S_r}{E_r}\right)\\ &= \tilde{z}\left(\frac{2}{3}V_{21}\frac{3y^2+S_r}{E_r} + \frac{1}{3}\frac{E_{r,x}}{E_r} - \frac{V_{21}y(y+\frac{A_r}{V_{21}})}{E_r}\right).\end{aligned} \tag{4.24}$$

Hence

$$\begin{aligned}\phi(P,x) &= \mu_j\left(-\frac{2}{3}\frac{\mu_{j,x}}{\tilde{z}-\mu_j} - \frac{1}{3}\frac{\mu_{j,x}}{\tilde{z}-\mu_j} + O(1)\right)\\ &= -\mu_j\frac{\mu_{j,x}}{\tilde{z}-\mu_j} + O(1), \quad \text{as } \tilde{z}\to\mu_j(x),\end{aligned} \tag{4.25}$$

where

$$y \to y(\hat{\mu}_j(x)) = -\frac{A_r(\mu_j(x))}{V_{21}(\mu_j(x))}, \quad \text{as } \tilde{z}\to\mu_j(x).$$

More concisely,

$$\phi(P,x) = \mu_j(x)\frac{\partial}{\partial x}\ln(\tilde{z}-\mu_j(x)) + O(1) \quad \text{for } P \text{ near } \hat{\mu}_j(x), \tag{4.26}$$

which together with (4.23) yields

$$\begin{aligned}\psi_2(P,x,x_0) &= \exp\left(\int_{x_0}^{x}dx'\left(\frac{\partial}{\partial x'}\ln(\tilde{z}-\mu_j(x')) + O(1)\right)\right)\\ &= \frac{\tilde{z}-\mu_j(x)}{\tilde{z}-\mu_j(x_0)}O(1)\\ &= \begin{cases}(\tilde{z}-\mu_j(x))O(1) & \text{for } P \text{ near } \hat{\mu}_j(x) \neq \hat{\mu}_j(x_0),\\ O(1) & \text{for } P \text{ near } \hat{\mu}_j(x) = \hat{\mu}_j(x_0),\\ (\tilde{z}-\mu_j(x_0))^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0) \neq \hat{\mu}_j(x),\end{cases}\end{aligned} \tag{4.27}$$

where $O(1) \neq 0$ in (4.27). Consequently, all zeros and poles of ψ_2 and Ψ on $\mathcal{K}_{r-2} \setminus \{P_{\infty_1}, P_0\}$ are simple and coincident. It remains to identify the behavior of ψ_2 and Ψ near P_{∞_1} and P_0.

(i) Near P_{∞_1}, from (4.1), we infer

$$\exp\left(\tilde{z}^{-1}\int_{x_0}^{x}dx'\phi(P,x')\right) \underset{\zeta\to 0}{=} 1 + \int_{x_0}^{x}\kappa_0(x')dx' + O(\zeta^2), \quad \text{as } P \to P_{\infty_1}. \tag{4.28}$$

Taking into account the expression (4.21) for Ψ then shows that ψ_2 and Ψ have identical exponential behavior near P_{∞_1}.

(ii) Near P_0, from (4.6), we arrive at

$$\tilde{z}^{-1}\int_{x_0}^{x} dx'\phi(P,x') \underset{\zeta\to 0}{=} -\int_{x_0}^{x} m^{\frac{1}{3}}(x')dx'\left(\zeta^{-2}+O(\zeta^2)\right), \quad \text{as } P\to P_0.$$

Taking into account the expression (4.21) for Ψ then shows that ψ_2 and Ψ have identical exponential behavior up to order $O(\zeta^2)$ near P_0.

The uniqueness result for Baker–Akhiezer functions [13], [15], [16], [18] then completes the proof $\psi_2 = \Psi$ as both functions share the same singularities and zeros. The extension of this result from $x \in \widetilde{\Omega}_\mu$ to $x \in \Omega_\mu$ then simply follows from the continuity of $\underline{\alpha}_{Q_0}$ and the hypothesis of $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ being nonspecial for $x \in \Omega_\mu$. ∎

The asymptotic behavior of $y(P)$ and S_r near P_{∞_1} is summarized as follows.

LEMMA 4.5.

$$y(P) \underset{\zeta\to 0}{=} -\frac{1}{3}\varrho\zeta^{-4n-3}(1+\alpha_0\zeta^2+\alpha_1\zeta^4+O(\zeta^6)), \quad \textit{as } P\to P_{\infty_1}, \tag{4.29}$$

$$S_r \underset{\zeta\to 0}{=} -\frac{1}{3}\zeta^{-8n-6}(1+\beta_0\zeta^2+\beta_1\zeta^4+O(\zeta^6)), \quad \textit{as } P\to P_{\infty_1}, \tag{4.30}$$

where $\varrho = -3\aleph_1$ *and* $\aleph_1$ *is the root of of algebraic equation* (2.32) *corresponding to the point* $P_{\infty_1} \in \mathcal{K}_{r-2}$.

Proof. From (3.1) and (3.2), we arrive at

$$y(P) = V_{21}\frac{(\tilde{z}^3-\phi\tilde{z}^2)}{m} + V_{22}\tilde{z} + V_{23}\phi. \tag{4.31}$$

Then, in terms of the local coordinate $\zeta = \tilde{z}^{-1}$, insertion of (2.11) and (4.1) into (4.31) yields

$$\begin{aligned} y(P) \;=\; & \frac{1}{m}\sum_{\ell=0}^{n} V_{21}^{(\ell)}(G_\ell)\zeta^{-4(n+1-\ell)}\left(\zeta^{-3}-\zeta^{-2}\sum_{j=0}^{\infty}\kappa_j\zeta^{j-1}\right) \\ & + \sum_{\ell=0}^{n} V_{22}^{(\ell)}(G_\ell)\zeta^{-4(n+1-\ell)-1} \\ & + \sum_{\ell=0}^{n} V_{23}^{(\ell)}(G_\ell)\zeta^{-4(n+1-\ell)}\sum_{j=0}^{\infty}\kappa_j\zeta^{j-1} \\ \underset{\zeta\to 0}{=} & -\frac{1}{3}\varrho\zeta^{-4n-3}(1+\alpha_0\zeta^2+\alpha_1\zeta^4+O(\zeta^6)), \quad \text{as } P\to P_{\infty_1}. \end{aligned} \tag{4.32}$$

Similarly, we recall the definition of S_r,

$$S_r = \tilde{z}^2(V_{11}V_{22}+V_{11}V_{33}+V_{22}V_{33}-V_{12}V_{21}-V_{13}V_{31}-V_{23}V_{32}). \tag{4.33}$$

Insertion of (4.1) into (4.33) leads to (4.30). ∎

A straightforward Laurent expansion of (4.11), (4.12), and (4.13) near P_{∞_1} yields the following results.

LEMMA 4.6. *Assume the curve* $\mathcal{K}_{r-2}$ *to be nonsingular. Then the vector of normalized holomorphic differentials* $\underline{\omega}$ *have the Laurent series*

$$\underline{\omega} = (\omega_1,\ldots,\omega_{r-2}) \underset{\zeta\to 0}{=} (\underline{\rho}_0 + \underline{\rho}_1\zeta + O(\zeta^2))d\zeta \tag{4.34}$$

near P_{∞_1} *with*

$$\underline{\rho}_0 = \frac{-3}{\varrho^2-1}\underline{e}(8n+5) + \frac{\varrho}{\varrho^2-1}\underline{e}(r-2),$$
$$\underline{\rho}_1 = \frac{-3}{\varrho^2-1}\underline{e}(8n+4) + \frac{\varrho}{\varrho^2-1}\underline{e}(r-3),$$

where $\varrho = -3\aleph_1$, $\aleph_1$ *is given in Lemma* 4.5.

Proof. Using (4.29) and (4.30), the local coordinate $\zeta = \tilde{z}^{-1}$ near P_{∞_1}, we obtain

$$(4.35)\quad 3y^2+S_r \underset{\zeta\to 0}{=} \frac{1}{3}\zeta^{-8n-6}[\varrho^2-1+(2\varrho^2\alpha_0-\beta_0)\zeta^2+(2\varrho^2\alpha_1+\varrho^2\alpha_0^2-\beta_1)\zeta^4+O(\zeta^6)].$$

Then

$$(4.36)\quad \frac{1}{3y^2+S_r} \underset{\zeta\to 0}{=} 3\zeta^{8n+6}\left[\frac{1}{\varrho^2-1} - \frac{2\varrho^2\alpha_0-\beta_0}{(\varrho^2-1)^2}\zeta^2 + \left(\frac{2\varrho^2\alpha_1+\varrho^2\alpha_0^2-\beta_1}{-(\varrho^2-1)^2} + \frac{(2\varrho^2\alpha_0-\beta_0)^2}{(\varrho^2-1)^3}\right)\zeta^4 + O(\zeta^6)\right].$$

From (4.11), (4.13), and (4.36), we have

$$\begin{aligned}
\omega_j &= \sum_{l=1}^{r-2} e_j(l)\eta_l = \sum_{l=1}^{8n+5} e_j(l)\frac{\tilde{z}^{l-1}d\tilde{z}}{3y^2+S_r} + \sum_{l=8n+6}^{r-2} e_j(l)\frac{y\tilde{z}^{l-8n-6}d\tilde{z}}{3y^2+S_r}\\
&= -\sum_{l=1}^{8n+5} e_j(l)\frac{\zeta^{-l-1}d\zeta}{3y^2+S_r} - \sum_{l=8n+6}^{r-2} e_j(l)\frac{y\zeta^{-l+8n+4}d\zeta}{3y^2+S_r}\\
&\underset{\zeta\to 0}{=} -\sum_{l=1}^{8n+5} 3e_j(l)\zeta^{-l+8n+5}\left[\frac{1}{\varrho^2-1} - \frac{2\varrho^2\alpha_0-\beta_0}{(\varrho^2-1)^2}\zeta^2\right.\\
&\qquad\left. + \left(\frac{2\varrho^2\alpha_1+\varrho^2\alpha_0^2-\beta_1}{-(\varrho^2-1)^2} + \frac{(2\varrho^2\alpha_0-\beta_0)^2}{(\varrho^2-1)^3}\right)\zeta^4 + O(\zeta^6)\right]d\zeta\\
&\quad + \sum_{l=8n+6}^{r-2} \varrho e_j(l)\zeta^{-l+r-2}\left[\frac{1}{\varrho^2-1} - \frac{2\varrho^2\alpha_0-\beta_0}{(\varrho^2-1)^2}\zeta^2\right.\\
&\qquad\left. + \left(\frac{2\varrho^2\alpha_1+\varrho^2\alpha_0^2-\beta_1}{-(\varrho^2-1)^2} + \frac{(2\varrho^2\alpha_0-\beta_0)^2}{(\varrho^2-1)^3}\right)\zeta^4 + O(\zeta^6)\right]\\
&\quad \times [1+\alpha_0\zeta^2+\alpha_1\zeta^4+O(\zeta^6)]d\zeta\\
&\underset{\zeta\to 0}{=} \left(\frac{-3}{\varrho^2-1}e_j(8n+5) + \frac{\varrho}{\varrho^2-1}e_j(r-2)\right.\\
&\qquad\left. + \left[\frac{-3}{\varrho^2-1}e_j(8n+4) + \frac{\varrho}{\varrho^2-1} \times\ e_j(r-3)\right]\zeta + O(\zeta^2)\right)d\zeta, \qquad (4.37)
\end{aligned}$$

which yields (4.34). ∎

THEOREM 4.7. *Assume that the curve* $\mathcal{K}_{r-2}$ *is nonsingular and let* $x, x_0 \in \mathbb{C}$. *Then*

$$(4.38)\quad \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) + \frac{1}{3}\underline{e}(r-2)(x-x_0) + \underline{e}(8n+5)\int_{x_0}^{x} dx'(\Psi_1(\underline{\mu})),$$

$$(4.39)\quad \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0)}) + \frac{1}{3}\underline{e}(r-2)(x-x_0) + \underline{e}(8n+5)\int_{x_0}^{x} dx'(\Psi_1(\underline{\mu})),$$

where $\Psi_1(\underline{\mu}) = -\sum_{j=1}^{12n+4}\mu_j(x)$. *In particular, the Abel map does not linearize the divisor* $\mathcal{D}_{\underline{\hat{\mu}}(\cdot)}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(\cdot)}$.

Proof. We prove only (4.38) as (4.39) can be obtained from (4.38) and Abel's theorem. Assume temporarily that

$$\mu_j(x) \neq \mu_{j'}(x) \quad \text{for } j \neq j' \text{ and } x \in \widetilde{\Omega}_\mu \subseteq \mathbb{C}, \tag{4.40}$$

where $\widetilde{\Omega}_\mu$ is open and connected. Then using (3.26), (4.11), and (4.13), one computes

$$\begin{aligned}
&\frac{d}{dx}\alpha_{Q_0,l}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) \\
&= \frac{d}{dx}\sum_{j=1}^{12n+4}\int_{Q_0}^{\hat{\mu}_j}\omega_l = \sum_{j=1}^{12n+4}\mu_{j,x}(x)\omega_l(\hat{\mu}_j(x)) \\
&= \sum_{j=1}^{12n+4}\mu_{j,x}(x)\sum_{k=1}^{r-2}e_l(k)\eta_k \\
&= \sum_{j=1}^{12n+4}\frac{-[S_r(\mu_j)+3y(\hat{\mu}_j)^2]V_{21}(\mu_j(x))}{u\prod_{\substack{p=1\\p\neq j}}^{12n+4}(\mu_j-\mu_p)}\left(\sum_{k=1}^{8n+5}e_l(k)\frac{\mu_j^{k-1}}{S_r(\mu_j)+3y(\hat{\mu}_j)^2}\right. \\
&\qquad\qquad\left.+\sum_{k=8n+6}^{r-2}e_l(k)\frac{y(\hat{\mu}_j)\mu_j^{k-8n-6}}{S_r(\mu_j)+3y(\hat{\mu}_j)^2}\right) \\
&= \sum_{j=1}^{12n+4}\frac{-V_{21}(\mu_j(x))}{u\prod_{\substack{p=1\\p\neq j}}^{12n+4}(\mu_j-\mu_p)}\sum_{k=1}^{8n+5}e_l(k)\mu_j^{k-1} + \sum_{j=1}^{12n+4}\frac{-V_{21}(\mu_j(x))y(\hat{\mu}_j)}{u\prod_{\substack{p=1\\p\neq j}}^{12n+4}(\mu_j-\mu_p)} \\
&\quad\times\sum_{k=8n+6}^{r-2}e_l(k)\mu_j^{k-8n-6} \\
&= \sum_{k=1}^{8n+5}e_l(k)\sum_{j=1}^{12n+4}\frac{-V_{21}(\mu_j(x))\mu_j^{k-1}}{u\prod_{\substack{p=1\\p\neq j}}^{12n+4}(\mu_j-\mu_p)} + \sum_{k=8n+6}^{r-2}e_l(k) \\
&\quad\times\sum_{j=1}^{12n+4}\frac{-V_{21}(\mu_j(x))y(\hat{\mu}_j)\mu_j^{k-8n-6}}{u\prod_{\substack{p=1\\p\neq j}}^{12n+4}(\mu_j-\mu_p)} \\
&= \sum_{k=1}^{8n+5}e_l(k)\sum_{j=1}^{12n+4}\frac{-(u\mu_j^{4n}+a_0\mu_j^{4n-2}+\cdots)\mu_j^{k-1}}{u\prod_{\substack{p=1\\p\neq j}}^{12n+4}(\mu_j-\mu_p)} + \sum_{k=8n+6}^{r-2}e_l(k) \\
&\quad\times\sum_{j=1}^{12n+4}\frac{(\frac{1}{3}u\mu_j^{8n+2}+b_0\mu_j^{8n}+\cdots)\mu_j^{k-8n-6}}{u\prod_{\substack{p=1\\p\neq j}}^{12n+4}(\mu_j-\mu_p)}.
\end{aligned}$$

Using the standard Largange interpolation argument then yields

$$\frac{d}{dx}\alpha_{Q_0,l}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \Psi_1(\underline{\mu})e_l(8n+5) + \frac{1}{3}e_l(r-2). \tag{4.41}$$

Then we have

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) + \frac{1}{3}\underline{e}(r-2)(x-x_0) + \underline{e}(8n+5)\int_{x_0}^{x}dx'(\Psi_1(\underline{\mu})). \tag{4.42}$$

The equality (4.39) follows from the linear equivalence

$$\mathcal{D}_{P_{\infty_1}\underline{\hat{\mu}}(x)} \sim \mathcal{D}_{P_0\underline{\hat{\nu}}(x)},$$

that is,

$$\underline{A}_{Q_0}(P_{\infty_1}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \underline{A}_{Q_0}(P_0) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x)}),$$

and (4.42). The extension of all these results from $x \in \widetilde{\Omega}_\mu$ to $x \in \mathbb{C}$ then simply follows from the continuity of $\underline{\alpha}_{Q_0}$ and the hypothesis of $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ being nonspecial on Ω_μ. ∎

Next, we provide an explicit representation for the stationary DP solutions u in terms of the Riemann theta function associated with $\mathcal{K}_{r-2}$, assuming the affine part of $\mathcal{K}_{r-2}$ to be nonsingular.

THEOREM 4.8. *Assume that u satisfies the nth stationary DP equation* (2.14)*, that is, $X_n(u) = 0$, and the curve $\mathcal{K}_{r-2}$ is nonsingular. Let $x \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$, or equivalently $\mathcal{D}_{\underline{\hat{\nu}}(x)}$, is nonspecial for $x \in \Omega_\mu$. Then*

$$u(x) = u(x_0)\frac{\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0)))\theta(\underline{\tilde{z}}(P_{\infty_1}, \underline{\hat{\mu}}(x)))}{\theta(\underline{\tilde{z}}(P_{\infty_1}, \underline{\hat{\mu}}(x_0)))\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x)))}. \tag{4.43}$$

Proof. Using Theorem 4.4, one can write ψ_2 near P_{∞_1} in the coordinate $\zeta = \tilde{z}^{-1}$, as

$$\begin{aligned} &\psi_2(P, x, x_0) \\ &\underset{\zeta\to 0}{=} \left(\sigma_0(x) + \sigma_1(x)\zeta + \sigma_2(x)\zeta^2 + O(\zeta^3)\right) \\ &\quad \times \exp\left(\left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x')dx'\right)\left(f_3^{(2)}(Q_0)\zeta^2 + O(\zeta^4)\right)\right), \quad \text{as } P \to P_{\infty_1}, \end{aligned} \tag{4.44}$$

where the terms $\sigma_0(x), \sigma_1(x)$ and $\sigma_2(x)$ in (4.44) come from the Taylor expansion about P_{∞_1} of the ratios of the theta functions in (4.21). That is,

$$\begin{aligned} &\frac{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x))\right)}{\theta\left(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x))\right)} \\ &\quad = \frac{\theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)})\right)}{\theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_0) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)})\right)} \\ &\quad = \frac{\theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_1}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) + \int_P^{P_{\infty_1}} \underline{\omega}\right)}{\theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_0) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)})\right)} \\ &\quad \underset{\zeta\to 0}{=} \frac{\theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_1}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) - \rho_{0,j}\zeta - \frac{1}{2}\rho_{1,j}\zeta^2 + O(\zeta^3)\right)}{\theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_0) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)})\right)} \\ &\quad \underset{\zeta\to 0}{=} \frac{1}{\theta_1}\left[\theta_0 - \sum_{j=1}^{r-2}\frac{\partial\theta_0}{\partial\tilde{z}_j}\rho_{0,j}\zeta - \frac{1}{2}\sum_{j=1}^{r-2}\left(\frac{\partial\theta_0}{\partial\tilde{z}_j}\rho_{1,j} - \sum_{k=1}^{r-2}\frac{\partial^2\theta_0}{\partial\tilde{z}_j\partial\tilde{z}_k}\rho_{0,j}\rho_{0,k}\right)\zeta^2 + O(\zeta^3)\right] \\ &\quad \underset{\zeta\to 0}{=} \frac{\theta_0 - \partial_x\theta_0\zeta + (\frac{1}{2}\partial_x^2\theta_0 - \partial_{\underline{U}_3^{(2)}}\theta_0)\zeta^2 + O(\zeta^3)}{\theta_1} \end{aligned}$$

$$\underset{\zeta\to 0}{=} \frac{\theta_0}{\theta_1} - \frac{\partial_x\theta_0}{\theta_1}\zeta + \frac{\frac{1}{2}\partial_x^2\theta_0 - \partial_{\underline{U}_3^{(2)}}\theta_0}{\theta_1}\zeta^2 + O(\zeta^3), \quad \text{as } P \to P_{\infty_1}, \tag{4.45}$$

where

$$\theta_0 = \theta_0(x) = \theta(\underline{\tilde{z}}(P_{\infty_1}, \underline{\hat{\mu}}(x))) = \theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_1}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)})\right),$$
$$\theta_1 = \theta_1(x) = \theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x))) = \theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_0) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)})\right),$$

and

$$\partial_{\underline{U}_3^{(2)}} = \sum_{j=1}^{r-2} U_{3,j}^{(2)} \frac{\partial}{\partial \tilde{z}_j}$$

denotes the directional derivative in the direction of the vector of b-periods $\underline{U}_3^{(2)}$, defined by

$$\underline{U}_3^{(2)} = (U_{3,1}^{(2)}, \ldots, U_{3,r-2}^{(2)}), \quad U_{3,j}^{(2)} = \frac{1}{2\pi i}\int_{b_j} \omega_{P_{\infty_1},3}^{(2)}, \quad j = 1, \ldots, r-2, \tag{4.46}$$

with $\omega_{P_{\infty_1},3}^{(2)}$ holomorphic on $\mathcal{K}_{r-2} \setminus \{P_{\infty_1}\}$ with a pole of order 3 at P_{∞_1},

$$\omega_{P_{\infty_1},3}^{(2)}(P) \underset{\zeta\to 0}{=} (\zeta^{-3} + O(1))d\zeta, \quad \text{as } P \to P_{\infty_1}. \tag{4.47}$$

Similarly, we have

$$\begin{aligned}
\frac{\theta\left(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0))\right)}{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x_0))\right)} &= \left(\frac{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x))\right)}{\theta\left(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x))\right)}\right)^{-1}\Bigg|_{x=x_0} \\
&\underset{\zeta\to 0}{=} \left(\frac{\theta_0}{\theta_1}\left(1 - \frac{\partial_x\theta_0}{\theta_0}\zeta + O(\zeta^2)\right)\right)^{-1}\Bigg|_{x=x_0} \\
&\underset{\zeta\to 0}{=} \frac{\theta_1}{\theta_0}\left(1 + \partial_x \ln\theta_0\,\zeta + O(\zeta^2)\right)\Big|_{x=x_0} \\
&\underset{\zeta\to 0}{=} \frac{\theta_1(x_0)}{\theta_0(x_0)}\left(1 + \partial_x \ln\theta_0(x)\Big|_{x=x_0}\zeta + O(\zeta^2)\right), \text{ as } P \to P_{\infty_1}.
\end{aligned} \tag{4.48}$$

Then the Taylor expansion about ψ_2 is as follows:

$$\begin{aligned}
\psi_2(P, x, x_0) &\underset{\zeta\to 0}{=} \frac{\theta(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x)))\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0)))}{\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x)))\theta(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x_0)))} \\
&\quad \times \exp\left(\left(\int_{x_0}^x 2m^{\frac{1}{3}}(x')dx'\right)\left(f_3^{(2)}(Q_0)\zeta^2 + O(\zeta^4)\right)\right) \\
&\underset{\zeta\to 0}{=} \left[\frac{\theta_1(x_0)}{\theta_0(x_0)}\frac{\theta_0(x)}{\theta_1(x)} + \left(\frac{\theta_1(x_0)}{\theta_0(x_0)}\frac{\theta_0(x)}{\theta_1(x)}\,\partial_x \ln\theta_0(x)\Big|_{x=x_0} - \frac{\theta_1(x_0)}{\theta_0(x_0)}\frac{\partial_x\theta_0(x)}{\theta_1(x)}\right)\zeta\right. \\
&\quad \left. + O(\zeta^2)\right] \times \exp\left(\left(\int_{x_0}^x 2m^{\frac{1}{3}}(x')dx'\right)\left(f_3^{(2)}(Q_0)\zeta^2 + O(\zeta^4)\right)\right) \\
&\underset{\zeta\to 0}{=} \left[\frac{\theta_1(x_0)}{\theta_0(x_0)}\frac{\theta_0(x)}{\theta_1(x)} + \frac{\theta_1(x_0)}{\theta_0(x_0)}\frac{\theta_0(x)}{\theta_1(x)}\left(\partial_x \ln\theta_0(x)\Big|_{x=x_0} - \partial_x\ln\theta_0(x)\right)\zeta\right. \\
&\quad \left. + O(\zeta^2)\right] \times \left(1 + \left(f_3^{(2)}(Q_0)\int_{x_0}^x 2m^{\frac{1}{3}}(x')dx'\right)\zeta^2 + O(\zeta^4)\right), \\
&\qquad \text{as } P \to P_{\infty_1}.
\end{aligned} \tag{4.49}$$

Hence, comparing the same powers of ζ in (4.44) and (4.49) gives

$$\begin{aligned} \sigma_0(x) &= \frac{\theta_1(x_0)}{\theta_0(x_0)}\frac{\theta_0(x)}{\theta_1(x)}, \\ \sigma_1(x) &= \frac{\theta_1(x_0)}{\theta_0(x_0)}\frac{\theta_0(x)}{\theta_1(x)}\left(\partial_x \ln\theta_0(x)\Big|_{x=x_0} - \partial_x \ln\theta_0(x)\right). \end{aligned} \tag{4.50}$$

If we set

$$\psi_2 \underset{\zeta\to 0}{=} (\sigma_0(x) + \sigma_1(x)\zeta + \sigma_2(x)\zeta^2 + O(\zeta^3))\exp(\Delta), \quad \text{as } P \to P_{\infty_1} \tag{4.51}$$

with

$$\begin{aligned} \exp(\Delta) &= \exp\left(\left(\int_{x_0}^x 2m^{\frac{1}{3}}(x')dx'\right)\left(f_3^{(2)}(Q_0)\zeta^2 + O(\zeta^4)\right)\right) \\ &= \left(1 + \left(f_3^{(2)}(Q_0)\int_{x_0}^x 2m^{\frac{1}{3}}(x')dx'\right)\zeta^2 + O(\zeta^4)\right), \end{aligned}$$

then we compute its x-derivatives as ($P \to P_{\infty_1}$)

$$\begin{aligned} \psi_{2,x} &\underset{\zeta\to 0}{=} \left(\sigma_{0,x} + \sigma_{1,x}\zeta + O(\zeta^2)\right)\exp(\Delta) + \left(\left(2f_3^{(2)}(Q_0)m^{\frac{1}{3}}\right)\zeta^2 + O(\zeta^4)\right)\psi_2 \\ &\underset{\zeta\to 0}{=} \sigma_{0,x} + \sigma_{1,x}\zeta + O(\zeta^2), \\ \psi_{2,xx} &\underset{\zeta\to 0}{=} \sigma_{0,xx} + \sigma_{1,xx}\zeta + O(\zeta^2), \\ \psi_{2,xxx} &\underset{\zeta\to 0}{=} \sigma_{0,xxx} + \sigma_{1,xxx}\zeta + O(\zeta^2). \end{aligned} \tag{4.52}$$

By eliminating ψ_1 and ψ_3 in (2.3), we arrive at

$$\psi_{2,xxx} = -m\tilde{z}^{-2} + \frac{m_x}{m}\psi_{2,xx} - \frac{m_x}{m}\psi_2 + \psi_{2,x}. \tag{4.53}$$

Substituting (4.52) into (4.53) and comparing the coefficients of ζ^0, we obtain

$$\sigma_{0,xxx} = \frac{m_x}{m}(\sigma_{0,xx} - \sigma_0) + \sigma_{0,x},$$

that is,

$$\frac{(\sigma_{0,xx} - \sigma_0)_x}{\sigma_{0,xx} - \sigma_0} = \frac{m_x}{m} = \frac{(u(x) - u_{xx}(x))_x}{u(x) - u_{xx}(x)},$$

which together with the first line of (4.50) leads to (4.43). ◻

Remark 4.9. We note the unusual fact that P_0, as opposed to P_{∞_i}, $i = 1, 2, 3$, is the essential singularity of ψ_2. What makes matters worse is the intricate x-dependence of the leading-order exponential term in ψ_2, near P_0, as displayed in (4.21). This is in sharp contrast to standard Baker–Akhiezer functions that typically feature a linear behavior with respect to x in connection with their essential singularities of the type $\exp((x - x_0)\zeta^{-2})$ near $\zeta = 0$. Therefore, in Theorem 4.7, the Abel map does not provide the proper change of variables to linearize the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ in the DP context, which is in sharp contrast to standard integrable soliton equations such as the KdV and AKNS hierarchies.

5. The time-dependent DP formalism. In this section we extend the results of section 3 to the time-dependent DP hierarchy. We employ the notation $\widetilde{G}_j$, $\widetilde{V}$, $\widetilde{V}_{ij}$, etc., in order to distinguish them from G_j, V, V_{ij}, etc. In addition, we indicate that the individual pth DP flow by a separate time variable $t_p \in \mathbb{C}$. In analogy to (3.1), we introduce the time-dependent vector Baker–Akhiezer function $\psi = (\psi_1, \psi_2, \psi_3)^t$ by

$$\begin{aligned} \psi_x(P, x, x_0, t_p, t_{0,p}) &= U(u(x, t_p), \tilde{z}(P))\psi(P, x, x_0, t_p, t_{0,p}), \\ \psi_{t_p}(P, x, x_0, t_p, t_{0,p}) &= \widetilde{V}(u(x, t_p), \tilde{z}(P))\psi(P, x, x_0, t_p, t_{0,p}), \\ \tilde{z}V(u(x, t_p), \tilde{z}(P))\psi(P, x, x_0, t_p, t_{0,p}) &= y(P)\psi(P, x, x_0, t_p, t_{0,p}), \\ \psi_2(P, x_0, x_0, t_{0,p}, t_{0,p}) &= 1, \quad x, t_p \in \mathbb{C}, \end{aligned} \tag{5.1}$$

where $\widetilde{V} = (\widetilde{V}_{ij})_{3\times 3}$, and

$$\widetilde{V}_{ij} = \sum_{l=0}^{p} \widetilde{V}_{ij}^{(l)}(\widetilde{G}_l)\tilde{z}^{4(p-l+1)} \quad i, j = 1, \dots, 3, \quad l = 0, \dots, p \tag{5.2}$$

with $\widetilde{V}_{ij}^{(l)}(\widetilde{G}_l)$ determined by $\widetilde{G}_l$, which is defined in (2.6) by substituting $\widetilde{G}_l$ for G_l.

The compatibility conditions of the first three expressions in (5.1) yield that

$$\begin{aligned} U_{t_p}(\tilde{z}) - \widetilde{V}_x(\tilde{z}) + [U(\tilde{z}), \widetilde{V}(\tilde{z})] &= 0, \\ -V_x(\tilde{z}) + [U(\tilde{z}), V(\tilde{z})] &= 0, \\ -V_{t_p}(\tilde{z}) + [\widetilde{V}(\tilde{z}), V(\tilde{z})] &= 0. \end{aligned} \tag{5.3}$$

A direct calculation shows that $yI - \tilde{z}V(\tilde{z})$ satisfies the last two equations in (5.3). Then the characteristic polynomial of Lax matrix $\tilde{z}V(\tilde{z})$ for the DP hierarchy is an independent constant of variables x and t_p with the expansion

$$\det(yI - \tilde{z}V) = y^3 + yS_r(\tilde{z}) - T_r(\tilde{z}), \tag{5.4}$$

where $S_r(\tilde{z})$ and $T_r(\tilde{z})$ are defined as in (2.20) and (2.21). Then the time-dependent DP curve $\mathcal{K}_{r-2}$ is defined by

$$\mathcal{K}_{r-2} : \mathcal{F}_r(\tilde{z}, y) = y^3 + yS_r(\tilde{z}) - T_r(\tilde{z}) = 0. \tag{5.5}$$

In analogy to (3.2), we can define the following meromorphic function $\phi(P, x, t_p)$ on $\mathcal{K}_{r-2}$, the fundamental ingredient for the construction of algebro-geometric solutions of the time-dependent DP hierarchy,

$$\phi(P, x, t_p) = \tilde{z}\frac{\partial_x \psi_2(P, x, x_0, t_p, t_{0,p})}{\psi_2(P, x, x_0, t_p, t_{0,p})}, \quad P \in \mathcal{K}_{r-2},\ x \in \mathbb{C}. \tag{5.6}$$

Using (5.1), a direct calculation shows that

$$\begin{aligned} \phi(P, x, t_p) &= \tilde{z}\frac{yV_{31}(\tilde{z}, x, t_p) + C_r(\tilde{z}, x, t_p)}{yV_{21}(\tilde{z}, x, t_p) + A_r(\tilde{z}, x, t_p)} \\ &= \tilde{z}\frac{F_r(\tilde{z}, x, t_p)}{y^2V_{31}(\tilde{z}, x, t_p) - yC_r(\tilde{z}, x, t_p) + D_r(\tilde{z}, x, t_p)} \\ &= \tilde{z}\frac{y^2V_{21}(\tilde{z}, x, t_p) - yA_r(\tilde{z}, x, t_p) + B_r(\tilde{z}, x, t_p)}{E_r(\tilde{z}, x, t_p)}, \end{aligned} \tag{5.7}$$

where $P = (\tilde{z}, y) \in \mathcal{K}_{r-2}$, $(x, t_p) \in \mathbb{C}^2$, and $A_r(\tilde{z}, x, t_p)$, $B_r(\tilde{z}, x, t_p)$, $C_r(\tilde{z}, x, t_p)$, $D_r(\tilde{z}, x, t_p)$, $E_r(\tilde{z}, x, t_p)$, $F_r(\tilde{z}, x, t_p)$, and $J_r(\tilde{z}, x, t_p)$ are defined as in (3.5) and (3.6). Hence the interrelationships among them, (3.7)–(3.10), also hold in the time-dependent case.

Similarly, we denote by $\{\mu_j(x, t_p)\}_{j=1,\ldots,r-5}$ and $\{\nu_j(x, t_p)\}_{j=1,\ldots,r-3}$ the zeros of $E_r(\tilde{z}, x, t_p)$ and $\tilde{z}^2 F_r(\tilde{z}, x, t_p)$, respectively. Thus, we may write

$$E_r(\tilde{z}, x, t_p) = u(x, t_p) \prod_{j=1}^{r-5} (\tilde{z} - \mu_j(x, t_p)), \tag{5.8}$$

$$F_r(\tilde{z}, x, t_p) = -u(x, t_p) u_x^2(x, t_p) \tilde{z}^{-2} \prod_{j=1}^{r-3} (\tilde{z} - \nu_j(x, t_p)). \tag{5.9}$$

Defining

$$\begin{aligned} \hat{\mu}_j(x, t_p) &= \Big(\mu_j(x, t_p), y(\hat{\mu}_j(x, t_p))\Big) \\ &= \left(\mu_j(x, t_p), -\frac{A_r(\mu_j(x, t_p), x, t_p)}{V_{21}(\mu_j(x, t_p), x, t_p)}\right) \in \mathcal{K}_{r-2}, \\ &\qquad j = 1, \ldots, r-5, \ (x, t_p) \in \mathbb{C}^2, \end{aligned} \tag{5.10}$$

$$\begin{aligned} \hat{\nu}_j(x, t_p) &= \Big(\nu_j(x, t_p), y(\hat{\nu}_j(x, t_p))\Big) \\ &= \left(\nu_j(x, t_p), -\frac{C_r(\nu_j(x, t_p), x, t_p)}{V_{31}(\nu_j(x, t_p), x, t_p)}\right) \in \mathcal{K}_{r-2}, \\ &\qquad j = 1, \ldots, r-3, \ (x, t_p) \in \mathbb{C}^2. \end{aligned} \tag{5.11}$$

One infers from (5.7) that the divisor $(\phi(P, x, t_p))$ of $\phi(P, x, t_p)$ is given by

$$(\phi(P, x, t_p)) = \mathcal{D}_{P_0, \underline{\hat{\nu}}(x, t_p)}(P) - \mathcal{D}_{P_{\infty_1}, \underline{\hat{\mu}}(x, t_p)}(P), \tag{5.12}$$

where

$$\underline{\hat{\nu}}(x, t_p) = \{\hat{\nu}_1(x, t_p), \ldots, \hat{\nu}_{r-3}(x, t_p)\},$$

$$\underline{\hat{\mu}}(x, t_p) = \{P_{\infty_2}, P_{\infty_3}, \hat{\mu}_1(x, t_p), \ldots, \hat{\mu}_{r-5}(x, t_p)\}.$$

That is, $P_0, \hat{\nu}_1(x, t_p), \ldots, \hat{\nu}_{r-3}(x, t_p)$ are the $r-2$ zeros of $\phi(P, x, t_p)$ and $P_{\infty_1}, P_{\infty_2}, P_{\infty_3}, \hat{\mu}_1(x, t_p), \ldots, \hat{\mu}_{r-5}(x, t_p)$ its $r-2$ poles.

Further properties of $\phi(P, x, t_p)$ are summarized as follows.

THEOREM 5.1. *Assume* (5.1), (5.6), *and* $P = (\tilde{z}, y) \in \mathcal{K}_{r-2} \setminus \{P_{\infty_i}, P_0\}$, $i = 1, 2, 3$, *and let* $(\tilde{z}, x, t_p) \in \mathbb{C}^3$. *Then*

$$\begin{aligned} &\phi_{xx}(P, x, t_p) + 3\tilde{z}^{-1}\phi(P, x, t_p)\phi_x(P, x, t_p) + \tilde{z}^{-2}\phi^3(P, x, t_p) \\ &\quad - \frac{m_x(x, t_p)}{m(x, t_p)}\phi_x(P, x, t_p) - \tilde{z}^{-1}\frac{m_x(x, t_p)}{m(x, t_p)}\phi^2(P, x, t_p) \\ &\quad - \phi(P, x, t_p) + m(x, t_p)\tilde{z}^{-1} + \frac{m_x(x, t_p)}{m(x, t_p)}\tilde{z} = 0, \end{aligned} \tag{5.13}$$

$$\phi_{t_p}(P,x,t_p) = \tilde{z}\partial_x\Bigg(\frac{\widetilde{V}_{21}(\tilde{z},x,t_p)}{m(x,t_p)}(\tilde{z}^2 - \tilde{z}\phi_x(P,x,t_p) - \phi^2(P,x,t_p)) + \widetilde{V}_{22}(\tilde{z},x,t_p) + \widetilde{V}_{23}(\tilde{z},x,t_p)\tilde{z}^{-1}\phi(P,x,t_p)\Bigg), \tag{5.14}$$

$$\phi(P,x,t_p)\phi(P^*,x,t_p)\phi(P^{**},x,t_p) = -\tilde{z}^3\frac{F_r(\tilde{z},x,t_p)}{E_r(\tilde{z},x,t_p)}, \tag{5.15}$$

$$\phi(P,x,t_p) + \phi(P^*,x,t_p) + \phi(P^{**},x,t_p) = \tilde{z}\frac{E_{r,x}(\tilde{z},x,t_p)}{E_r(\tilde{z},x,t_p)}, \tag{5.16}$$

$$\frac{1}{\phi(P,x,t_p)} + \frac{1}{\phi(P^*,x,t_p)} + \frac{1}{\phi(P^{**},x,t_p)} = \frac{F_{r,x}(\tilde{z},x,t_p)}{\tilde{z}F_r(\tilde{z},x,t_p)} - \frac{m(x,t_p)J_r(\tilde{z},x,t_p)}{\tilde{z}F_r(\tilde{z},x,t_p)} - \frac{2m(x,t_p)V_{33}(\tilde{z},x,t_p)}{\tilde{z}^3V_{31}(\tilde{z},x,t_p)}, \tag{5.17}$$

$$y(P)\phi(P,x,t_p) + y(P^*)\phi(P^*,x,t_p) + y(P^{**})\phi(P^{**},x,t_p) = \tilde{z}\frac{3T_r(\tilde{z})V_{21}(\tilde{z},x,t_p) + 2S_r(\tilde{z})A_r(\tilde{z},x,t_p)}{E_r(\tilde{z},x,t_p)}. \tag{5.18}$$

Proof. Equation (5.13) follows from (5.1) and (5.7). Relation (5.14) can be proved as follows. Differentiating (5.6) with respect to t_p and using (5.1), we have

$$\begin{aligned}(\phi)_{t_p} &= \tilde{z}\partial_x\frac{\widetilde{V}_{21}\psi_1 + \widetilde{V}_{22}\psi_2 + \widetilde{V}_{23}\psi_3}{\psi_2}\\ &= \tilde{z}\partial_x\left[\widetilde{V}_{21}\frac{(-\psi_{2,xx}+\psi_2)\tilde{z}^2}{m\psi_2} + \widetilde{V}_{22} + \widetilde{V}_{23}\frac{\psi_3}{\psi_2}\right]\\ &= \tilde{z}\partial_x\left[\frac{\widetilde{V}_{21}}{m}(-\tilde{z}\phi_x - \phi^2 + \tilde{z}^2) + \widetilde{V}_{22} + \widetilde{V}_{23}\tilde{z}^{-1}\phi\right].\end{aligned} \tag{5.19}$$

Moreover, (5.15)–(5.18) can be derived as in Theorem 3.2. ▯

Next, we consider the t_p-dependence of E_r and F_r.

Lemma 5.2. *Assume* (5.1) *and* (5.3) *and let* $(\tilde{z},x,t_p) \in \mathbb{C}^3$. *Then*

$$E_{r,t_p}(\tilde{z},x,t_p) = E_{r,x}(\tilde{z},x,t_p)\left(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right) + E_r(\tilde{z},x,t_p)3\left(\widetilde{V}_{22} - \frac{\widetilde{V}_{21}}{V_{21}}V_{22}\right), \tag{5.20}$$

$$F_{r,t_p}(\tilde{z},x,t_p) = F_{r,x}(\tilde{z},x,t_p)\widetilde{V}_{32} - J_r(\tilde{z},x,t_p)(\tilde{z}^2\widetilde{V}_{31} + m\widetilde{V}_{32}) + F_r(\tilde{z},x,t_p)\left(3\widetilde{V}_{22} + 3\widetilde{V}_{23,x} - \frac{2mV_{33}}{\tilde{z}^2V_{31}}\left(\frac{\tilde{z}^2}{m}\widetilde{V}_{31} + \widetilde{V}_{32}\right)\right). \tag{5.21}$$

Proof. From (5.1) and (5.6), we obtain

$$\tilde{z}\phi_x + \phi^2 = \frac{m}{V_{21}}(-\tilde{z}^{-1}y + V_{22} + \tilde{z}^{-1}V_{23}\phi) + \tilde{z}^2. \tag{5.22}$$

Hence, one can compute

$$
\begin{aligned}
&\tilde{z}\phi_x(P,x,t_p)+\tilde{z}\phi_x(P^*,x,t_p)+\tilde{z}\phi_x(P^{**},x,t_p)\\
&\quad+\phi^2(P,x,t_p)+\phi^2(P^*,x,t_p)+\phi^2(P^{**},x,t_p)\\
&=\frac{m}{V_{21}}(-\tilde{z}^{-1}y_0+V_{22}+\tilde{z}^{-1}V_{23}\phi(P))+\tilde{z}^2\\
&\quad+\frac{m}{V_{21}}(-\tilde{z}^{-1}y_1+V_{22}+\tilde{z}^{-1}V_{23}\phi(P^*))+\tilde{z}^2\\
&\quad+\frac{m}{V_{21}}(-\tilde{z}^{-1}y_2+V_{22}+\tilde{z}^{-1}V_{23}\phi(P^{**}))+\tilde{z}^2\\
&=-\frac{m\tilde{z}^{-1}(y_0+y_1+y_2)}{V_{21}}+3\frac{mV_{22}}{V_{21}}+3\tilde{z}^2\\
&\quad+\frac{m\tilde{z}^{-1}V_{23}}{V_{21}}(\phi(P)+\phi(P^*)+\phi(P^{**}))\\
&=3\frac{mV_{22}}{V_{21}}+\frac{m\tilde{z}^{-1}V_{23}}{V_{21}}(\phi(P)+\phi(P^*)+\phi(P^{**}))+3\tilde{z}^2 \qquad (5.23)
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_{t_p}(\phi(P,x,t_p)+\phi(P^*,x,t_p)+\phi(P^{**},x,t_p))\\
&=\partial_{t_p}\left(\tilde{z}\frac{E_{r,x}(\tilde{z},x,t_p)}{E_r(\tilde{z},x,t_p)}\right)\\
&=\tilde{z}\partial_{t_p}\partial_x(\ln E_r(\tilde{z},x,t_p))\\
&=\tilde{z}\partial_x\partial_{t_p}(\ln E_r(\tilde{z},x,t_p)). \qquad (5.24)
\end{aligned}
$$

On the other hand, from (5.14), we can see that

$$
\begin{aligned}
&\partial_{t_p}(\phi(P,x,t_p)+\phi(P^*,x,t_p)+\phi(P^{**},x,t_p))\\
&=\tilde{z}\partial_x\bigg(\frac{\widetilde{V}_{21}}{m}(\tilde{z}^2-\tilde{z}\phi_x(P,x,t_p)-\phi^2(P,x,t_p))\\
&\qquad+\widetilde{V}_{22}+\widetilde{V}_{23}\tilde{z}^{-1}\phi(P,x,t_p)\bigg)\\
&\quad+\tilde{z}\partial_x\bigg(\frac{\widetilde{V}_{21}}{m}(\tilde{z}^2-\tilde{z}\phi_x(P^*,x,t_p)-\phi^2(P^*,x,t_p))\\
&\qquad+\widetilde{V}_{22}+\widetilde{V}_{23}\tilde{z}^{-1}\phi(P^*,x,t_p)\bigg)\\
&\quad+\tilde{z}\partial_x\bigg(\frac{\widetilde{V}_{21}}{m}(\tilde{z}^2-\tilde{z}\phi_x(P^{**},x,t_p)-\phi^2(P^{**},x,t_p))\\
&\qquad+\widetilde{V}_{22}+\widetilde{V}_{23}\tilde{z}^{-1}\phi(P^{**},x,t_p)\bigg). \qquad (5.25)
\end{aligned}
$$

Without loss of generality, we take the integration constants as zero and then obtain

$$\begin{aligned}
\partial_{t_p}(\ln E_r(\tilde{z},x,t_p)) &= -\frac{\widetilde{V}_{21}}{m}\tilde{z}(\phi_x(P,x,t_p)+\phi_x(P^*,x,t_p)+\phi_x(P^{**},x,t_p)) \\
&\quad -\frac{\widetilde{V}_{21}}{m}(\phi^2(P,x,t_p)+\phi^2(P^*,x,t_p)+\phi^2(P^{**},x,t_p)) \\
&\quad +\tilde{z}^{-1}\widetilde{V}_{23}(\phi(P,x,t_p)+\phi(P^*,x,t_p)+\phi(P^{**},x,t_p)) \\
&\quad +3\widetilde{V}_{22}+3\frac{\widetilde{V}_{21}}{m}\tilde{z}^2 \\
&= \tilde{z}^{-1}\left(\widetilde{V}_{23}-\frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)(\phi(P)+\phi(P^*)+\phi(P^{**})) \\
&\quad +3\widetilde{V}_{22}-3\frac{\widetilde{V}_{21}}{V_{21}}V_{22} \\
&= \tilde{z}^{-1}\left(\widetilde{V}_{23}-\frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)\left(\tilde{z}\frac{E_{r,x}}{E_r}\right)+3\widetilde{V}_{22}-3\frac{\widetilde{V}_{21}}{V_{21}}V_{22} \\
&= \left(\widetilde{V}_{23}-\frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)\left(\frac{E_{r,x}}{E_r}\right)+3\widetilde{V}_{22}-3\frac{\widetilde{V}_{21}}{V_{21}}V_{22}, \qquad (5.26)
\end{aligned}$$

which implies

$$E_{r,t_p}(\tilde{z},x,t_p) = E_{r,x}\left(\widetilde{V}_{23}-\frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)+E_r3\left(\widetilde{V}_{22}-\frac{\widetilde{V}_{21}}{V_{21}}V_{22}\right). \qquad (5.27)$$

Relation (5.21) can be proved as follows. Using (5.3), (5.13), (5.15), (5.17), and (5.23), we have

$$\begin{aligned}
&\partial_{t_p}\left(-\tilde{z}^3\frac{F_r(\tilde{z},x,t_p)}{E_r(\tilde{z},x,t_p)}\right) \\
&\quad = \partial_{t_p}[\phi(P,x,t_p)\phi(P^*,x,t_p)\phi(P^{**},x,t_p)] \\
&\quad = \phi_{t_p}(P)\phi(P^*)\phi(P^{**})+\phi(P)\phi_{t_p}(P^*)\phi(P^{**})+\phi(P)\phi(P^*)\phi_{t_p}(P^{**}) \\
&\quad = \phi(P^*)\phi(P^{**})\left(\tilde{z}\partial_x\left[\frac{\widetilde{V}_{21}}{m}(-\tilde{z}\phi_x(P)-\phi^2(P)+\tilde{z}^2)+\widetilde{V}_{22}+\widetilde{V}_{23}\tilde{z}^{-1}\phi(P)\right]\right) \\
&\qquad +\phi(P)\phi(P^{**})\left(\tilde{z}\partial_x\left[\frac{\widetilde{V}_{21}}{m}(-\tilde{z}\phi_x(P^*)-\phi^2(P^*)+\tilde{z}^2)+\widetilde{V}_{22}+\widetilde{V}_{23}\tilde{z}^{-1}\phi(P^*)\right]\right) \\
&\qquad +\phi(P)\phi(P^*)\left(\tilde{z}\partial_x\left[\frac{\widetilde{V}_{21}}{m}(-\tilde{z}\phi_x(P^{**})-\phi^2(P^{**})+\tilde{z}^2)+\widetilde{V}_{22}+\widetilde{V}_{23}\tilde{z}^{-1}\phi(P^{**})\right]\right) \\
&\quad = \phi(P)\phi(P^*)\phi(P^{**})\Bigg[-\frac{\tilde{z}^2}{m}\widetilde{V}_{31}\partial_x\ln\phi(P)\phi(P^*)\phi(P^{**}) \\
&\qquad -\frac{\tilde{z}\widetilde{V}_{21,x}}{m}(\phi(P)+\phi(P^*)+\phi(P^{**}))+\tilde{z}\left(\frac{\tilde{z}^2\widetilde{V}_{21,x}}{m}+\widetilde{V}_{21}+\widetilde{V}_{22,x}\right) \\
&\qquad \left(\frac{1}{\phi(P)}+\frac{1}{\phi(P^*)}+\frac{1}{\phi(P^{**})}\right)-3\frac{\tilde{z}^2\widetilde{V}_{21}}{m}+3\widetilde{V}_{23,x} \\
&\qquad +\frac{\widetilde{V}_{21}}{m}\bigg(\tilde{z}\phi_x(P)+\phi^2(P)+\tilde{z}\phi_x(P^*)+\phi^2(P^*) \\
&\qquad\qquad +\tilde{z}\phi_x(P^{**})+\phi^2(P^{**})\bigg)\Bigg]
\end{aligned}$$

$$= \phi(P)\phi(P^*)\phi(P^{**})\Bigg[-\frac{\tilde{z}^2}{m}\widetilde{V}_{31}\partial_x \ln\phi(P)\phi(P^*)\phi(P^{**})$$

$$-\frac{\tilde{z}\widetilde{V}_{21,x}}{m}(\phi(P)+\phi(P^*)+\phi(P^{**})) + \tilde{z}\left(\frac{\tilde{z}^2\widetilde{V}_{21,x}}{m} + \widetilde{V}_{21} + \widetilde{V}_{22,x}\right)$$

$$\left(\frac{1}{\phi(P)} + \frac{1}{\phi(P^*)} + \frac{1}{\phi(P^{**})}\right) - 3\frac{\tilde{z}^2\widetilde{V}_{21}}{m} + 3\widetilde{V}_{23,x}$$

$$+\frac{\widetilde{V}_{21}}{m}\left(\frac{3mV_{22}}{V_{21}} + \frac{\tilde{z}^{-1}V_{23}}{V_{21}}(\phi(P)+\phi(P^*)+\phi(P^{**})) + 3\tilde{z}^2\right)\Bigg]$$

$$= -\tilde{z}^3\frac{F_r}{E_r}\Bigg[-\frac{\tilde{z}^2}{m}\widetilde{V}_{31}\left(\frac{F_{r,x}}{F_r} - \frac{E_{r,x}}{E_r}\right) - \frac{\tilde{z}^2\widetilde{V}_{21,x}}{m}\frac{E_{r,x}}{E_r} + 3\frac{\widetilde{V}_{21}}{V_{21}}V_{22} + \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\frac{E_{r,x}}{E_r}$$

$$+\left(\frac{\tilde{z}^2}{m}\widetilde{V}_{21,x} + \widetilde{V}_{21} + \widetilde{V}_{22,x}\right)\left(\frac{F_{r,x}}{F_r} - \frac{mJ_r}{F_r} - \frac{2mV_{33}}{\tilde{z}^2V_{31}}\right) + 3\widetilde{V}_{23,x}\Bigg], \tag{5.28}$$

which implies that

$$\frac{F_{r,t_p}}{E_r} - \frac{F_rE_{r,t_p}}{E_r^2}$$

$$= \frac{F_r}{E_r}\Bigg[-\frac{\tilde{z}^2}{m}\widetilde{V}_{31}\left(\frac{F_{r,x}}{F_r} - \frac{E_{r,x}}{E_r}\right) - \frac{\tilde{z}^2\widetilde{V}_{21,x}}{m}\frac{E_{r,x}}{E_r} + \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\frac{E_{r,x}}{E_r} + 3\frac{\widetilde{V}_{21}}{V_{21}}V_{22}$$

$$+\left(\frac{\tilde{z}^2}{m}\widetilde{V}_{21,x} + \widetilde{V}_{21} + \widetilde{V}_{22,x}\right)\left(\frac{F_{r,x}}{F_r} - \frac{mJ_r}{F_r} - \frac{2mV_{33}}{\tilde{z}^2V_{31}}\right) + 3\widetilde{V}_{23,x}\Bigg]$$

$$= \frac{F_{r,x}}{E_r}\left(-\frac{\tilde{z}^2}{m}\widetilde{V}_{31} + \left(\frac{\tilde{z}^2}{m}\widetilde{V}_{21,x} + \widetilde{V}_{21} + \widetilde{V}_{22,x}\right)\right)$$

$$+\frac{E_{r,x}F_r}{E_r^2}\left(\frac{\tilde{z}^2}{m}\widetilde{V}_{31} - \frac{\tilde{z}^2\widetilde{V}_{21,x}}{m} + \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)$$

$$+\frac{F_r}{E_r}\left(3\frac{V_{22}}{V_{21}}\widetilde{V}_{21} + 3\widetilde{V}_{23,x} - \frac{2mV_{33}}{\tilde{z}^2V_{31}}(\frac{\tilde{z}^2}{m}\widetilde{V}_{21,x} + \widetilde{V}_{21} + \widetilde{V}_{22,x})\right)$$

$$-\frac{mJ_r}{E_r}\left(\frac{\tilde{z}^2}{m}\widetilde{V}_{21,x} + \widetilde{V}_{21} + \widetilde{V}_{22,x}\right). \tag{5.29}$$

Then substituting (5.27) and the formulas

$$\widetilde{V}_{21,x} = \widetilde{V}_{31} + \tilde{z}^{-2}m\widetilde{V}_{23},$$

$$\widetilde{V}_{22,x} = \widetilde{V}_{32} - \widetilde{V}_{21} - \widetilde{V}_{23}$$

into (5.29), we obtain (5.21). ∎

The properties of $\psi_2(P,x,x_0,t_p,t_{0,p})$ are summarized as follows.

THEOREM 5.3. *Assume* (5.1), (5.6), *and* $P = (\tilde{z}, y) \in \mathcal{K}_{r-2} \setminus \{P_{\infty_i}, P_0\}$, $i = 1, 2, 3$, *and let* $(\tilde{z}, x, x_0, t_p, t_{0,p}) \in \mathbb{C}^5$. *Then*

$$\psi_{2,t_p}(P, x, x_0, t_p, t_{0,p}) = \left(\frac{\widetilde{V}_{21}(\tilde{z}, x, t_p)}{m(x, t_p)}(\tilde{z}^2 - \tilde{z}\phi_x(P, x, t_p) - \phi^2(P, x, t_p)) + \widetilde{V}_{22}(\tilde{z}, x, t_p) + \widetilde{V}_{23}(\tilde{z}, x, t_p)\tilde{z}^{-1}\phi(P, x, t_p)\right)\psi_2(P, x, x_0, t_p, t_{0,p}), \tag{5.30}$$

$$\psi_2(P, x, x_0, t_p, t_{0,p}) = \exp\Bigg(\tilde{z}^{-1}\int_{x_0}^{x}\phi(P, x', t_p)dx' + \int_{t_{0,p}}^{t_p}\Bigg[\frac{\tilde{z}^{-1}y(P) - V_{22}(\tilde{z}, x_0, t')}{V_{21}(\tilde{z}, x_0, t')} \times \widetilde{V}_{21}(\tilde{z}, x_0, t') + \left(\widetilde{V}_{23}(\tilde{z}, x_0, t') - \frac{\widetilde{V}_{21}(\tilde{z}, x_0, t')}{V_{21}(\tilde{z}, x_0, t')}V_{23}(\tilde{z}, x_0, t')\right) \tilde{z}^{-1}\phi(P, x_0, t') + \widetilde{V}_{22}(\tilde{z}, x_0, t')\Bigg]dt'\Bigg), \tag{5.31}$$

$$\psi_2(P, x, x_0, t_p, t_{0,p})\psi_2(P^*, x, x_0, t_p, t_{0,p})\psi_2(P^{**}, x, x_0, t_p, t_{0,p}) = \frac{E_r(\tilde{z}, x, t_p)}{E_r(\tilde{z}, x_0, t_{0,p})}, \tag{5.32}$$

$$\psi_{2,x}(P, x, x_0, t_p, t_{0,p})\psi_{2,x}(P^*, x, x_0, t_p, t_{0,p})\psi_{2,x}(P^{**}, x, x_0, t_p, t_{0,p}) = -\frac{F_r(\tilde{z}, x, t_p)}{E_r(\tilde{z}, x_0, t_{0,p})}, \tag{5.33}$$

$$\psi_2(P, x, x_0, t_p, t_{0,p}) = \left(\frac{E_r(\tilde{z}, x, t_p)}{E_r(\tilde{z}, x_0, t_{0,p})}\right)^{1/3} \times \exp\Bigg\{\int_{x_0}^{x}\left(\frac{y(P)^2V_{21}(\tilde{z}, x', t_p) - y(P)A_r(\tilde{z}, x', t_p) + \frac{2}{3}S_r(\tilde{z})V_{21}(\tilde{z}, x', t_p)}{E_r(\tilde{z}, x', t_p)}\right)dx' + \int_{t_{0,p}}^{t_p}\left(\frac{y(P)^2V_{21}(\tilde{z}, x_0, t') - y(P)A_r(\tilde{z}, x_0, t') + \frac{2}{3}S_r(\tilde{z})V_{21}(\tilde{z}, x_0, t')}{E_r(\tilde{z}, x_0, t')}\right) \times \left(\widetilde{V}_{23}(\tilde{z}, x_0, t') - \frac{\widetilde{V}_{21}(\tilde{z}, x_0, t')}{V_{21}(\tilde{z}, x_0, t')}V_{23}(\tilde{z}, x_0, t')\right) + \tilde{z}^{-1}y(P)\frac{\widetilde{V}_{21}(\tilde{z}, x_0, t')}{V_{21}(\tilde{z}, x_0, t')}dt'\Bigg\}. \tag{5.34}$$

Proof. Relation (5.30) can be proved as follows. Using (5.1) and (5.6), we have

$$\begin{aligned}\psi_{2,t_p}(P, x, x_0, t_p, t_{0,p}) &= \widetilde{V}_{21}\psi_1 + \widetilde{V}_{22}\psi_2 + \widetilde{V}_{23}\psi_3 \\ &= \widetilde{V}_{21}\left(\frac{\psi_2 - \psi_{2,xx}}{m}\right)\tilde{z}^2 + \widetilde{V}_{22}\psi_2 + \widetilde{V}_{23}\tilde{z}^{-1}\phi\psi_2 \\ &= \widetilde{V}_{21}\left(\frac{\tilde{z}^2 - \tilde{z}\phi_x - \phi^2}{m}\right)\psi_2 + \widetilde{V}_{22}\psi_2 + \widetilde{V}_{23}\tilde{z}^{-1}\phi\psi_2 \\ &= \left[\widetilde{V}_{21}\left(\frac{\tilde{z}^2 - \tilde{z}\phi_x - \phi^2}{m}\right) + \widetilde{V}_{22} + \widetilde{V}_{23}\tilde{z}^{-1}\phi\right]\psi_2.\end{aligned} \tag{5.35}$$

Then using (5.22), we obtain

$$
\begin{aligned}
&\psi_2(P,x,x_0,t_p,t_{0,p})\\
&\quad=\exp\Bigg(\int_{x_0}^{x}\tilde z^{-1}\phi(P,x',t_p)dx'\\
&\qquad\quad+\int_{t_{0,p}}^{t_p}\bigg[\widetilde V_{21}(\tilde z,x_0,t')\times\bigg(\frac{\tilde z^2-\tilde z\phi_x(P,x_0,t')-\phi^2(P,x_0,t')}{m}\bigg)\\
&\qquad\qquad\quad+\widetilde V_{22}(\tilde z,x_0,t')+\widetilde V_{23}(\tilde z,x_0,t')\tilde z^{-1}\phi(P,x_0,t')\bigg]dt'\Bigg)\\
&\quad=\exp\Bigg(\int_{x_0}^{x}\tilde z^{-1}\phi(P,x',t_p)dx'\\
&\qquad\quad+\int_{t_{0,p}}^{t_p}\bigg[\widetilde V_{21}(\tilde z,x_0,t')\times\bigg(\frac{\tilde z^{-1}y(P)-V_{22}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}\bigg)+\widetilde V_{22}(\tilde z,x_0,t')\\
&\qquad\qquad\quad+\bigg(\widetilde V_{23}(\tilde z,x_0,t')-\frac{\widetilde V_{21}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}V_{23}(\tilde z,x_0,t')\bigg)\times\ \tilde z^{-1}\phi(P,x_0,t')\bigg]dt'\Bigg),
\end{aligned}
\tag{5.36}
$$

which is (5.31).

Hence

$$
\begin{aligned}
&\psi_2(P,x,x_0,t_p,t_{0,p})\psi_2(P^*,x,x_0,t_p,t_{0,p})\psi_2(P^{**},x,x_0,t_p,t_{0,p})\\
&\quad=\exp\Bigg(\tilde z^{-1}\int_{x_0}^{x}\phi(P,x',t_p)dx'\\
&\qquad\quad+\int_{t_{0,p}}^{t_p}\bigg[\frac{\tilde z^{-1}y(P)-V_{22}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}\widetilde V_{21}(\tilde z,x_0,t')\\
&\qquad\qquad\quad+\bigg(\widetilde V_{23}(\tilde z,x_0,t')-\frac{\widetilde V_{21}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}V_{23}(\tilde z,x_0,t')\bigg)\tilde z^{-1}\phi(P,x_0,t')\\
&\qquad\qquad\quad+\widetilde V_{22}(\tilde z,x_0,t')\bigg]dt'\Bigg)\\
&\qquad\times\exp\Bigg(\tilde z^{-1}\int_{x_0}^{x}\phi(P^*,x',t_p)dx'\\
&\qquad\quad+\int_{t_{0,p}}^{t_p}\bigg[\frac{\tilde z^{-1}y(P^*)-V_{22}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}\widetilde V_{21}(\tilde z,x_0,t')\\
&\qquad\qquad\quad+\bigg(\widetilde V_{23}(\tilde z,x_0,t')-\frac{\widetilde V_{21}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}V_{23}(\tilde z,x_0,t')\bigg)\tilde z^{-1}\phi(P^*,x_0,t')\\
&\qquad\qquad\quad+\widetilde V_{22}(\tilde z,x_0,t')\bigg]dt'\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\times \exp\Bigg(\tilde{z}^{-1}\int_{x_0}^{x}\phi(P^{**},x',t_p)dx' \\
&\qquad + \int_{t_{0,p}}^{t_p}\Bigg[\frac{\tilde{z}^{-1}y(P^{**})-V_{22}(\tilde{z},x_0,t')}{V_{21}(\tilde{z},x_0,t')}\widetilde{V}_{21}(\tilde{z},x_0,t') \\
&\qquad\qquad + \left(\widetilde{V}_{23}(\tilde{z},x_0,t')-\frac{\widetilde{V}_{21}(\tilde{z},x_0,t')}{V_{21}(\tilde{z},x_0,t')}V_{23}(\tilde{z},x_0,t')\right)\tilde{z}^{-1}\phi(P^{**},x_0,t') \\
&\qquad\qquad + \widetilde{V}_{22}(\tilde{z},x_0,t')\Bigg]dt'\Bigg) \\
&= \exp\Bigg(\int_{x_0}^{x}\tilde{z}^{-1}[\phi(P,x',t_p)+\phi(P^{*},x',t_p)+\phi(P^{**},x',t_p)]dx' \\
&\qquad + \int_{t_{0,p}}^{t_p}\Bigg[3\left(\widetilde{V}_{22}(\tilde{z},x_0,t')-\frac{\widetilde{V}_{21}(\tilde{z},x_0,t')}{V_{21}(\tilde{z},x_0,t')}V_{22}(\tilde{z},x_0,t')\right) \\
&\qquad\qquad + \tilde{z}^{-1}\left(\widetilde{V}_{23}(\tilde{z},x_0,t')-\frac{\widetilde{V}_{21}(\tilde{z},x_0,t')}{V_{21}(\tilde{z},x_0,t')}V_{23}(\tilde{z},x_0,t')\right) \\
&\qquad\qquad \times[\phi(P,x_0,t')+\phi(P^{*},x_0,t')+\phi(P^{**},x_0,t')]\Bigg]dt'\Bigg) \\
&= \exp\Bigg(\int_{x_0}^{x}\frac{E_{r,x'}(\tilde{z},x',t_p)}{E_r(\tilde{z},x',t_p)}dx' \\
&\qquad + \int_{t_{0,p}}^{t_p}\Bigg[3\left(\widetilde{V}_{22}(\tilde{z},x_0,t')-\frac{\widetilde{V}_{21}(\tilde{z},x_0,t')}{V_{21}(\tilde{z},x_0,t')}V_{22}(\tilde{z},x_0,t')\right) \\
&\qquad\qquad + \left(\widetilde{V}_{23}(\tilde{z},x_0,t')-\frac{\widetilde{V}_{21}(\tilde{z},x_0,t')}{V_{21}(\tilde{z},x_0,t')}V_{23}(\tilde{z},x_0,t')\right)\left(\frac{E_{r,x}(\tilde{z},x_0,t')}{E_r(\tilde{z},x_0,t')}\right)\Bigg]dt'\Bigg) \\
&= \exp\left(\int_{x_0}^{x}\partial_{x'}(\ln E_r(\tilde{z},x',t_p))dx' + \int_{t_{0,p}}^{t_p}\partial_{t'}(\ln E_r(\tilde{z},x_0,t'))dt'\right) \\
&= \frac{E_r(\tilde{z},x,t_p)}{E_r(\tilde{z},x_0,t_{0,p})}.
\end{aligned}
\tag{5.37}
$$

Then the relation (5.33) follows from (5.37) and (5.15), that is,

$$
\begin{aligned}
&\psi_{2,x}(P,x,x_0,t_p,t_{0,p}) \times \psi_{2,x}(P^{*},x,x_0,t_p,t_{0,p}) \times \psi_{2,x}(P^{**},x,x_0,t_p,t_{0,p}) \\
&\quad = \tilde{z}^{-1}\phi(P,x,t_p)\psi_2(P,x,x_0,t_p,t_{0,p}) \times \tilde{z}^{-1}\phi(P^{*},x,t_p)\psi_2(P^{*},x,x_0,t_p,t_{0,p}) \\
&\qquad \times \tilde{z}^{-1}\phi(P^{**},x,t_p)\psi_2(P^{**},x,x_0,t_p,t_{0,p}) \\
&\quad = -\frac{F_r(\tilde{z},x,t_p)}{E_r(\tilde{z},x_0,t_{0,p})}.
\end{aligned}
\tag{5.38}
$$

Moreover, using (5.20), we arrive at

$$\begin{aligned}
&\psi_2(P,x,x_0,t_p,t_{0,p})\\
&= \exp\Bigg(\int_{x_0}^{x} \tilde z^{-1}\phi(P,x',t_p)dx'\\
&\qquad + \int_{t_{0,p}}^{t_p}\Bigg[\widetilde V_{23}(\tilde z,x_0,t')\tilde z^{-1}\phi(P,x_0,t')\\
&\qquad\qquad + \widetilde V_{21}(\tilde z,x_0,t')\left(\frac{\tilde z^2-\tilde z\phi_x(P,x_0,t')-\phi^2(P,x_0,t')}{m}\right)+\widetilde V_{22}(\tilde z,x_0,t')\Bigg]dt'\Bigg)\\
&= \exp\Bigg(\int_{x_0}^{x} \tilde z^{-1}\phi(P,x',t_p)dx'\\
&\qquad + \int_{t_{0,p}}^{t_p}\Bigg[\widetilde V_{21}(\tilde z,x_0,t')\left(\frac{\tilde z^{-1}y(P)-V_{22}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}\right)+\widetilde V_{22}(\tilde z,x_0,t')\\
&\qquad\qquad + \left(\widetilde V_{23}(\tilde z,x_0,t')-\frac{\widetilde V_{21}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}V_{23}(\tilde z,x_0,t')\right)\tilde z^{-1}\phi(P,x_0,t')\Bigg]dt'\Bigg)\\
&= \exp\Bigg(\int_{x_0}^{x}\left(\frac{y(P)^2V_{21}(\tilde z,x',t_p)-y(P)A_r(\tilde z,x',t_p)+B_r(\tilde z,x',t_p)}{E_r(\tilde z,x',t_p)}\right)dx'\\
&\qquad + \int_{t_{0,p}}^{t_p} M(\tilde z,x_0,t')dt'\Bigg),
\end{aligned}$$

where

$$\begin{aligned}
M(\tilde z,x_0,t') &= \widetilde V_{21}(\tilde z,x_0,t')\left(\frac{\tilde z^{-1}y(P)-V_{22}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}\right)+\widetilde V_{22}(\tilde z,x_0,t')\\
&\quad +\left(\widetilde V_{23}(\tilde z,x_0,t')-\frac{\widetilde V_{21}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}V_{23}(\tilde z,x_0,t')\right)\tilde z^{-1}\phi(P,x_0,t').
\end{aligned}\tag{5.39}$$

From (5.20), it is easy to see that

$$\begin{aligned}
&\left(\widetilde V_{22}(\tilde z,x_0,t')-\frac{\widetilde V_{21}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}V_{22}(\tilde z,x_0,t')\right)\\
&\quad = \frac{1}{3}\frac{E_{r,t'}(\tilde z,x_0,t')}{E_r(\tilde z,x_0,t')}\\
&\qquad - \frac{1}{3}\left(\widetilde V_{23}(\tilde z,x_0,t')-\frac{\widetilde V_{21}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}V_{23}(\tilde z,x_0,t')\right)\frac{E_{r,x}(\tilde z,x_0,t')}{E_r(\tilde z,x_0,t')}.
\end{aligned}\tag{5.40}$$

Inserting (5.40) into (5.39), we arrive at

$$\begin{aligned}
M(\tilde z,x_0,t') &= \left(\widetilde V_{23}(\tilde z,x_0,t')-\frac{\widetilde V_{21}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}V_{23}(\tilde z,x_0,t')\right)\\
&\quad\times\left(\tilde z^{-1}\phi(P,x_0,t')-\frac{1}{3}\frac{E_{r,x}(\tilde z,x_0,t')}{E_r(\tilde z,x_0,t')}\right)+\frac{1}{3}\frac{E_{r,t'}(\tilde z,x_0,t')}{E_r(\tilde z,x_0,t')}\\
&\quad + \tilde z^{-1}y(P)\frac{\widetilde V_{21}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}.
\end{aligned}\tag{5.41}$$

Substituting (5.41) into the above representation of ψ_2, we have

$$\begin{aligned}
&\psi_2(P,x,x_0,t_p,t_{0,p})\\
&\quad=\left(\frac{E_r(\tilde z,x,t_p)}{E_r(\tilde z,x_0,t_p)}\right)^{1/3}\\
&\qquad\times\exp\left(\int_{x_0}^{x}\left(\frac{y(P)^2V_{21}(\tilde z,x',t_p)-y(P)A_r(\tilde z,x',t_p)+\frac{2}{3}V_{21}(\tilde z,x',t_p)S_r(\tilde z)}{E_r(\tilde z,x',t_p)}\right)dx'\right)\\
&\qquad\times\left(\frac{E_r(\tilde z,x_0,t_p)}{E_r(\tilde z,x_0,t_{0,p})}\right)^{1/3}\\
&\qquad\times\exp\left(\int_{t_{0,p}}^{t_p}\left(\frac{y(P)^2V_{21}(\tilde z,x_0,t')-y(P)A_r(\tilde z,x_0,t')+\frac{2}{3}S_r(\tilde z)V_{21}(\tilde z,x_0,t')}{E_r(\tilde z,x_0,t')}\right)\right.\\
&\qquad\qquad\left.\times\left(\widetilde V_{23}(\tilde z,x_0,t')-\frac{\widetilde V_{21}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}V_{23}(\tilde z,x_0,t')\right)+\tilde z^{-1}y(P)\frac{\widetilde V_{21}(\tilde z,x_0,t')}{V_{21}(\tilde z,x_0,t')}dt'\right),
\end{aligned}$$

which implies (5.34). ∎

The stationary Dubrovin-type equations in Lemma 3.3 have analogues for each DP_p flow (indexed by the parameter t_p), which govern the dynamics of $\mu_j(x,t_p)$ and $\nu_j(x,t_p)$ with respect to variations of x and t_p. In this context the stationary case simply corresponds to the special case $p=0$ as described in the following result.

LEMMA 5.4. *Assume* (5.1)–(5.7).

(i) *Suppose the zeros* $\{\mu_j(x,t_p)\}_{j=1,\dots,r-5}$ *of* $E_r(\tilde z,x,t_p)$ *remain distinct for* $(x,t_p)\in\Omega_\mu$, *where* $\Omega_\mu\subseteq\mathbb{C}^2$ *is open and connected. Then* $\{\mu_j(x,t_p)\}_{j=1,\dots,r-5}$ *satisfy the system of differential equations,*

$$\mu_{j,x}(x,t_p)=-\frac{[S_r(\mu_j(x,t_p))+3y(\hat\mu_j(x,t_p))^2]V_{21}(\mu_j(x,t_p),x,t_p)}{u(x,t_p)\prod_{k=1,k\neq j}^{r-5}(\mu_j(x,t_p)-\mu_k(x,t_p))},$$

$$j=1,\dots,r-5, \tag{5.42}$$

$$\begin{aligned}
\mu_{j,t_p}(x,t_p)=&-[V_{21}(\mu_j(x,t_p),x,t_p)\widetilde V_{23}(\mu_j(x,t_p),x,t_p)\\
&-\widetilde V_{21}(\mu_j(x,t_p),x,t_p)V_{23}(\mu_j(x,t_p),x,t_p)]\\
&\times\frac{[S_r(\mu_j(x,t_p))+3y(\hat\mu_j(x,t_p))^2]}{u(x,t_p)\prod_{k=1,k\neq j}^{r-5}(\mu_j(x,t_p)-\mu_k(x,t_p))},
\end{aligned}$$

$$j=1,\dots,r-5, \tag{5.43}$$

with initial conditions

$$\{\hat\mu_j(x_0,t_{0,p})\}_{j=1,\dots,r-5}\in\mathcal{K}_{r-2} \tag{5.44}$$

for some fixed $(x_0,t_{0,p})\in\Omega_\mu$. *The initial value problem* (5.43), (5.44) *has a unique solution satisfying*

$$\hat\mu_j\in C^\infty(\Omega_\mu,\mathcal{K}_{r-2}),\quad j=1,\dots,r-5. \tag{5.45}$$

(ii) *Suppose the zeros* $\{\nu_j(x,t_p)\}_{j=1,\dots,r-3}$ *of* $F_r(\tilde z,x,t_p)$ *remain distinct for* $(x,t_p)\in\Omega_\nu$, *where* $\Omega_\nu\subseteq\mathbb{C}^2$ *is open and connected. Then* $\{\nu_j(x,t_p)\}_{j=1,\dots,r-3}$ *satisfy the sys-*

tem of differential equations,

$$\begin{aligned}\nu_{j,x}(x,t_p) &= \nu_j(x,t_p)^2\Big([S_r(\nu_j(x,t_p)) + 3y(\hat{\nu}_j(x,t_p))^2]\\ &\qquad \times V_{31}(\nu_j(x,t_p),x,t_p) + m(x,t_p)J_r(\nu_j(x,t_p),x,t_p)\Big)\\ &\quad\times \frac{1}{u(x,t_p)u_x^2(x,t_p)\prod_{k=1,k\neq j}^{r-3}(\nu_j(x,t_p)-\nu_k(x,t_p))},\\ &\qquad j=1,\dots,r-3,\end{aligned} \tag{5.46}$$

$$\begin{aligned}\nu_{j,t_p}(x,t_p) &= \nu_j(x,t_p)^2\Big([S_r(\nu_j(x,t_p)) + 3y(\hat{\nu}_j(x,t_p))^2]\\ &\qquad \times V_{31}(\nu_j(x,t_p),x,t_p)\widetilde{V}_{32}(\nu_j(x,t_p),x,t_p)\\ &\qquad - \nu_j(x,t_p)^2 J_r(\nu_j(x,t_p),x,t_p)\widetilde{V}_{31}(\nu_j(x,t_p),x,t_p)\Big)\\ &\quad\times \frac{1}{u(x,t_p)u_x^2(x,t_p)\prod_{k=1,k\neq j}^{r-3}(\nu_j(x,t_p)-\nu_k(x,t_p))},\\ &\qquad j=1,\dots,r-3,\end{aligned} \tag{5.47}$$

with initial conditions

$$\{\hat{\nu}_j(x_0,t_{0,p})\}_{j=1,\dots,r-3} \in \mathcal{K}_{r-2} \tag{5.48}$$

for some fixed $(x_0,t_{0,p}) \in \Omega_\nu$. *The initial value problem* (5.47), (5.48) *has a unique solution satisfying*

$$\hat{\nu}_j \in C^\infty(\Omega_\nu,\mathcal{K}_{r-2}), \quad j=1,\dots,r-3. \tag{5.49}$$

Proof. For obvious reasons it suffices to focus on (5.42) and (5.43). But the proof of (5.42) is identical to that in Lemma 3.3. We now prove (5.43). From (5.8), we have

$$E_{r,t_p}(\tilde{z},x,t_p)|_{\tilde{z}=\mu_j(x,t_p)} = -u(x,t_p)\mu_{j,t_p}(x,t_p)\prod_{\substack{k=1\\k\neq j}}^{r-5}(\mu_j(x,t_p)-\mu_k(x,t_p)). \tag{5.50}$$

On the other hand, using (5.20) and (5.42), one computes

$$\begin{aligned}E_{r,t_p}(\tilde{z},x,t_p)|_{\tilde{z}=\mu_j(x,t_p)} &= E_{r,x}(\mu_j(x,t_p),x,t_p)\left(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)\\ &= -u(x,t_p)\mu_{j,x}(x,t_p)\prod_{\substack{k=1\\k\neq j}}^{r-5}(\mu_j(x,t_p)-\mu_k(x,t_p))\\ &\quad \times\left(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)\\ &= V_{21}[S_r(\mu_j(x,t_p)) + 3y(\hat{\mu}_j(x,t_p))^2]\left(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)\\ &= [S_r(\mu_j(x,t_p)) + 3y(\hat{\mu}_j(x,t_p))^2](V_{21}\widetilde{V}_{23} - \widetilde{V}_{21}V_{23}),\end{aligned} \tag{5.51}$$

which together with (5.50) yields (5.43). ∎

The analog of Remark 3.4 directly extends to the current time-dependent setting.

6. Time-dependent algebro-geometric solutions. In the final section, we extend the results of section 4 from the stationary DP hierarchy to the time-dependent case. In particular, we obtain Riemann theta function representations for the Baker–Akhiezer function, the meromorphic function ϕ, and the algebro-geometric solutions for the DP hierarchy.

We start with the theta function representation of the meromorphic function $\phi(P,x,t_p)$.

THEOREM 6.1. *Assume that the curve $\mathcal{K}_{r-2}$ is nonsingular. Let $P=(\tilde z,y)\in\mathcal{K}_{r-2}\backslash\{P_{\infty_1},P_0\}$ and let $(x,t_p),(x_0,t_{0,p})\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat\mu}(x,t_p)}$, or equivalently $\mathcal{D}_{\underline{\hat\nu}(x,t_p)}$, is nonspecial for $(x,t_p)\in\Omega_\mu$. Then*

$$\begin{aligned}\phi(P,x,t_p)&=-m^{\frac13}(x,t_p)\frac{\theta(\underline{\tilde z}(P,\underline{\hat\nu}(x,t_p)))\theta(\underline{\tilde z}(P_0,\underline{\hat\mu}(x,t_p)))}{\theta(\underline{\tilde z}(P_0,\underline{\hat\nu}(x,t_p)))\theta(\underline{\tilde z}(P,\underline{\hat\mu}(x,t_p)))}\\&\quad\times\exp\left(e^{(3)}(Q_0)-\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1}P_0}\right).\end{aligned}\tag{6.1}$$

Proof. The proof of (6.1) is analogous to the stationary case in Theorem 4.3. ∎

Motivated by (5.30), we define the meromorphic function $I_s(P,x,t_p)$ on $\mathcal{K}_{r-2}\times\mathbb{C}^2$ by

$$\begin{aligned}I_s(P,x,t_p)=&\frac{\widetilde V_{21}(\tilde z,x,t_p)}{m(x,t_p)}(\tilde z^2-\tilde z\phi_x(P,x,t_p)-\phi^2(P,x,t_p))+\widetilde V_{22}(\tilde z,x,t_p)\\&+\widetilde V_{23}(\tilde z,x,t_p)\tilde z^{-1}\phi(P,x,t_p).\end{aligned}\tag{6.2}$$

The asymptotic properties of $I_s(P,s,t_p)$ are summarized as follows.

THEOREM 6.2. *Let $s=4p+2$, $p\in\mathbb{N}_0$, $(x,t_p)\in\mathbb{C}^2$. Then*[3]

$$I_s(P,x,t_p)\underset{\zeta\to0}{=}\frac23\zeta^{-s}+\sum_{j=0}^{\frac{s-4}{2}}\tilde\alpha_j\zeta^{-(s-2j-2)}+\chi_{\frac{s-2}{2}}+O(\zeta^2),\tag{6.3}$$
$$\zeta=\tilde z^{-1},\quad as\ P\to P_{\infty_1},$$

$$I_2(P,x,t_0)\underset{\zeta\to0}{=}u(x,t_0)m^{1/3}(x,t_0)\zeta^{-2}+O(\zeta^2),\tag{6.4}$$
$$\zeta=\tilde z^{1/3},\quad as\ P\to P_0,$$

where $\{\tilde\alpha_j\}_{j=0,\dots,\frac{s-4}{2}}\in\mathbb{C}$, and

$$\chi_0=-\frac{u}{m}(\kappa_2+2\kappa_0\kappa_{2,x})-u\kappa_0,$$
$$\chi_{\frac{s-2}{2}}=m^{-1}\sum_{k=-2}^{s-2}\vartheta_kV_{21}^{([\frac{2j-k+4}{4}],\frac{s+2-k}{4}-[\frac{s+2-k}{4}])}+\sum_{\ell=0}^{s-2}\kappa_\ell V_{23}^{([\frac{s-\ell}{4}],\frac{s-\ell}{4}-[\frac{s-\ell}{4}])},\quad s>2,$$

the function $[\cdot]$ returns the value of a number rounded downward to the nearest integer.

Proof. Treating t_p as a parameter, we note that the asymptotic expansions of $\phi(P)$ near P_{∞_1} and near P_0 in (4.1) and (4.6) still apply in the present time-dependent

[3] Here sums with upper limits strictly less than their lower limits are interpreted as zero.

context. In terms of local coordinate $\zeta = \tilde{z}^{-1}$ near P_{∞_1}, from (4.1), (5.2), and (6.2) we easily get

$$
\begin{aligned}
I_2(P,x,t_0) &= \frac{\widetilde{V}_{21}(\tilde{z},x,t_0)}{m(x,t_0)}\left(\tilde{z}^2 - \tilde{z}\phi_x(P,x,t_0) - \phi^2(P,x,t_0)\right) + \widetilde{V}_{22}(\tilde{z},x,t_0) \\
&\quad + \widetilde{V}_{23}(\tilde{z},x,t_0)\tilde{z}^{-1}\phi(P,x,t_0) \\
&= \frac{u(x,t_0)}{m(x,t_0)}\Big[\left(1-\kappa_{0,x}-\kappa_0^2\right)\zeta^{-2} - (\kappa_{2,x}+2\kappa_0\kappa_{2,x})\Big] - \frac{1}{3}\zeta^{-2} \\
&\quad - u(x,t_0)\kappa_0 + O(\zeta^2) \\
&= \frac{2}{3}\zeta^{-2} + \chi_0 + O(\zeta^2), \qquad \text{as } P \to P_{\infty_1}, \qquad (6.5)
\end{aligned}
$$

where

$$\chi_0 = -\frac{u}{m}\left(\kappa_2 + 2\kappa_0\kappa_{2,x}\right) - u\kappa_0,$$

and

$$\kappa_0 = \frac{u_x}{u}, \quad \kappa_{0,x} = \frac{u_{xx}}{u} - \left(\frac{u_x}{u}\right)^2.$$

Therefore (6.3) holds for $s = 2$. For $s > 2$, recall the definitions of $\widetilde{V}_{21}, \widetilde{V}_{22}, \widetilde{V}_{23}$ in (5.2); we may write

$$
\begin{aligned}
\widetilde{V}_{21} &= V_{21}^{(0,1)}\tilde{z}^{4p} + V_{21}^{(1,0)}\tilde{z}^{4p-2} + V_{21}^{(1,1)}\tilde{z}^{4p-4} + \cdots + V_{21}^{(p,0)}\tilde{z}^2 + V_{21}^{(p,1)} \\
&= \sum_{j=0}^{2p} V_{21}^{([\frac{j+1}{2}],j+1-[\frac{j+1}{2}])}\zeta^{-(4p-2j)} \\
&= \sum_{j=0}^{\infty} V_{21}^{([\frac{j+1}{2}],j+1-[\frac{j+1}{2}])}\zeta^{-(4p-2j)}, \\
\widetilde{V}_{22} &= V_{22}^{(0,0)}\tilde{z}^{4p+2} + V_{21}^{(1,0)}\tilde{z}^{4p-2} + V_{22}^{(2,0)}\tilde{z}^{4p-6} + \cdots + V_{22}^{(p,0)}\tilde{z}^2 \\
&= \sum_{j=0}^{2p} V_{22}^{([\frac{j}{2}],j-[\frac{j}{2}])}\zeta^{-(4p+2-2j)} \\
&= \sum_{j=0}^{\infty} V_{22}^{([\frac{j}{2}],j-[\frac{j}{2}])}\zeta^{-(4p+2-2j)}, \\
\widetilde{V}_{23} &= V_{23}^{(0,1)}\tilde{z}^{4p} + V_{23}^{(1,0)}\tilde{z}^{4p-2} + V_{23}^{(1,1)}\tilde{z}^{4p-4} + \cdots + V_{21}^{(p,0)}\tilde{z}^2 + V_{23}^{(p,1)} \\
&= \sum_{j=0}^{2p} V_{23}^{([\frac{j+1}{2}],j+1-[\frac{j+1}{2}])}\zeta^{-(4p-2j)},
\end{aligned}
$$

where

$$
\begin{aligned}
&V_{21}^{(\beta_1,\beta_2)} = V_{23}^{(\beta_1,\beta_2)} = 0 \quad \text{for} \quad \beta_1 \geq p+1, \quad \beta_1,\beta_2 \in \mathbb{N}, \\
&V_{22}^{(\beta_1,\beta_2)} = 0 \quad \text{for} \quad \beta_1 \geq p+1 \quad \text{or} \quad \beta_2 = 1, \quad \beta_1,\beta_2 \in \mathbb{N}.
\end{aligned}
$$

Moreover, from (4.1), we find

$$\begin{aligned}\tilde{z}^2 - \tilde{z}\phi_x(P,x,t_p) - \phi^2(P,x,t_p) &\underset{\zeta\to 0}{=} \sum_{j=-1}^{\infty} \vartheta_{2j}\zeta^{2j},\\ &\underset{\zeta\to 0}{=} \sum_{j=-1}^{\infty}\left(\vartheta_{2j}\zeta^{2j} + \vartheta_{2j+1}\zeta^{2j+1}\right), \quad \text{as } P \to P_{\infty_1},\end{aligned}$$

with

$$\begin{aligned}\vartheta_{-2} &= 1 - \kappa_{0,x} - \kappa_0^2 = \frac{m(x,t_p)}{u(x,t_p)},\\ \vartheta_{2j} &= -\kappa_{2j+2} - \sum_{i=0}^{2j+2}\kappa_i\kappa_{2j+2-i},\\ \vartheta_{2j+1} &= 0, \quad j \in \mathbb{N}_0.\end{aligned}$$

Therefore, in terms of local coordinate $\zeta = \tilde{z}^{-1}$ near P_{∞_1}, we obtain

$$\begin{aligned}I_s(P,x,t_p) &= \frac{\widetilde{V}_{21}(\tilde{z},x,t_p)}{m(x,t_p)}(\tilde{z}^2 - \tilde{z}\phi_x(P,x,t_p) - \phi^2(P,x,t_p)) + \widetilde{V}_{22}(\tilde{z},x,t_p)\\ &\quad + \widetilde{V}_{23}(\tilde{z},x,t_p)\tilde{z}^{-1}\phi(P,x,t_p)\\ &= m^{-1}(x,t_p)\left(\sum_{j=0}^{2p} V_{21}^{([\frac{j+1}{2}],j+1-[\frac{j+1}{2}])}\zeta^{-(4p-2j)}\right)\left(\sum_{s=-1}^{\infty}\vartheta_{2s}\zeta^{2s}\right)\\ &\quad + \sum_{j=0}^{2p} V_{22}^{([\frac{j}{2}],j-[\frac{j}{2}])}\zeta^{-(4p+2-2j)} + \left(\sum_{j=0}^{2p} V_{23}^{([\frac{j+1}{2}],j+1-[\frac{j+1}{2}])}\zeta^{-(4p-2j)}\right)\\ &\quad \times \zeta\left(\frac{1}{\zeta}\sum_{j=0}^{\infty}\kappa_j\zeta^j\right)\\ &= m^{-1}\sum_{j=-1}^{\infty}\left(\sum_{k=-2}^{2j}\vartheta_k V_{21}^{([\frac{2j-k+4}{4}],\frac{2j-k+4}{4}-[\frac{2j-k+4}{4}])}\right)\zeta^{-4p+2j}\\ &\quad + \sum_{j=-1}^{2p-1} V_{22}^{([\frac{j+1}{2}],j+1-[\frac{j+1}{2}])}\zeta^{-4p+2j}\\ &\quad + \sum_{j=0}^{\infty}\left(\sum_{\ell=0}^{2j}\kappa_\ell V_{23}^{([\frac{2j-\ell+2}{4}],\frac{2j-\ell+2}{4}-[\frac{2j-\ell+2}{4}])}\right)\zeta^{-4p+2j}\\ &= \frac{2}{3}\zeta^{-(4p+2)} + \sum_{j=0}^{2p-1}\chi_j\zeta^{-4p+2j} + \chi_{2p} + \sum_{j=2p+1}^{\infty}\chi_j\zeta^{-4p+2j}, \qquad (6.6)\end{aligned}$$

where

$$\chi_j = m^{-1}\sum_{k=-2}^{2j}\vartheta_k V_{21}^{([\frac{2j-k+4}{4}],\frac{2j-k+4}{4}-[\frac{2j-k+4}{4}])} + V_{22}^{([\frac{j+1}{2}],j+1-[\frac{j+1}{2}])} + \sum_{\ell=0}^{2j}\kappa_\ell V_{23}^{([\frac{2j-\ell+2}{4}],\frac{2j-\ell+2}{4}-[\frac{2j-\ell+2}{4}])} \quad \text{for} \quad 0 \le j \le 2p-1, \quad j \in \mathbb{N}_0,$$

$$\chi_j = m^{-1}\sum_{k=-2}^{2j}\vartheta_k V_{21}^{([\frac{2j-k+4}{4}],\frac{2j-k+4}{4}-[\frac{2j-k+4}{4}])} + \sum_{\ell=0}^{2j}\kappa_\ell V_{23}^{([\frac{2j-\ell+2}{4}],\frac{2j-\ell+2}{4}-[\frac{2j-\ell+2}{4}])} \quad \text{for} \quad j \ge 2p, \quad j \in \mathbb{N}_0.$$

Then inserting (6.6) into (5.14) and comparing the coefficients of the same powers of ζ^ℓ ($\ell < 0$) yields

$$\chi_{j,x} = 0 \quad \text{for} \quad 0 \le j \le 2p-1, \quad j \in \mathbb{N}_0.$$

Hence, we conclude that

$$\begin{aligned} \chi_0 &= \gamma_0(t_p), \\ \chi_1 &= \gamma_1(t_p), \\ &\vdots \\ \chi_{2p-1} &= \gamma_{2p-1}(t_p), \end{aligned}$$

where $\gamma_j(t_p)$ $(j = 1, 2, \dots)$ are integration constants. Next we note that the coefficients κ_j $(j = 0, 1, \dots)$ of the power series for $\phi(P, x, t_p)$ in the coordinate ζ near P_{∞_1} are the ratios of two functions closely related to u. Meanwhile, the coefficients of the homogeneous polynomials $\widetilde{V}_{ij}$ $(i, j = 1, 2, 3)$ are differential polynomials in u. From these considerations it follows that $\gamma_j = \tilde{\alpha}_j \in \mathbb{C}$. Hence, we obtain (6.3). Finally, (6.4) follows from (4.6) and (6.2). ∎

Let $\omega^{(2)}_{P_{\infty_1},j}$, $j = 4l+2$, $l \in \mathbb{N}_0$, be the Abel differentials of the second kind normalized by the vanishing of all their a-periods,

$$\int_{a_k} \omega^{(2)}_{P_{\infty_1},j} = 0, \quad k = 1, \dots, r-2$$

and holomorphic on $\mathcal{K}_{r-2} \setminus \{P_{\infty_1}\}$, with a pole of order j at P_{∞_1},

$$\omega^{(2)}_{P_{\infty_1},j}(P) \underset{\zeta\to 0}{=} (\zeta^{-j} + O(1))d\zeta, \quad \text{as } P \to P_{\infty_1}. \tag{6.7}$$

Furthermore, define the normalized differential of the second kind by

$$\widetilde{\Omega}^{(2)}_{P_{\infty_1},s+1} = \frac{2}{3}s\omega^{(2)}_{P_{\infty_1},s+1} + \sum_{j=0}^{\frac{s-4}{2}}(s-2j-2)\tilde{\alpha}_j\omega^{(2)}_{P_{\infty_1},s-2j-3} \tag{6.8}$$

and

$$\widetilde{\Omega}^{(2)}_{P_0,3} = 2\omega^{(2)}_{P_0,3}, \tag{6.9}$$

where $s = 4p+2$, $p \in \mathbb{N}_0$. Thus, one infers

$$\int_{a_k} \widetilde{\Omega}^{(2)}_{P_{\infty_1},s+1} = 0, \qquad \int_{a_k} \widetilde{\Omega}^{(2)}_{P_0,3} = 0, \qquad k = 1, \dots, r-2.$$

In addition, we define the vector of b-periods of the differential of the second kind $\widetilde{\Omega}^{(2)}_{P_{\infty_1},s+1}$,

$$\underline{\widetilde{U}}^{(2)}_{s+1}=(\widetilde{U}^{(2)}_{s+1,1},\dots,\widetilde{U}^{(2)}_{s+1,r-2}),\quad \widetilde{U}^{(2)}_{s+1,j}=\frac{1}{2\pi i}\int_{b_j}\widetilde{\Omega}^{(2)}_{P_{\infty_1},s+1},\quad j=1,\dots,r-2 \tag{6.10}$$

with $s=4p+2$, $p\in\mathbb{N}_0$. Integrating (6.8) and (6.9) yields

$$\begin{aligned}\int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_{\infty_1},s+1}&\underset{\zeta\to0}{=}\frac{2}{3}s\int_{\zeta_0}^{\zeta}\omega^{(2)}_{P_{\infty_1},s+1}+\sum_{j=0}^{\frac{s-4}{2}}(s-2j-2)\tilde{\alpha}_j\int_{\zeta_0}^{\zeta}\omega^{(2)}_{P_{\infty_1},s-2j-3}\\&\underset{\zeta\to0}{=}\frac{2}{3}s\int_{\zeta_0}^{\zeta}\frac{1}{\zeta^{s+1}}d\zeta+\sum_{j=0}^{\frac{s-4}{2}}(s-2j-2)\tilde{\alpha}_j\int_{\zeta_0}^{\zeta}\frac{1}{\zeta^{s-2j-3}}d\zeta+O(1)\\&\underset{\zeta\to0}{=}-\frac{2}{3}\zeta^{-s}-\sum_{j=0}^{\frac{s-4}{2}}\tilde{\alpha}_j\frac{1}{\zeta^{s-2j-2}}+\hat{e}^{(2)}_{s+1}(Q_0)+O(\zeta),\\&\qquad\text{as } P\to P_{\infty_1},\end{aligned}\tag{6.11}$$

and

$$\int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_0,3}\underset{\zeta\to0}{=}-\zeta^{-2}+\tilde{e}^{(2)}_{3}(Q_0)+O(\zeta),\quad\text{as } P\to P_0,\tag{6.12}$$

where $\hat{e}^{(2)}_{s+1}(Q_0),\tilde{e}^{(2)}_{3}(Q_0)$ are constants that arise from evaluating all the integrals at their lowers limits Q_0, and summing accordingly. Combining (6.3), (6.4), (6.11), and (6.12) yields

$$\begin{aligned}\int_{t_{0,p}}^{t_p}I_s(P,x,\tau)d\tau&\underset{\zeta\to0}{=}(t_p-t_{0,p})\left(\hat{e}^{(2)}_{s+1}(Q_0)-\int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_{\infty_1},s+1}\right)\\&\quad+\int_{t_{0,p}}^{t_p}\chi_{\frac{s-2}{2}}(x,\tau)d\tau+O(\zeta),\quad\text{as } P\to P_{\infty_1},\end{aligned}\tag{6.13}$$

and

$$\begin{aligned}\int_{t_{0,0}}^{t_0}I_2(P,x,\tau)d\tau&\underset{\zeta\to0}{=}\int_{t_{0,0}}^{t_0}\left(u(x,\tau)m^{\frac{1}{3}}(x,\tau)\left(\tilde{e}^{(2)}_{3}(Q_0)-\int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_0,3}\right)\right)d\tau\\&\quad+O(\zeta),\qquad\text{as } P\to P_0.\end{aligned}\tag{6.14}$$

Given these preparations, the theta function representation of $\psi_2(P,x,x_0,t_p,t_{0,p})$ reads as follows.

THEOREM 6.3. *Assume that the curve $\mathcal{K}_{r-2}$ is nonsingular. Let $P=(\tilde{z},y)\in\mathcal{K}_{r-2}\backslash\{P_{\infty_1},P_0\}$ and let $(x,t_p),(x_0,t_{0,p})\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}$, or equivalently $\mathcal{D}_{\underline{\hat{\nu}}(x,t_p)}$, is nonspecial for $(x,t_p)\in\Omega_\mu$. Then* *for* $s = 2$

$$\begin{aligned}(6.15)\qquad &\psi_2(P, x, x_0, t_0, t_{0,0})\\ &= \frac{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x, t_0))\right)\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0, t_{0,0})))}{\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x, t_0)))\theta(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x_0, t_{0,0})))}\\ &\quad\times \exp\Bigg(\int_{x_0}^{x} 2m^{1/3}(x', t_0)dx'\Bigg(\int_{Q_0}^{P}\omega^{(2)}_{P_0,3} - e^{(2)}_3(Q_0)\Bigg)\Bigg)\\ &\quad + (t_0 - t_{0,0})\Bigg(\hat{e}^{(2)}_3(Q_0) - \int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_{\infty_1},3}\Bigg) + \int_{t_{0,0}}^{t_0}\chi_0(x_0, \tau)d\tau\Bigg)\\ &\quad\times \int_{t_{0,0}}^{t_0}\Bigg(u(x_0, \tau)m^{1/3}(x_0, \tau)\Bigg(\tilde{e}^{(2)}_3(Q_0) - \int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_0,3}\Bigg)\Bigg)d\tau,\end{aligned}$$

and for $s > 2$

$$\begin{aligned}(6.16)\quad &\psi_2(P, x, x_0, t_p, t_{0,p})\\ &= \frac{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x, t_p))\right)\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0, t_{0,p})))}{\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x, t_p)))\theta(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x_0, t_{0,p})))}\\ &\quad\times \exp\Bigg(\int_{x_0}^{x} 2m^{1/3}(x', t_p)dx'\Bigg(\int_{Q_0}^{P}\omega^{(2)}_{P_0,3} - e^{(2)}_3(Q_0)\Bigg)\Bigg)\\ &\quad + (t_p - t_{0,p})\Bigg(\hat{e}^{(2)}_{s+1}(Q_0) - \int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_{\infty_1},s+1}\Bigg) + \int_{t_{0,p}}^{t_p}\chi_{\frac{s-2}{2}}(x_0, \tau)d\tau\Bigg).\end{aligned}$$

Proof. We present only the proof of the time variation here, since the proof of the space variation is analogous to the stationary case in Theorem 4.4. Let $\psi_2(P, x, x_0, t_p, t_{0,p})$ be defined as in (5.31). For $s > 2$, we denote the right-hand side of (6.16) by $\Psi(P, x, x_0, t_p, t_{0,p})$. While $s = 2$, for our convenience, we also denote the right-hand side of (6.15) by $\Psi(P, x, x_0, t_p, t_{0,p})$. Temporarily assume that

$$(6.17)\qquad \mu_j(x, t_p) \neq \mu_k(x, t_p) \quad \text{for } j \neq k \text{ and } (x, t_p) \in \widetilde{\Omega}_\mu \subseteq \Omega_\mu\ ,$$

where $\widetilde{\Omega}_\mu$ is open and connected. In order to prove that $\psi_2 = \Psi$, by using (5.20) and (5.22), we compute

$$\begin{aligned}I_s(P, x, t_p) &= \frac{\widetilde{V}_{21}}{m}(\tilde{z}^2 - \tilde{z}\phi_x - \phi^2) + \widetilde{V}_{22} + \widetilde{V}_{23}\tilde{z}^{-1}\phi\\ &= \widetilde{V}_{21}\frac{\tilde{z}^{-1}y - V_{22}}{V_{21}} + \widetilde{V}_{22} + \left(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)\tilde{z}^{-1}\phi\\ &= \left(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)\tilde{z}^{-1}\phi + \widetilde{V}_{22} - \frac{\widetilde{V}_{21}}{V_{21}}V_{22} + \tilde{z}^{-1}y\frac{\widetilde{V}_{21}}{V_{21}}\\ &= \left(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)\left(\frac{y^2V_{21} - yA_r + \frac{2}{3}S_rV_{21} + \frac{1}{3}E_{r,x}}{E_r}\right)\\ &\quad + \widetilde{V}_{22} - \frac{\widetilde{V}_{21}}{V_{21}}V_{22} + \tilde{z}^{-1}y\frac{\widetilde{V}_{21}}{V_{21}}\end{aligned}$$

$$= \frac{1}{3}\frac{E_{r,t_p}}{E_r} + \left(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)\left(\frac{y^2V_{21} - yA_r + \frac{2}{3}S_rV_{21}}{E_r}\right) + \tilde{z}^{-1}y\frac{\widetilde{V}_{21}}{V_{21}}$$

$$(6.18) \qquad = \frac{1}{3}\frac{E_{r,t_p}}{E_r} + \left(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\right)\left[\frac{2}{3}\frac{V_{21}(3y^2+S_r)}{E_r} - \frac{yV_{21}(y+\frac{A_r}{V_{21}})}{E_r}\right] + \tilde{z}^{-1}y\frac{\widetilde{V}_{21}}{V_{21}}.$$

Hence,

$$I_s(P,x,t_p) = -\frac{1}{3}\frac{\mu_{j,t_p}}{\tilde{z}-\mu_j} - \frac{2}{3}\frac{\mu_{j,t_p}}{\tilde{z}-\mu_j} + O(1)$$

$$(6.19) \qquad = -\frac{\mu_{j,t_p}}{\tilde{z}-\mu_j} + O(1), \qquad \text{as } \tilde{z} \to \mu_j(x,t_p).$$

More concisely,

$$(6.20) \qquad I_s(P,x_0,\tau) = \frac{\partial}{\partial\tau}\ln(\tilde{z}-\mu_j(x_0,\tau)) + O(1) \quad \text{for } P \text{ near } \hat{\mu}_j(x_0,t_p).$$

Therefore

$$\exp\left(\int_{t_{0,p}}^{t_p} d\tau \left(\frac{\partial}{\partial\tau}\ln(\tilde{z}-\mu_j(x_0,\tau)) + O(1)\right)\right)$$

$$= \frac{\tilde{z}-\mu_j(x_0,t_p)}{\tilde{z}-\mu_j(x_0,t_{0,p})}O(1)$$

$$(6.21) \qquad = \begin{cases} (\tilde{z}-\mu_j(x_0,t_p))O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0,t_p) \neq \hat{\mu}_j(x_0,t_{0,p}), \\ O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0,t_p) = \hat{\mu}_j(x_0,t_{0,p}), \\ (\tilde{z}-\mu_j(x_0,t_{0,p}))^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0,t_{0,p}) \neq \hat{\mu}_j(x_0,t_p), \end{cases}$$

where $O(1) \neq 0$ in (6.21). Consequently, all zeros and poles of ψ_2 and Ψ on $\mathcal{K}_{r-2} \setminus \{P_{\infty_1}, P_0\}$ are simple and coincident. It remains to identify the essential singularity of ψ_2 and Ψ at P_{∞_1} and P_0 with respect to the time variation. By (6.13) and (6.14), we see that the singularities in the exponential terms of ψ_2 and Ψ with respect to the time variation coincide. The uniqueness result for Baker–Akhiezer functions [13], [15], [16], [18] completes the proof that $\psi_2 = \Psi$ on $\widetilde{\Omega}_\mu$. The extension of this result from $(x,t_p) \in \widetilde{\Omega}_\mu$ to $(x,t_p) \in \Omega_\mu$ then simply follows from the continuity of $\underline{\alpha}_{Q_0}$ and the hypothesis of $\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}$ being nonspecial for $(x,t_p) \in \Omega_\mu$. ∎

Remark 6.4. We provided two explicit representations for the Baker–Akhiezer function ψ_2 in terms of the Riemann theta function, corresponding to the case $s = 2$ and $s > 2$, respectively. By (6.4), $I_2 = O(\zeta^{-2})$, P_0 is an essential singularity of ψ_2 for $s = 2$. However, for $s > 2$, $I_s = O(\zeta^2)$ near P_0, and there are no singularities in this case. Thus, we investigated these two situations, respectively, in Theorem 6.2 and Theorem 6.3. What we want to emphasize is that these results will not take any trouble for us to obtain the solution $u(x,t_p)$. We can deal with the two expressions (6.15) and (6.16) uniformly. The more details will be given in Theorem 6.6.

The straightening out of the DP flows by the Abel map is contained in our next result.

THEOREM 6.5. *Assume that the curve $\mathcal{K}_{r-2}$ is nonsingular, and let (x,t_p), $(x_0,t_{0,p})\in\mathbb{C}^2$. Then for $s>2$,*

$$\begin{aligned}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,p})}) - \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x',t_p)dx'\right)\underline{\hat{U}}_3^{(2)}\\ &\quad + \underline{\widetilde{U}}_{s+1}^{(2)}(t_p-t_{0,p}), \end{aligned} \tag{6.22}$$

$$\begin{aligned}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_p)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,p})}) - \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x',t_p)dx'\right)\underline{\hat{U}}_3^{(2)}\\ &\quad + \underline{\widetilde{U}}_{s+1}^{(2)}(t_p-t_{0,p}), \end{aligned} \tag{6.23}$$

and for $s=2$,

$$\begin{aligned}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_0)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,0})}) - \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x',t_0)dx'\right)\underline{\hat{U}}_3^{(2)}\\ &\quad + \underline{\widetilde{U}}_{s+1}^{(2)}(t_0-t_{0,0}) + \left(\int_{t_{0,0}}^{t_0} 2u(x_0,\tau)m^{\frac{1}{3}}(x_0,\tau)d\tau\right)\underline{\hat{U}}_3^{(2)}, \end{aligned} \tag{6.24}$$

$$\begin{aligned}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_0)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,0})}) - \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x',t_0)dx'\right)\underline{\hat{U}}_3^{(2)}\\ &\quad + \underline{\widetilde{U}}_{s+1}^{(2)}(t_0-t_{0,0}) + \left(\int_{t_{0,0}}^{t_0} 2u(x_0,\tau)m^{\frac{1}{3}}(x_0,\tau)d\tau\right)\underline{\hat{U}}_3^{(2)}. \end{aligned} \tag{6.25}$$

Proof. As in the context of Theorem 4.7, it suffices to prove (6.22). Temporarily assume that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}$ is nonspecial for $(x,t_p)\in\Omega_\mu\subseteq\mathbb{C}^2$, where Ω_μ is open and connected. We introduce the meromorphic differential

$$\Omega(x,x_0,t_p,t_{0,p}) = \frac{\partial}{\partial\tilde{z}}\ln(\psi_2(\cdot,x,x_0,t_p,t_{0,p}))d\tilde{z}. \tag{6.26}$$

From the representation (6.16), one infers

$$\begin{aligned}\Omega(x,x_0,t_p,t_{0,p}) &= \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x',t_p)dx'\right)\omega_{P_0,3}^{(2)} - (t_p-t_{0,p})\widetilde{\Omega}_{P_{\infty_1},s+1}^{(2)}\\ &\quad - \sum_{j=1}^{r-5}\omega^{(3)}_{\hat{\mu}_j(x_0,t_{0,p}),\hat{\mu}_j(x,t_p)} + \hat{\omega}, \end{aligned} \tag{6.27}$$

where $\hat{\omega}$ denotes a holomorphic differential on $\mathcal{K}_{r-2}$, that is, $\hat{\omega}=\sum_{j=1}^{r-2}e_j\omega_j$ for some $e_j\in\mathbb{C}$ and ω_j $(j=1,\ldots,r-2)$ denote the normalized holomorphic differentials (see (4.13)). Since $\psi_2(\cdot,x,x_0,t_p,t_{0,p})$ is single-valued on $\mathcal{K}_{r-2}$, all a- and b-periods of Ω are integer multiples of $2\pi i$ and hence

$$2\pi i m_k = \int_{a_k}\Omega(x,x_0,t_p,t_{0,p}) = \int_{a_k}\hat{\omega} = e_k, \quad k=1,\ldots,r-2, \tag{6.28}$$

for some $m_k \in \mathbb{Z}$. Similarly, for some $n_k \in \mathbb{Z}$,

$$\begin{aligned} 2\pi i n_k &= \int_{b_k} \Omega(x, x_0, t_p, t_{0,p}) \\ &= \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x', t_p)dx'\right) \int_{b_k} \omega_{P_0,3}^{(2)} - (t_p - t_{0,p}) \int_{b_k} \widetilde{\Omega}_{P_{\infty_1},s+1}^{(2)} \\ &\quad - \sum_{j=1}^{r-5} \int_{b_k} \omega_{\hat{\mu}_j(x_0,t_{0,p}),\hat{\mu}_j(x,t_p)}^{(3)} + 2\pi i \sum_{j=1}^{r-2} m_j \int_{b_k} \omega_j \\ &= 2\pi i \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x', t_p)dx'\right) \hat{U}_{3,k}^{(2)} - 2\pi i (t_p - t_{0,p}) \widetilde{U}_{s+1,k}^{(2)} \\ &\quad - 2\pi i \sum_{j=1}^{r-5} \int_{\hat{\mu}_j(x,t_p)}^{\hat{\mu}_j(x_0,t_{0,p})} \omega_k + 2\pi i \sum_{j=1}^{r-2} m_j \int_{b_k} \omega_j \\ &= 2\pi i \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x', t_p)dx'\right) \hat{U}_{3,k}^{(2)} - 2\pi i (t_p - t_{0,p}) \widetilde{U}_{s+1,k}^{(2)} \\ &\quad + 2\pi i \alpha_{Q_0,k}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}) - 2\pi i \alpha_{Q_0,k}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,p})}) + 2\pi i \sum_{j=1}^{r-2} m_j \Gamma_{j,k}, \end{aligned} \tag{6.29}$$

where we have used the formula

$$\int_{b_k} \omega_{Q_1,Q_2}^{(3)} = 2\pi i \int_{Q_2}^{Q_1} \omega_k, \qquad k = 1, \ldots, r-2. \tag{6.30}$$

By symmetry of Γ this is equivalent to

$$\begin{aligned} \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,p})}) - \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x', t_p)dx'\right) \underline{\hat{U}}_3^{(2)} \\ + \underline{\widetilde{U}}_{s+1}^{(2)}(t_p - t_{0,p}) \end{aligned} \tag{6.31}$$

for $(x, t_p) \in \Omega_\mu$, which leads to (6.22). Since $\mathcal{D}_{P_0\underline{\hat{\nu}}}$ and $\mathcal{D}_{P_{\infty_1}\underline{\hat{\mu}}}$ are linearly equivalent, that is,

$$\underline{A}_{Q_0}(P_0) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_p)}) = \underline{A}_{Q_0}(P_{\infty_1}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}),$$

(6.23) holds. Similarly, one can prove (6.24) and (6.25). Finally, this result extends from $(x, t_p) \in \Omega_\mu$ to $(x, t_p) \in \mathbb{C}^2$ using the continuity of $\underline{\alpha}_{Q_0}$ and the fact that positive nonspecial divisors are dense in the space of divisors. ∎

Our main result, the theta function representation of time-dependent algebro-geometric solutions for the DP hierarchy, now quickly follows from the materials prepared above.

THEOREM 6.6. *Assume that u satisfies the pth DP equation* (2.14), *that is, $DP_p(u) = m_{t_p} - X_p = 0$, and the curve $\mathcal{K}_{r-2}$ is nonsingular. Let $(x, t_p) \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}^2$ is open and connected. Suppose also that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}$, or equivalently $\mathcal{D}_{\underline{\hat{\nu}}(x,t_p)}$, is nonspecial for $(x, t_p) \in \Omega_\mu$. Then*

$$u(x, t_p) = u(x_0, t_{0,p}) \frac{\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0, t_{0,p})))\theta(\underline{\tilde{z}}(P_{\infty_1}, \underline{\hat{\mu}}(x, t_p)))}{\theta(\underline{\tilde{z}}(P_{\infty_1}, \underline{\hat{\mu}}(x_0, t_{0,p})))\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x, t_p)))}. \tag{6.32}$$

Proof. In the time-dependent context, we will use the same strategy as was used in Theorem 4.8 in the stationary case, treating t_p as a parameter. Taking a closer look at Theorem 6.3, we note that the two expressions of ψ_2 in (6.15) $(p = 0, s = 2)$ and (6.16) $(p > 0, s > 2)$ can be written uniformly as the following form near P_{∞_1}:

$$\begin{aligned}\psi_2 \underset{\zeta\to 0}{=} & \left(\sigma_0 + \sigma_1\zeta + \sigma_2\zeta^2 + O(\zeta^3)\right)\\ & \times \exp\left(\left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x', t_p)dx'\right)\left(f_3^{(2)}(Q_0)\zeta^2 + O(\zeta^4)\right)\right)\\ & \times \exp\left((t_p - t_{0,p})\left(\frac{2}{3}\zeta^{-s} + \sum_{j=0}^{\frac{s-4}{2}}\tilde{\alpha}_j\frac{1}{\zeta^{s-2j-2}} + \int_{t_{0,p}}^{t_p}\chi_{\frac{s-2}{2}}(x_0, t'_p)dt'_p\right) + O(\zeta^2)\right),\end{aligned}\tag{6.33}$$

$$\zeta = \tilde{z}^{-1}, \quad \text{as } P \to P_{\infty_1},$$

where the terms $\sigma_i = \sigma_i(x, t_p)$ $(i = 1, 2, 3)$ come from the Taylor expansion about P_{∞_1} of the ratios of the theta functions in (6.15) $(p = 0)$ or (6.16) $(p > 0)$ (see (4.45)). That is,

$$\frac{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x, t_p))\right)}{\theta\left(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x, t_p))\right)} \underset{\zeta\to 0}{=} \frac{\theta_0}{\theta_1} - \frac{\partial_x\theta_0}{\theta_1}\zeta + \frac{\frac{1}{2}\partial_x^2\theta_0 - \partial_{\underline{U}_3^{(2)}}\theta_0}{\theta_1}\zeta^2 + O(\zeta^3), \quad \text{as } P \to P_{\infty_1},$$

with

$$\begin{aligned}\theta_0 &= \theta_0(x, t_p) = \theta(\underline{\tilde{z}}(P_{\infty_1}, \underline{\hat{\mu}}(x, t_p))) = \theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_1}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)})\right),\\ \theta_1 &= \theta_1(x, t_p) = \theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x, t_p))) = \theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_0) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)})\right),\end{aligned}$$

and

$$\partial_{\underline{U}_3^{(2)}} = \sum_{j=1}^{r-2} U_{3,j}^{(2)}\frac{\partial}{\partial\tilde{z}_j}.$$

Similarly, we have

$$\begin{aligned}\frac{\theta\left(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0, t_{0,p}))\right)}{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x_0, t_{0,p}))\right)} &= \left(\frac{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x, t_p))\right)}{\theta\left(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x, t_p))\right)}\right)^{-1}\Big|_{(x,t_p)=(x_0,t_{0,p})}\\ &\underset{\zeta\to 0}{=} \left(\frac{\theta_0}{\theta_1}\left(1 - \frac{\partial_x\theta_0}{\theta_0}\zeta + O(\zeta^2)\right)\right)^{-1}\Big|_{(x,t_p)=(x_0,t_{0,p})}\\ &\underset{\zeta\to 0}{=} \frac{\theta_1}{\theta_0}\left(1 + \partial_x\ln\theta_0\,\zeta + O(\zeta^2)\right)\Big|_{(x,t_p)=(x_0,t_{0,p})}\\ &\underset{\zeta\to 0}{=} \frac{\theta_1(x_0, t_{0,p})}{\theta_0(x_0, t_{0,p})}\left(1 + \partial_x\ln\theta_0(x, t_p)\Big|_{(x,t_p)=(x_0,t_{0,p})}\zeta + O(\zeta^2)\right),\\ &\qquad \text{as } P \to P_{\infty_1}.\end{aligned}$$

Then we will give the Taylor expansion about ψ_2,

$$\psi_2 \underset{\zeta\to 0}{=} \frac{\theta(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x, t_p)))\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0, t_{0,p})))}{\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x, t_p)))\theta(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x_0, t_{0,p})))}$$

$$\times \exp\left(\left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x',t_p)dx'\right)\left(f_3^{(2)}(Q_0)\zeta^2 + O(\zeta^4)\right)\right)$$
$$\times \exp\left((t_p - t_{0,p})\left(\frac{2}{3}\zeta^{-s} + \sum_{j=0}^{\frac{s-4}{2}} \tilde{\alpha}_j \frac{1}{\zeta^{s-2j-2}} + \int_{t_{0,p}}^{t_p} \chi_{\frac{s-2}{2}}(x_0,t_p')dt_p'\right) + O(\zeta^2)\right)$$
$$\underset{\zeta\to 0}{=} \left[\frac{\theta_1(x_0,t_{0,p})}{\theta_0(x_0,t_{0,p})}\frac{\theta_0(x,t_p)}{\theta_1(x,t_p)} + \frac{\theta_1(x_0,t_{0,p})}{\theta_0(x_0,t_{0,p})}\frac{\theta_0(x,t_p)}{\theta_1(x,t_p)}\right.$$
$$\left.\times\left(\partial_x \ln\theta_0(x,t_p)\Big|_{(x,t_p)=(x_0,t_{0,p})} - \partial_x \ln\theta_0(x,t_p)\right)\zeta + O(\zeta^2)\right]$$
$$\times \exp\left(\left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x',t_p)dx'\right)\left(f_3^{(2)}(Q_0)\zeta^2 + O(\zeta^4)\right)\right)$$
$$\times \exp\left((t_p - t_{0,p})\left(\frac{2}{3}\zeta^{-s} + \sum_{j=0}^{\frac{s-4}{2}} \tilde{\alpha}_j \frac{1}{\zeta^{s-2j-2}} + \int_{t_{0,p}}^{t_p} \chi_{\frac{s-2}{2}}(x_0,t_p')dt_p'\right) + O(\zeta^2)\right),$$
$$\text{as } P \to P_{\infty_1}. \tag{6.34}$$

Hence, comparing the same powers of ζ in (6.33) and (6.34) gives

$$\sigma_0(x,t_p) = \frac{\theta_1(x_0,t_{0,p})}{\theta_0(x_0,t_{0,p})}\frac{\theta_0(x,t_p)}{\theta_1(x,t_p)}, \tag{6.35}$$
$$\sigma_1(x,t_p) = \left(\partial_x \ln\theta_0(x,t_p)\Big|_{(x,t_p)=(x_0,t_{0,p})} - \partial_x \ln\theta_0(x,t_p)\right)$$
$$\times \frac{\theta_1(x_0,t_{0,p})}{\theta_0(x_0,t_{0,p})}\frac{\theta_0(x,t_p)}{\theta_1(x,t_p)}. \tag{6.36}$$

If we set

$$\psi_2 \underset{\zeta\to 0}{=} \left(\sigma_0(x,t_p) + \sigma_1(x,t_p)\zeta + \sigma_2(x,t_p)\zeta^2 + O(\zeta^2)\right)\exp(\Delta)\exp\left(\tilde{\Delta}\right),$$
$$\text{as } P \to P_{\infty_1},$$

with

$$\exp(\Delta) = \exp\left(\left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x',t_p)dx'\right)\left(f_3^{(2)}(Q_0)\zeta^2 + O(\zeta^4)\right)\right)$$

and

$$\exp(\tilde{\Delta}) = \exp\left((t_p - t_{0,p})\left(\frac{2}{3}\zeta^{-s} + \sum_{j=0}^{\frac{s-4}{2}} \tilde{\alpha}_j \frac{1}{\zeta^{s-2j-2}} + \int_{t_{0,p}}^{t_p} \chi_{\frac{s-2}{2}}(x_0,t_p')dt_p'\right) + O(\zeta^2)\right),$$

then we can show as $(P \to P_{\infty_1})$

$$\psi_{2,x} \underset{\zeta\to 0}{=} \left(\sigma_{0,x} + \sigma_{1,x}\zeta + O(\zeta^2)\right)\exp\left(\tilde{\Delta}\right),$$
$$\psi_{2,xx} \underset{\zeta\to 0}{=} \left(\sigma_{0,xx} + \sigma_{1,xx}\zeta + O(\zeta^2)\right)\exp\left(\tilde{\Delta}\right), \tag{6.37}$$
$$\psi_{2,xxx} \underset{\zeta\to 0}{=} \left(\sigma_{0,xxx} + \sigma_{1,xxx}\zeta + O(\zeta^2)\right)\exp\left(\tilde{\Delta}\right).$$

By eliminating ψ_1 and ψ_3 in (2.3), we arrive at

$$\psi_{2,xxx} = -m\tilde{z}^{-2} + \frac{m_x}{m}\psi_{2,xx} - \frac{m_x}{m}\psi_2 + \psi_{2,x}. \tag{6.38}$$

Substituting (6.37) into (6.38) and comparing the coefficients of ζ^0 yields

$$\sigma_{0,xxx} = \frac{m_x}{m}(\sigma_{0,xx} - \sigma_0) + \sigma_{0,x},$$

that is,

$$\frac{(\sigma_{0,xx} - \sigma_0)_x}{\sigma_{0,xx} - \sigma_0} = \frac{m_x}{m} = \frac{(u(x,t_p) - u_{xx}(x,t_p))_x}{u(x,t_p) - u_{xx}(x,t_p)},$$

which together with (6.35) leads to (6.32). ◻

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