FOUR NEW COMPLETELY INTEGRABLE FINITE-DIMENSIONAL HAMILTONIAN SYSTEMS

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ABSTRACT Based on the Cao's method[1], by nonlinearization of four eigenvalue problems four new completely integrable finite-dimensional Hamiltonian systems in the Liouville's sense are presented. Particularly, in part 4 of Section II a new eigenvalue problem is given.

KEY WORDS nonlinearization, eigenvalue problems, completely integrable system

I. INTRODUCTION

It is well-known that many completely integrable finite-dimensional Hamiltonian systems in the Liouville's sense in explicit form can be successfully obtained by nonlinearization of eigenvalue problems[1-8]. The present paper is written in line with Cao's thought[1], i.e. it is a continuation of ref.[1], and is also devoted to the application of the nonlinearization method[1,2] proposed by professor Cao Cewen to the investigation of completely integrable Hamiltonian systems. Following the procedure offered in ref.[1], we obtain four new completely integrable Hamiltonian systems in the Liouville's sense in this paper. Thus, the scope of finite-dimensional integrable Hamiltonian systems is further enlarged.

II. FOUR NEW COMPLETELY INTEGRABLE FINITE-DIMENSIONAL HAMILTONIAN SYSTEMS

Following the thought of ref.[1], we continue to use the concerned symbols of ref.[1] in this paper, consider four eigenvalue problems below and nonlinearize them under some certain constraints.

1. Introduced the eigenvalue problem (the special case of the WKI eigenvalue problem[9])

$$y_x = My\,, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad M = \begin{pmatrix} -i\lambda & (u-1)\lambda \\ -\lambda & i\lambda \end{pmatrix} \tag{1}$$

where $i^2 = -1$, $y_x = \partial y/\partial x$, u is scalar potential, λ is a constant eigenvalue.

Let $\lambda_1, \cdots, \lambda_N$ be N different eigenvalues of (1), then the functional gradient $\nabla\lambda_j$ of the eigenvalue λ_j with regard to the potential u is

$$\nabla\lambda_j = \delta\lambda_j/\delta u = \lambda_j p_j^2 \tag{2}$$

where $y = (q_j, p_j)^T$ is the eigenfunction corresponding to the eigenvalue λ_j, i.e.

$$\left.\begin{aligned} q_{j,x} &= -i\lambda_j q_j + (u-1)\lambda_j p_j \\ p_{j,x} &= -i\lambda_j q_j + i\lambda_j p_j \end{aligned}\right\} \tag{3}$$

The pair of Lenard's operators associated with (1) are

$$K = \partial^3\,, \quad J = -2(\partial u + u\partial)\,, \quad \partial = \partial/\partial x \tag{4}$$

$\nabla\lambda_j$ satisfies $K\nabla\lambda_j = \lambda_j^2 \cdot J\nabla\lambda_j$

The Lenard's gradient sequence G_{2j} of (1) can be recursively decided by $KG_{2(j-1)} = JG_{2j}$, $G_{-2} = u^{-1/2} \in \mathrm{Ker}J$, $j = 0, 1, 2, \cdots$. The soliton hierarchy $u_{t_m} = JG_{2m}$ $(m = 0, 1, 2, \cdots)$ is given as the isospectral equations of (1) with the representative equation

$$u_{t_0} = JG_0 = KG_{-2} = (u^{-1/2})_{xxx} \tag{5}$$

which is exactly the well-known Harry-Dym (HD) equation. Thus, the isospectral evolution equations $u_{t_m} = JG_{2m}$ of (1) yields the HD hierarchy of equations. The HD hierarchy $u_{t_m} = JG_{2m}$ has the Lax representation[10], the eigenvalue problem (1) and the auxiliary problem

$$y_{t_m} = \sum_{j=0}^{m} V(G_{2(j-1)})\lambda^{2(m-j)+1} y, \quad m = 0, 1, 2, \cdots, \tag{6}$$

$$V(G_{2(j-1)}) = \begin{pmatrix} \lambda G_{2(j-1),x} - 2i\lambda^2 G_{2(j-1)} & G_{2(j-1),xx} - 2i\lambda G_{2(j-1),x} - 2\lambda^2(1-u)G_{2(j-1)} \\ -2\lambda^2 G_{2(j-1)} & -\lambda G_{2(j-1),x} + 2i\lambda^2 G_{2(j-1)} \end{pmatrix}$$

The Bargmann constraint is given by $G_{-2} = \sum_{j=0}^{N} \nabla\lambda_j$, which is equivalent to

$$u = <\Lambda p, p>^{-2} \tag{7}$$

where $p = (p_1, \cdots, p_N)^T$, $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, $<\cdot,\cdot>$ stands for the standard inner-product in R^N. The nonlinearized (1) under (7) $(q = (q_1, \cdots, q_N)^T)$

$$\left.\begin{aligned} q_x &= -i\Lambda q + (<\Lambda p, p>^{-2} - 1)\Lambda p = \partial H/\partial p \\ p_x &= -\Lambda q + i\Lambda p = -\partial H/\partial q \end{aligned}\right\} \tag{8}$$

is a completely integrable finite-dimensional Hamiltonian system $(R^{2N}, d_q\Lambda d_q, H)$ in the Liouville's sense with

$$H = -i <\Lambda p, q> + \frac{1}{2} <\Lambda q, q> - \frac{1}{2} <\Lambda p, p> - \frac{1}{2} <\Lambda p, p>^{-1} \tag{9}$$

whose involutive system of conserved integrals is

$$\begin{aligned} F_m = {} & <\Lambda^{2m+3}p, p><\Lambda p, p>^{-1} \\ & + \sum_{j=0}^{m} \begin{vmatrix} <\Lambda^{2j+2}p, p> & <\Lambda^{2j+1}p, p> \\ <\Lambda^{2(m-j)+3}p, p> & <\Lambda^{2(m-j)+2}p, p> \end{vmatrix} \\ & + \begin{vmatrix} <\Lambda^{2j+3}q, q> & <\Lambda^{2j+2}p, q> \\ <\Lambda^{2(m-j)+2}p, q> & <\Lambda^{2(m-j)+1}p, p> \end{vmatrix} \\ & + 2i \sum_{j=0}^{m} \begin{vmatrix} <\Lambda^{2j+2}p, q> & <\lambda^{2j+3}p, q> \\ <\Lambda^{2(m-j)+1}p, p> & <\Lambda^{2(m-j)+2}p, p> \end{vmatrix}, \quad m = 0, 1, 2, \cdots \end{aligned} \tag{10}$$

Through a lengthy calculations, by using properties of the Poisson bracket we can prove

$$(H, F_m) = 0, \quad (F_m, F_n) = 0, \quad \forall_{m,n} \in Z^+ \tag{11}$$

Moreover, it can be shown that (10) is exactly produced by nonlinearization of the time part (6) of the Lax pair under (7).

2. Heisenberg eigenvalue problem[11]

$$y_x = My, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad M = \begin{pmatrix} -i\lambda w & -i\lambda u \\ -i\lambda v & i\lambda w \end{pmatrix}, \quad w^2 + uv = 1, \quad i^2 = -1 \tag{12}$$

$$\nabla\lambda_j = \begin{pmatrix} \delta\lambda_j/\delta u \\ \delta\lambda_j/\delta v \end{pmatrix} = \begin{pmatrix} -\lambda_j p_j^2 + \lambda_j(v/w)p_j q_j \\ \lambda_j q_j^2 + \lambda_j(u/w)p_j q_j \end{pmatrix}, \quad j = 1, 2, \cdots, N \tag{13}$$

Choise the pair of Lenard's operators

$$K = i\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad J = \begin{pmatrix} -\partial(u/w)\partial^{-1}u & 2w + \partial(u/w)\partial^{-1}v \\ -2w - \partial(v/w)\partial^{-1}u & \partial(v/w)\partial^{-1}v \end{pmatrix}$$
$$\partial = \partial/\partial x, \quad \partial\partial^{-1} = \partial^{-1}\partial = 1 \tag{14}$$

then $\nabla\lambda_j$ satisfies $K\nabla\lambda_j = \lambda_j \cdot J\nabla\lambda_j$. $G_0 = (v_x/w, -u_x/w)^T \in \mathrm{Ker}J$, hence the Bargmann constraint $G_0 = \sum_{j=0}^{N} \lambda_j \nabla\lambda_j$, i.e. (note $2ww_x + (uv)_x = 0$ and (12))

$$\left.\begin{aligned} v_x/w &= -<\Lambda^2 p, p> + (v/w) <\Lambda^2 p, q> = i <\Lambda p, p>_x /2w \\ u_x/w &= <\Lambda^2 q, q> + (u/w) <\Lambda^2 p, q> = i <\Lambda q, q>_x /2w \\ 2w_x &= v <\Lambda^2 q, q> + u <\Lambda^2 p, p> = i <\Lambda p, q>_x \end{aligned}\right\} \tag{15}$$

yields

$$u = -\frac{i}{2} <\Lambda q, q>, \quad v = \frac{i}{2} <\Lambda p, p>, \quad w = \frac{i}{2} <\Lambda p, q> \tag{16}$$

The Bargmann system {(12),(16)}

$$\left.\begin{aligned} q_x &= \frac{1}{2} <\Lambda p, q> \Lambda q - \frac{1}{2} <\Lambda q, q> \Lambda p = \partial H/\partial p \\ p_x &= \frac{1}{2} <\Lambda p, p> \Lambda q - \frac{1}{2} <\Lambda p, q> \Lambda p = -\partial H/\partial q \end{aligned}\right\} \tag{17}$$

has the Hamiltonian $H = F_1$ and the conserved integrals involutive in pairs F_m

$$\left.\begin{aligned} F_1 &= \frac{1}{4} <\Lambda p, q>^2 - \frac{1}{4} <\Lambda q, q><\Lambda p, p> \\ F_m &= \frac{1}{4} \sum_{j=0}^{m-1} \left(<\Lambda^{j+1} p, q><\Lambda^{m-j} p, q> - <\Lambda^{j+1} q, q><\Lambda^{m-j} p, p> \right) \end{aligned}\right\} \tag{18}$$

3. WKI eigenvalue problem[9]

$$y_x = My, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad M = \begin{pmatrix} -i\lambda & \lambda u \\ \lambda v & i\lambda \end{pmatrix}, \quad i^2 = -1 \tag{19}$$

$$\nabla\lambda_j = \begin{pmatrix} \delta\lambda_j/\delta u \\ \delta\lambda_j/\delta v \end{pmatrix} = \begin{pmatrix} \lambda_j\, p_j^2 \\ -\lambda_j\, q_j^2 \end{pmatrix}, \quad j = 1, 2, \cdots, N \tag{20}$$

$\nabla\lambda_j$ satisfies the linear equation

$$K\nabla\lambda_j = \lambda_j \cdot J\nabla\lambda_j$$

where K and J are the pair of Lenard's operators

$$K = \frac{1}{4i}\begin{pmatrix} -\partial^2(u/w)\partial^{-1}(u/w)\partial^2 & 2\partial^3 + \partial^2(u/w)\partial^{-1}(v/w)\partial^2 \\ 2\partial^3 + \partial^2(v/w)\partial^{-1}(u/w)\partial^2 & -\partial^2(v/w)\partial^{-1}(v/w)\partial^2 \end{pmatrix}$$
$$J = \begin{pmatrix} 0 & -\partial^2 \\ \partial^2 & 0 \end{pmatrix}, \quad w^2 + uv = 1$$

which are skew-symmetric.

$G_{-1} = (1,1)^T \in \mathrm{Ker}J$, $G_0 = J^{-1}KG_{-1} = (iv/w, iu/w)^T$, recursively define the Lenard's gradient sequence[12] $G_m : KG_{m-1} = JG_m$, $m = 0,1,2,\cdots$. The WKI hierarchy of soliton equations is expressed by

$$(u,v)^T_{t_m} = JG_m\ , \quad m = 0,1,2,\cdots \tag{21}$$

with the representative equation $(u,v)^T_{t_1} = JG_1$ which can be reduced the well-known HD equation

$$s_{t_1} = -(s^{-1/2})_{xxx} \tag{22}$$

as $v = -1$ and $1 + u = s$. The Lax representation of (21) has been secured in ref.[12].

The Bargmann constraint $G_0 = \sum_{j=0}^{N} \nabla\lambda_j$ gives

$$u = \frac{i < \Lambda q, q >}{\sqrt{1+ < \Lambda q, q >< \Lambda p, p >}}\ , \qquad v = \frac{-i < \Lambda p, p >}{\sqrt{1+ < \Lambda q, q >< \Lambda p, p >}} \tag{23}$$

The nonlinearized (19) under (23)

$$\left.\begin{aligned} q_x &= -i\Lambda q + \frac{i < \Lambda q, q >}{\sqrt{1+ < \Lambda q, q >< \Lambda p, p >}}\Lambda p = \partial H/\partial p \\ p_x &= i\Lambda p - \frac{i < \Lambda p, p >}{\sqrt{1+ < \Lambda q, q >< \Lambda p, p >}}\Lambda q = -\partial H/\partial q \end{aligned}\right\} \tag{24}$$

is a completely integrable finite-dimensional Hamiltonian system $(R^{2N}, dp\Lambda dq, H)$ in the Liouville's sense with

$$H = -i < \Lambda p, q > + i\sqrt{1+ < \Lambda q, q >< \Lambda p, p >} \tag{25}$$

whose involutive system of conserved integrals is

$$\begin{aligned} F_m =& - < \Lambda p, q >< \Lambda^m p, q > + \sqrt{1+ < \Lambda q, q >< \Lambda p, p >} < \Lambda^m p, q > \\ &+ \frac{1}{2}\sum_{j=0}^{m-1} \begin{vmatrix} < \Lambda^{j+1} q, q > & < \Lambda^{j+1} p, q > \\ < \Lambda^{m-j} p, q > & < \Lambda^{m-j} p, p > \end{vmatrix} \end{aligned} \tag{26}$$

Remark: $(H, F_m) = 0$ and $(F_m, F_n) = 0$ $(\forall m, n \in Z^+)$ can be shown through a series of caloulations.

4. Consider the eigenvalue problem

$$y_x = My\ , \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\ , \quad M = \begin{pmatrix} -iu & \lambda + v - (u^2 - v^2) \\ -\lambda + v + (u^2 + v^2) & iu \end{pmatrix}\ , \quad i^2 = -1 \tag{27}$$

which is a simple extension of the Dirac eigenvalue problem

$$y_x = My\ , \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\ , \quad M = \begin{pmatrix} -iu & \lambda + v \\ -\lambda + v & iu \end{pmatrix}$$

$$\nabla\lambda_j = \begin{pmatrix} \delta\lambda_j/\delta u \\ \delta\lambda_j/\delta v \end{pmatrix} = \begin{pmatrix} 2u(q_j^2 + p_j^2) + 2ip_jq_j \\ (q_j^2 - p_j^2) - 2v(q_j^2 + p_j^2) \end{pmatrix}\ , \quad j = 1,2,\cdots,N \tag{28}$$

$G_0 = (u, -v)^T$, hence the Bargmann constraint $G_0 = \sum_{j=0}^{N} \nabla \lambda_j$ yields

$$u = \frac{2i < p, q >}{1 - 2(< p, p > + < q, q >)}, \qquad v = \frac{< p, p > - < q, q >}{1 - 2(< p, p > + < q, q >)} \tag{29}$$

Under (29), (27) is nonlinearized to be as

$$\left.\begin{aligned} q_x =& \Lambda p + \frac{2 < p, q > q + (< p, p > - < q, q >)p}{1 - 2(< p, p > + < q, q >)} \\ &+ \frac{4 < p, q >^2 + (< p, p > - < q, q >)^2}{[1 - 2(< p, p > + < q, q >)]^2} p = \partial H / \partial p \\ p_x =& -\Lambda q + \frac{-2 < p, q > p + (< p, p > - < q, q >)q}{1 - 2(< p, p > + < q, q >)} \\ &- \frac{4 < p, q >^2 + (< p, p > - < q, q >)^2}{[1 - 2(< p, p > + < q, q >)]^2} q = -\partial H / \partial q \end{aligned}\right\} \tag{30}$$

which is an integrable Hamiltonian system $(R^{2N}, dp \wedge dq, H)$ in the Liouville's sense with

$$H = \frac{1}{2} < \Lambda p, p > + \frac{1}{2} < \Lambda q, q > + \frac{4 < p, q >^2 + (< p, p > - < q, q >)^2}{4 - 8(< p, p > + < q, q >)} \tag{31}$$

whose involutive system of conserved integrals is

$$\begin{aligned} F_m =& \frac{1}{2}(1 - 2(< p, p > + < q, q >))(< \Lambda^m p, p > + < \Lambda^m q, q >) \\ &+ \sum_{j=0}^{m} \begin{vmatrix} < \Lambda^{j-1} q, q > & < \Lambda^{j-1} p, q > \\ < \Lambda^{m-j} p, q > & < \Lambda^{m-j} p, p > \end{vmatrix} \quad m = 0, 1, 2, \cdots \end{aligned} \tag{32}$$

Remark $(H, F_m) = 0$, $(F_m, F_n) = 0$, $\forall m, n \in Z^+$.

Note: On the involutive solutions of the above four hierarchy of soliton equations, we shall solve them in the future paper.

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