Iteratively Compensating for Multiple Scattering in SAR Imaging

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ABSTRACT

The Born approximation is a common approach taken in modeling the physics of SAR imaging. In essence it says that radiation only scatters once when in space. This is a reasonable assumption for targets that lie far apart or that are far from the transmit and receive antennas, but it introduces error into the imaging process. The goal of this paper is to iteratively compensate for this error by using estimates of the target distribution to estimate multiple scattering phenomena. We will use a noise reduction technique at each iteration on the corrected data as well as the estimated image to control any excess error caused by the estimated multiple scattering phenomena. The physical model for our work will be based on the wave equation. We will briefly derive the important features of the model as well as account for the error brought by common approximations that are made. Typically one does not get an image that is approximately the target distribution, but rather an image that is approximately proportional to the target distribution. This means that there is a scaling parameter that must be chosen when using target distribution estimates to correct data. We will discuss methods for choosing this parameter. We will provide a few basic SAR imaging methods and perform simulation using the Gotcha Data set in combination with the iterative technique. At the end of the paper we will outline future work involving this method.

Keywords: Born approximation, multiple scattering, Neumann series, SAR imaging, iterative imaging.

1. INTRODUCTION

Traditional RADAR is an imaging modality that has grown considerably since WWII. As the acronym RAdio Detection And Ranging suggest, its primary interest early on was in detecting and finding the distance to targets.[1] Roughly speaking, in an active system one would focus a narrow beam of radiation into some region of interest. A receiving antenna measures radiation and if the power is above some noise threshold or if the amount time taken for a signal to return was short enough, it determined target presence at a certain range. Traditional radar equations are concerned with the maximum detectable range. If transmitted radiation lies in some cone emanating from the transmitter, one can discern that any target must lie in this cone. Eventually this evolved to include finding the azimuth of a certain target by designing antennas which restrict the beam of radiation to a certain cone of illumination. It turns out that larger antenna apertures lead to smaller cones (beam width) of illumination. This would seem to solve the $2-\mathrm{d}$ imaging problem if large antennas were practical. They are not. An alternative is to use multiple sensors to estimate the direction of arrival of the radiation.[2] This requires several smaller antennas and is an active area of research.[2,3] With the advent of powerful computers the trend has been to move the work of imaging from hardware to software. That is, we collect data from multiple transmissions at several points in space and use the data coherently to develop an estimation of our targets. The basic idea of Synthetic Aperture RADAR (SAR) is to move a single (or several) small antenna(s) through space and collect data which is combined to provide an estimate of our targets. This is slower than traditional RADAR but allows for finer azimuth resolution.[4] Radiation scattered from a set of targets is determined by a nonlinear integral equation with kernel dependent on the target distribution. The mathematics of such a scenario can be quite complicated and as such several approximations are taken. First, one usually assumes that the antenna is moving slow enough that it can be considered stationary[5] while radiation propagates and returns to the antenna. Then one may make the assumption that the targets are only in a small region of space

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that is far from the antenna. Finally, to linearize the integral equation one takes the Born approximation[6,7] and assumes that after a target scatters radiation, the resulting field does not scatter again in a way that is noticeable. Our primary concern in this study is to compensate for errors in image formation that are caused by the Born approximation. The idea is that we may use estimates of our targets to estimate higher order scattering and remove it from the collected data. In order for this method to be robust we do not concern ourselves with image formation, but rather image improvement using some given image formation algorithm. In general we consider a linear imaging operator that acts on our data to produce an image estimate. The structure of the remainder of this paper is as follows. In section 2 we introduce the model used. Next, in section 3 we define a scattering operator, high order scattering and the Born approximation. After this, in section 4 we examine the suitability of the Born approximation. This is followed by our iterative algorithm for removing artifacts introduced by this approximation and a computational analysis of the performance of our algorithm in sections 4 and 5 respectively. In the conclusion we summarize our results and discuss further research in this direction. A rudimentary analysis of the scattering operator is given in the appendix.

2. BACKGROUND AND MODEL FORMULATION

One can abstract away much of the physical aspects of SAR imaging in order to focus directly on recreating images. In this section we address the notation and mathematics used in the rest of the paper. In a fairly general sense, we deal with the action of two operators on two Hilbert spaces: a scattering operator $\mathcal{L}$, and an imaging operator $\mathcal{I}$. The imaging operator acts on our field space $\mathcal{F}$ to produce an approximate image from our image space $\mathcal{V}$. Making these Hilbert spaces gives us a way to measure error in an approximation as well as discuss convergence of sequences of approximations. With this space comes the induced norm $\|\cdot\| = \sqrt{<\cdot,\cdot>}$ which in turn can be used to define a distance function by $d(\cdot,\cdot) = \|\cdot - \cdot\|$. Our primary concern and analysis with regard to the scattering operator. A bilinear operator is a map from two linear spaces to a third linear that is linear each of its arguments. If we consider $\mathcal{F}$ to be a linear space of possible fields, and $\mathcal{V}$ a linear space of possible reflectivity functions, then our scattering operator is bilinear from $\mathcal{V} \times \mathcal{F}$ to $\mathcal{F}$. A bilinear operator may also be viewed as a family of linear operators indexed by one of its arguments. That is, for every fixed image our scattering operator is simply a linear operator. We consider the scalar wave model[3,6–11] in the frequency domain (Helmholtz equation) and its respective scattering operator. In this model, which is a special case of Maxwell's equations, we assume that any component of electro magnetic (EM) radiation satisfies the wave equation. The EM field u is broken up into an unknown scattered u_{sc}, and a known incident portion u_{in} such that

$$u(x,w) = u^{sc}(x,w) + u^{in}(x,w), \tag{1}$$

$$\left(\nabla^2 + \frac{w^2}{c(x)^2}\right) u = -j(x,w), \tag{2}$$

and

$$\left(\nabla + \frac{w^2}{c_0^2}\right) u^{in} = -j(x,w) \tag{3}$$

where c_0 is the speed of light in free space, $c(x,w)$ is the speed of light at some point $x \in \mathbb{R}^3$ and frequency w, and $j(x,w)$ is the source of the field.

We define our reflectivity function[3,6–8,12] by

$$v(x,w) = \frac{1}{c_0^2} - \frac{1}{c(x,w)^2}. \tag{4}$$

The reflectivity function gives us an idea of what our target distribution looks like. It is zero when there is no target at the given point, and largest in absolute value when the wave speed is minimum. The goal of RADAR imaging in the wave model is then to solve for (4). The free space Helmholtz equation may be solved by the method of Green's functions. That is, we can solve the equation by integrating against a certain delayed version

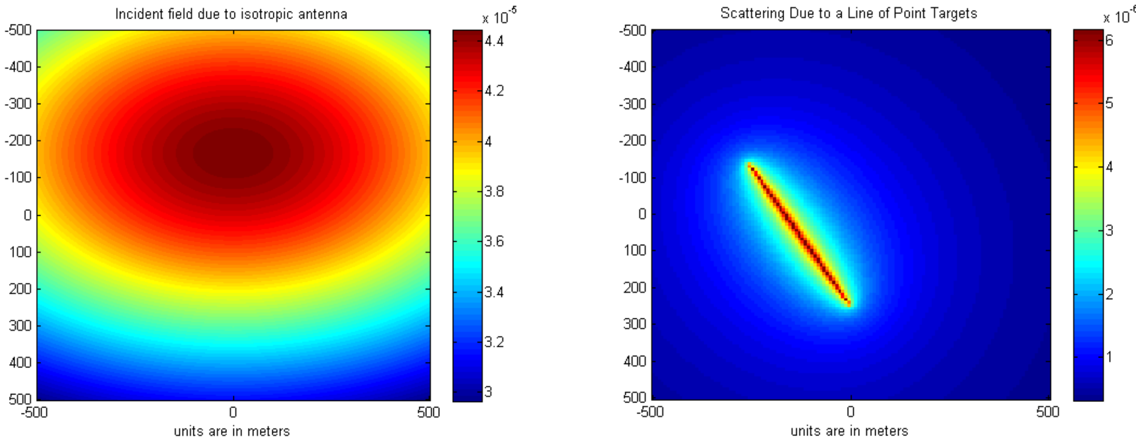

Figure 1. Incident and Born approximated scattered field on left and right, respectively.

of some known function(convolving with) called a Green's function. The Green's[8] function corresponding to the free space Helmholtz equation is

$$G(x, w) = \frac{e^{-iw\frac{\|x\|}{c_0}}}{4\pi\|x\|}. \tag{5}$$

Combining (1), (2), and (3), and assuming that our targets are stationary, we can find the scattered field as the solution to a free space Helmholtz equation[8] which combined with (5) can gives us the integral formulation for the scattered field

$$u^{sc}(x, w) = -w^2 \int_{\mathbb{R}^3} \frac{e^{-iw\frac{\|x-z\|}{c_0}} \nu(z) u(z, w)}{4\pi\|x - z\|}\, dz. \tag{6}$$

This form is a variant of the Lippmann Schwinger equation.[8] If we know the total field u, then finding an imaging operator is equivalent to finding an approximate left inverse to this equation[8] and can take the form of an oscillatory integral.[4] Sadly, we do not know u since it is dependent on u^{sc} by (1). The purpose of the Born approximation is to remove this non-linearity by approximating the total field by the incident field. To be formal about this we must define our scattering operator.

3. SCATTERING OPERATOR AND HIGH ORDER SCATTERING

Let $\psi \in \mathcal{V}$ and $\phi \in \mathcal{F}$. Then we define the scattering operator as the map from $\mathcal{V} \times \mathcal{F}$ to $\mathcal{F}$ by

$$L(\psi, \phi) = -w^2 \int_{\mathbb{R}^3} \frac{e^{-iw\frac{\|x-z\|}{c_0}} \psi(z) \phi(z, w)}{4\pi\|x - z\|}\, dz \tag{7}$$

If the scattering operator behaves nicely (see appendix) then we can solve for the scattered field explicitly in terms of the incident field via the Neumann series[8]

$$u^{sc} = \sum_{n=1}^{\infty} L^n\{\nu(x), u^{in}\} \tag{8}$$

where $L^n\{\nu(x), u^{in}\} = L\{\nu(x), L^{n-1}\{u^{in}\}\}$ and $L^0 = I$ is the identity operator. Physically, the first term in the series corresponds to radiation interacting with the targets and getting scattered. The second term corresponds to this first scattered field again interacting with targets and getting scattered again. In general the n-th term is the $n-1$-th term being scattered. Adding all of these together we get the scattered field. The scattered field is then the sum of these rescattered fields.

We discuss the singularity present in the appendix. Throughout the rest of the paper convergence of the series is assumed, meaning we restrict our field and image space to those that allow the series to make sense. With this, we have an integral representation for the scattered field in which the only unknown is the reflectivity function and the scattered field. The idea in SAR is to move through space and measure the scattered field to solve for the reflectivity function. In this form the problem is still fairly difficult even if we completely know the scattered field. For this reason one typically assumes the higher order terms in the sum are negligible. This is called the Born approximation. The incident field is dependent on the source in the original model. This in turn is dependent on a known input signal from the space $\mathcal{S}$. One then uses an imaging operator $I : \mathcal{F} \times \mathcal{S} \to \mathcal{V}$ on the Born approximated data and the input signal to estimate the reflectivity function. Our goal is to use this estimated image to estimate and remove the higher order terms from the scattered data up to some finite number N. From experimentation it is evident that estimated reflectivity functions tend to be scaled improperly. That is to say, the positions of targets are accurate, and their relative values are reasonable, but the estimated reflectivity values are off. To resolve this we simply scale the estimated high order scattering. In addition, estimates tend to look fairly noisy, so we must perform some denoising operator $\mathcal{N}$ before using the image estimate. Formally we create a sequence of sequence of estimates $\{\hat{V}_k^N\}$ in $\mathcal{V}$ by

$$\hat{V}_{k+1}^N = I\left\{d - \sum_{n=2}^{N} L^n\{\lambda_k N\{V_k^N\}, U^{in}\}\right\} \tag{9}$$

where λ_k is a scaling parameter. The fact that L is linear in its first argument allows us to either scale the estimated image first then scatter it, or to scale the scattered unscaled image. The order does matter when round off error is taken into account, so we scale before scattering, but after noise reduction. We will discuss the specifics later, but first let's consider when it is appropriate to use this method. We do not establish convergence of this method, but test it computationally.

4. SUITABILITY OF METHOD

Here, we discuss when high order scattering may lead to artifacts during image formation. We give some idea of the error introduced with the truncated sum. Recall that the index of refraction is defined by

$$n(x, w) = \frac{c_0}{c(x, w)} \tag{10}$$

and is related to our reflectivity function by

$$\frac{1 - n^2(x)}{c_0^2} = v(x). \tag{11}$$

Since indices tend to be small compared to the speed of light, the reflectivity function will be small. One may consult a table to see how small $v(x)$ will be. For this reason we assume that $|v(x)|$ is bounded and that every value is less than the least upper bound $\sup(v(x))$ of $v(x)$. Further, we assume that it is only non-zero outside of some finite radius region denoted $\mathrm{supp}(v(x))$, and that $u^{in}(x, w)$ has finite bandwidth B around some center frequency w_0. For any field, the scattering operator has the following bound

$$|L\{v(x), \phi\}(x, w)| \leqslant (w_0 + \frac{B}{2})^2 |\sup(v(z))| \sup|\phi| C(\mathrm{supp}(v(x))) \tag{12}$$

for some constant C dependent on the size of the reflectivity function. Then the n-th order term in the scattered field sum can be bounded by

$$|L^n\{v(x), u^{in}\}(x, w)| \leqslant ((w_0 + \frac{B}{2})^2 |\sup(v(z))| C(\mathrm{supp}(v(x))))^n \sup\{u^{in}\}. \tag{13}$$

If $|\sup(v(z))| C(\mathrm{supp}(v(x))) < \frac{1}{(w_0 + \frac{B}{2})^2}$, the terms in the series decrease. So, we have that the error in the Born approximation is largest for high frequencies, and large reflective targets.

5. COMPUTATION OF THE SCATTERING OPERATOR AND NEUMANN SERIES

There are several efficient methods to evaluate the oscillatory integral that defines the scattering operator.[4,13] For simplicity we use a Riemann sum to evaluate the scattering operator at this point. We can efficiently evaluate a truncation of the scattering series by applying Horner's method to calculate the scattered field by:[14]

$$u_N^{sc} = L\{u^{in} + L\{u^{in} + \ldots L\{u^{in}\}\}\}$$

At this point we did not use a noise reduction method. In a rough approach, the Born approximated scattered field turns out to be the spatial Fourier transform of the reflectivity function sampled at varying frequencies. One can form an estimate then by taking the inverse transform of the scattered field.[15,16] This is the approach we use. We will pick a fixed scaling parameter λ for each step in the iteration.

6. CONCLUSION

There were several problems with the experiment. To push things forward it seems we need to use some fast Fourier integral method for scattering, and image formation. We were able to produce high order scattering data, but image formation was a bottle neck. There is still much work to be done in this direction. This work mostly constituted a numerical experiment, and several assumptions were made. The point of this work was to examine the use of high order terms to clear up images. At this point, our implementation was not optimized enough to examine this problem effectively. This was the cause of most of the trouble in the experiment. The change from continuous to discrete leaves open the possibility for round off error and such. This was an additional cause of trouble in the experiment. A rigorous analysis of this method would help clear up much of this trouble. The question of whether or not the method converges has not been answered. We were successful in producing high order scattering in the frequency domain, and have some idea of when it is appropriate to use a truncated Neumann series to approximate the scattering function. We would like to make explicit the relation between the error in truncation the number of terms in sum, and the final error in image formation. We will expand on these fronts in the future, though at least one author[17] seems to have found a way around the Born approximation in a compressive sensing setting.

APPENDIX A. CONVERGENCE OF THE NEUMANN SERIES

Suppose that we have a continuous linear operator L defined on a Hilbert space $\mathcal{H}$, and we want to solve

$$u^{sc} = L\{u^{sc}\} + u^{in} \tag{14}$$

or the equivalent

$$(I - L)\{u^{sc}\} = u^{in} \tag{15}$$

for u^{sc} given u^{in}. Similar to the geometric series, we have a solution in the form

$$u^{sc} = (I - L)^{-1}u^{in} = \sum_{n=0}^{\infty} L^n u^{in}. \tag{16}$$

when the series converges. Such a series is called a Neumann series[8] and is a model for multiple scattering phenomena. A sufficient condition for convergence is $\|L\| < 1$, but this is not necessary. It has been shown that the Neumann series (16) of a continuous linear operator L defined on a Hilbert space $\mathcal{H}$ converges if the norm of $L^n\{u^{in}\}$ tends to 0.[18] This is weaker than the previous condition. Our scattering operator can be seen as a family of linear operators acting on our field space by fixing a reflectivity function. Since (7) is linear in its second argument, it suffices to show boundedness to have continuity.[19] This was already shown above if we assume that the singularity can be taken care of and that any field and reflectivity functions are bounded. Let $\mathcal{V}$ and $\mathcal{F}$ be subsets of $L^1 \bigcap L^2$. That is, they are functions that are both integrable, and square integrable. This means that the reflectivity, and field functions are bounded except possibly on sets that are negligible. If we substitute $y = x - z$, and change to spherical coordinates, then the singularity goes away as long as $x \neq y$. The weight of this point is negligible so the integral is bounded so that the operator is continuous. We showed above that each term in the scattering series is less than the previous if our targets are small enough, and our input signal is sufficiently band limited. This means that the series should converge if we restrict our image and signal space appropriately. Since it is required for the series to converge to make sense of the Born approximation, and the Born approximation can be a reasonable estimate, we assume that our images and signals behave nice enough for the Neumann series to converge.

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