A Unisonant r-Matrix Structure of Integrable Systems and Its Reductions

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A new method is presented to generate finite dimensional integrable systems. Our starting point is a generalized Lax matrix instead of usual Lax pair. Then a unisonant r-matrix structure and a set of generalized Hamiltonian functions are constructed. It can be clearly seen that various constrained integrable flows by nonlinearization method, such as the c-AKNS, c-MKdV, c-Toda, etc., are derived from the reduction of this structure. Furthermore, some new integrable flows are produced.

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It is well-known that the nonlinearization technique is a powerful tool to produce finite dimensional integrable systems. With the help of this method, many new completely integrable systems were found. Each integrable system is generated through making nonlinearized procedure for a concrete spectral problem or Lax pair, and it has its own individuality. Then a natural question is whether or not there are a unified structure such that it can contain those individual integrable systems generated by nonlinearization method. In the present letter, we give an affirmative answer. We propose a new procedure to generate finite dimensional integrable systems from a generalized Lax matrix instead of usual Lax pair. To do so, we construct a unisonant r-matrix structure and a set of generalized integrable Hamiltonian functions through studying the fundamental Poisson bracket.

It can be clearly seen that various constrained integrable flows by nonlinearization method, such as the c-AKNS, c-MKdV, c-Toda, etc., are derived from the reduction of the structure. Moreover, some new integrable flows are produced from this structure. Let us first give some necessary notation in this letter: dp∧dq stands for the standard symplectic structure in Euclidean space $R^{2N} = \{(p,q)|p = (p_1,\ldots,p_N), q = (q_1,\ldots,q_N)\}$, $(\cdot,\cdot)$ the standard inner product in $R^N$; in $(R^{2N}, dp∧dq)$ the Poisson bracket of two Hamiltonian functions $F,G$ is defined by

$$
\{F,G\} = \sum_{i=1}^{N} \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)
$$

$$
= \left( \frac{\partial F}{\partial q} \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial q} \right). \tag{1}
$$

And $\lambda_1,\ldots,\lambda_N$ are $N$ arbitrarily given distinct constants; $\lambda$ and $\mu$ are the two different spectral parameters; $A = \text{diag}(\lambda_1,\ldots,\lambda_N)$, $I_0 = \{q_i\}$, $J_0 = \{p_q\}$, $K_0 = \{p_p\}$, $I_1 = \{Ap,p\}(Aq,q)$, $J_1 = \{Ap,q\}$; $a_0, a_1 = \text{const}.$

Denote all infinitely times differentiable functions on real field $R$ by $C^\infty(R)$.

Consider the following matrix (called Lax matrix)

$$
L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}, \tag{2}
$$

where

$$
A(\lambda) = a_{-2}(I_1, J_1)\lambda^{-2} + a_{-1}(J_0)\lambda^{-1} + a_0 + a_1\lambda + \sum_{j=1}^{N} \frac{q_j}{\lambda - \lambda_j}, \tag{3}
$$

$$
B(\lambda) = b_{-1}(I_0, J_0)\lambda^{-1} + b_0(J_0) - \sum_{j=1}^{N} \frac{q_j^2}{\lambda - \lambda_j}, \tag{4}
$$

$$
C(\lambda) = c_{-1}(J_0, K_0)\lambda^{-1} + c_0(J_0) + \sum_{j=1}^{N} \frac{p_j^2}{\lambda - \lambda_j}. \tag{5}
$$

Now, we make an Assumption (P): $\{A(\lambda), A(\mu), A(\lambda), B(\mu), A(\lambda), C(\mu), B(\lambda), B(\mu), B(\lambda), C(\lambda), C(\mu)\}$ are all expressed as some linear combinations of $A(\lambda), A(\mu), B(\lambda), B(\mu), C(\lambda), C(\mu)$, then we have:

Proposition 1: If the Assumption (P) holds, then $L(\lambda)$ only contains the following cases:

1. As $a_{-2} \neq \text{const}$, $a_0 = b_0 = c_0 = a_1 = 0$, $a_{-1} = -b_0 = c_0 = -K_0 = a_{-2}$ satisfies the relation $I_1 = (J_1 + a_{-2})^2 + f(a_{-2})$, $\forall f(a_{-2}) \in C^\infty(R)$; as $a_{-2} = \text{const} \neq 0$, $a_0 = b_0 = c_0 = a_1 = 0$, $a_{-1} = \text{const}$, $b_{-1} = l_0 + f(J_0)$, $c_{-1} = -K_0 + g(J_0)$, and $f(J_0), g(J_0) \in C^\infty(R)$ satisfy the relation $f(J_0)g(J_0) = -J_0^2 - 2a_{-1}J_0 + \text{const}$, or

$\lambda_1,\ldots,\lambda_N$ are $N$ arbitrarily given distinct constants; $\lambda$ and $\mu$ are the two different spectral parameters; $A = \text{diag}(\lambda_1,\ldots,\lambda_N)$, $I_0 = \{q_i\}$, $J_0 = \{p_q\}$, $K_0 = \{p_p\}$, $I_1 = \{Ap,p\}(Aq,q)$, $J_1 = \{Ap,q\}$; $a_0, a_1 = \text{const}.$

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2. \( a_{-2} = a_{-1} = b_{-1} = c_{-1} = b_0 = c_0 = a_1 = 0, a_0 = \text{const.} \)

3. \( a_{-2} = b_{-1} = a_0 = b_0 = c_0 = a_1 = 0, c_{-1} = -K_0, a_{-1} = a_0(J_0), \forall a_{-1}(J_0) \in C^\infty(R), \text{ but } d a_{-1}/d J_0 \neq 0. \)

4. \( a_{-2} = a_{-1} = b_{-1} = c_{-1} = b_0 = a_1 = 0, a_0 = \text{const.}, c_0 \neq 0, c_0(J_0) \in C^\infty(R). \)

5. \( a_{-2} = c_{-1} = b_0 = a_1 = 0, a_0 = \text{const.}, b_{-1} = I_0 + f(J_0), c_0 = c_0(J_0) \text{ satisfies } d(J_0)/(d(J_0) \cdot f(J_0)) = -2a_0, \forall f(J_0) \in C^\infty(R), \text{ as } c_0(J_0) \cdot f(J_0) = \text{const.}, \text{ choose } a_0 = 0. \)

6. \( a_{-2} = c_{-1} = a_0 = b_0 = c_0 = a_1 = 0, a_{-1} = J_0 + \text{const.}, b_{-1} = I_0. \)

7. As \( a_{-2} = a_0 = b_0 = c_0 = a_1 = 0, \) there are the following five subcases:

   (7.1) \( a_{-1} = \text{const.}, c_{-1} = -K_0 + f(J_0), b_{-1} = I_0 + g(J_0), \forall f(J_0), g(J_0) \in C^\infty(R); \)

   (7.2) \( a_{-1} = J_0, b_{-1} = I_0, c_{-1} = K_0; \)

   (7.3) \( a_{-1} = J_0 + \text{const.}, (d/d J_0)(b_{-1} = c_{-1}) = 2a_0, \forall b_{-1} = c_{-1} = a_{-1} = J_0 \in C^\infty(R); \)

   (7.4) \( a_{-1} = -J_0 + \text{const.}, b_{-1} = I_0, c_{-1} = -K_0 + \text{const.}, c_{-1} = a_{-1} = J_0 \in C^\infty(R); \)

   (7.5) \( a_{-1} = -J_0 + \text{const.}, c_{-1} = -K_0, b_{-1} = a_{-1} = J_0 \in C^\infty(R). \)

8. \( a_{-2} = a_{-1} = b_{-1} = c_{-1} = 0, a_0 = \text{const.}, a_1 = \text{const.}, b_0 \neq 0, c_0 \neq 0, \) which satisfy the relation 

\[
(d/d J_0)(b_0 c_0) = -2a_1, \forall b_0 = b_0(J_0), c_0 = c_0(J_0) \in C^\infty(R).
\]

9. \( a_{-2} = a_{-1} = b_{-1} = c_{-1} = 0, a_0 = \text{const.}, b_0 \neq 0, \forall b_0 = b_0(J_0) \in C^\infty(R). \)

10. \( a_{-2} = b_{-1} = c_0 = a_1 = 0, a_{-1} = \text{const.}, a_0 = \text{const.}, a_1 = -K_0 + f(J_0), b_0 = b_0(J_0) \) satisfies the relation 

\[
(d/d J_0)(b_0(J_0) \cdot f(J_0)) = -2a_0, \forall f(J_0) \in C^\infty(R), \text{ as } b_0(J_0) \cdot f(J_0) = \text{const.}, \text{ choose } a_0 = 0.
\]

Let \( L_1(\lambda) = L(\lambda) \otimes I, L_2(\mu) = I \otimes L(\mu), \) where \( I \) is the \( 2 \times 2 \) unit matrix, \( \otimes \) is the tensor product of matrix. In the following, we search for a general \( 4 \times 4 \) \( r \)-matrix structure \( r_{12}(\lambda, \mu) \) such that the fundamental Poisson bracket:

\[
\{L(\lambda) \otimes L(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] - [r_{21}(\lambda, \mu), L_2(\mu)]
\]

holds, where \( r_{21}(\lambda, \mu) = Pr_{12}(\lambda, \mu)P = (1/2) \sum_{i=0}^{3} \sigma_i \otimes \sigma_i, \sigma_i \) is the standard Pauli matrices.

Theorem 1: Under the Assumption (P),

\[
r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P + S
\]

is an \( r \)-matrix structure satisfying Eq. (6), where

\[
S = \begin{pmatrix}
\frac{2 \lambda \partial a_{-2}}{\mu^2} \frac{\partial B_{-1}}{\partial J_0} + \frac{2 \partial a_{-1}}{\mu} \frac{\partial B_{-1}}{\partial J_0} & \frac{2 \lambda \partial B_{-1}}{\mu^2} \frac{\partial a_{-2}}{\partial J_1} & 0 \\
\frac{2 \partial B_{-1}}{\mu} \frac{\partial C_{-1}}{\partial K_0} & \frac{2 \partial C_{-1}}{\mu} \frac{\partial B_{-1}}{\partial J_0} & -\frac{2 \lambda}{\mu^2} \frac{\partial a_{-2}}{\partial J_1} \\
0 & -\frac{2 \partial B_{-1}}{\mu} \frac{\partial a_{-2}}{\partial J_0} & \frac{2 \partial C_{-1}}{\mu^2} \frac{\partial a_{-2}}{\partial J_1} + \frac{2 \partial a_{-1}}{\mu} \frac{\partial B_{-1}}{\partial J_0}
\end{pmatrix}
\]

The proof of Proposition 1 and Theorem 1 can be seen in Ref. 5.

Through considering the determinant of \( L(\lambda) \) and combining it with Eq. (6), we can easily obtain the following theorem.

Theorem 2: Under the assumption (P), the following equalities

\[
\{E_i, E_j\} = 0, \{H_l, E_j\} = 0, \{F_m, E_j\} = 0, \quad i, j = 1, 2, \ldots, N, \quad l = -4, \ldots, 2, \quad m = 0, 1, 2, \ldots;
\]

hold. Hence, the Hamiltonian systems \( (H_l) \) and \( (F_m) \)

\[
(H_l): q_l = \frac{\partial H_l}{\partial p}, p_l = -\frac{\partial H_l}{\partial q}, \quad l = -4, \ldots, 2;
\]

\[
(F_m): q_m = \frac{\partial F_m}{\partial p}, p_m = -\frac{\partial F_m}{\partial q}, \quad m = 0, 1, 2, \ldots
\]

are completely integrable in Liouville's sense. Here, the expressions \( E_j \) \( (j = 1, 2, \ldots, N), \) \( H_l \) \( (l = -4, \ldots, 2), \) and \( F_m \) \( (m = 0, 1, 2, \ldots) \) are the same as ones in Ref. 5.

Because Lax matrix (2) includes various cases displayed in Proposition 1, we call matrix (2) as a generalized Lax matrix. In the following it can be seen that various constrained flows are reduced from the unisonant \( r \)-matrix (7).

The following numbers of title coincide with the ones in Proposition 1, i.e. the corresponding conditions are coincidental. For simplicity, here we only give a new reduction and several well-known examples from the unisonant \( r \)-matrix (7). Other cases are similar. In what follows we take the prime \( \prime \) for \( d/d J_0. \)

\[
r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P
\]
This is nothing but the r-matrix of the well-known constrained AKNS (c-AKNS) system.\(^1\)

\[ r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P + \frac{2}{\mu} S, \]

where

\[ S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12} \]

This is a new r-matrix.

\[ r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P + S, \]

\[ S = b_0(J_0) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{13} \]

As \( b_0 = -J_0 \), Eq. (13) is reduced to the common r-matrix of the constrained Toda (c-Toda) system and the constrained CKdV (c-CKdV) system.\(^2\)

\[ r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P + S, \]

\[ S = \begin{pmatrix} 1 & 0 & b_0(J_0) & 0 \\ 0 & b_0'(J_0) & 0 & -2 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & -1 / \mu & b_0'(J_0) \end{pmatrix}. \tag{14} \]

As \( f(J_0) = \text{const} \), \( b_0(J_0) = 0 \), Eq. (14) reads the r-matrix\(^3\) of the constrained MKdV (c-MKdV) system.

The reduction procedure of the above four cases and the r-matrices related to other cases in Theorem 1 are detailedly presented in Ref. 5.

Here, we simply give the integrable system arising from the new r-matrix (12). The corresponding involutive systems are

\[ E_j^1 = 2((p, q) + c)\lambda_j^{-1}p_jq_j + (q, q)\lambda_j^{-1}p_j^2 - \Gamma_j, \quad j = 1, \ldots, N, \]

\[ \Gamma_j = \sum_{k=1, k \neq j}^{N} \frac{(p_jq_k - p_kq_j)^2}{\lambda_j - \lambda_k}, \]

where \( c \) is an arbitrarily given constant. Thus, the finite dimensional Hamiltonian systems (\( F_{m}^{1} \)) defined by

\[ F_{m}^{1} = \sum_{j=1}^{N} \lambda_j^{m} E_j^1, \quad m = 0, \ldots, \]

are completely integrable. Particularly, as \( m = 2 \) the Hamiltonian system (\( F_{2}^{1} \)):

\[ \begin{aligned}
q_{z} &= \frac{\partial F_{2}^{1}}{\partial p} = 2c\lambda q - 2\langle \lambda q, q \rangle p \\
+4\langle Ap, q \rangle q + 4\langle p, q \rangle Aq, \\
p_{x} &= -\frac{\partial F_{2}^{1}}{\partial q} = -2cAp + 2\langle p, p \rangle Aq \\
-4\langle Ap, q \rangle p - 4\langle p, q \rangle Ap,
\end{aligned} \tag{17} \]

is a new finite dimensional integrable system, which can be changed as the following spectral problem

\[ \phi_{x} = \begin{pmatrix} (2c + 4\nu)\lambda + 4u \\
-2w \\
2s \lambda \\
-(2c + 4\nu)\lambda - 4u \end{pmatrix} \phi \tag{18} \]

with the constraint condition \( u = \langle Ap, q \rangle, \quad v = \langle p, q \rangle, \quad w = \langle Ap, q \rangle, \quad s = \langle p, p \rangle, \) and \( \lambda = \lambda_j, \quad \phi = (\xi_1, \xi_2)^{T}, \quad j = 1, \ldots, N. \) Apparently, spectral problem (18) is a new one and has never been studied before.

As we see as above, r-matrix indeed plays a very important role in guaranteeing integrability as well as in unifying each concrete integrable system by nonlinearity method. Recently, we found two different (even a continuous and a discrete) constrained flows share a common r-matrix, Lax matrix, and even involutive systems.\(^7,8\) This, more or less, will be helpful to the classification of finite dimensional integrable systems. In addition, the above new integrable system also implies an interesting procedure how to connect r-matrix with the spectral problem.

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